LINEAR PROGRAMMING FORMULATION OF THE PROBLEMS IN HW 5

1. (Basketweavers University) Let us assign a decision variable x_i to course i, with the meaning that $x_i = 1$ if we take course i and $x_i = 0$ if we don't take course i. (I number the courses in the order of appearance in the problem, i.e. the first one is Calculus, the second one is Operations Research, etc.) The algebraic formulation is

$$
\begin{aligned}
\text{min} \quad & x_1 + x_2 + \cdots + x_7 \\
\text{s.t.} \quad & x_1 + x_2 + x_3 + x_4 + x_7 \geq 2 \qquad \text{(math requirement)} \\
& x_2 + x_4 + x_5 + x_7 \geq 2 \qquad \text{(OR requirement)} \\
& x_3 + x_5 + x_6 \geq 2 \qquad \text{(computer requirement)} \\
& x_1 \geq x_4 \qquad \text{(prerequisites)} \\
& x_6 \geq x_3 \qquad \vdots \\
& x_6 \geq x_5 \qquad \vdots \\
& x_4 \geq x_7 \qquad \text{(prerequisites)} \\
& x_i \text{ is binary} \qquad i = 1, 2, \ldots, 7.
\end{aligned}
$$

2. (Simon's Mall) Let us declare the following decision variables:

$$
x_{i,j} = \begin{cases} 1 & \text{if the number of stores of type } i \text{ is } j \\ 0 & \text{otherwise} \end{cases}.
$$

Here $i = 1, \ldots, 5$ means jewelry store, ..., clothing store, respectively, and j can be 1, 2, or 3. The algebraic formulation is

$$
\begin{aligned}
\text{min} \quad & 0.05 \cdot 10000\big(9x_{1,1} + 2 \cdot 8x_{1,2} + 3 \cdot 7x_{1,3} + 10 \cdot x_{2,1} + 2 \cdot 9x_{2,2} + \cdots + 3 \cdot 10x_{5,3}\big) \\
\text{s.t.} \quad & x_{i,1} + x_{i,2} + x_{i,3} = 1 \qquad i = 1, 2, 3, 5 \qquad \text{(bounds on number of stores)} \\
& x_{4,1} + x_{4,2} + x_{4,3} \leq 1 \qquad \text{(bound on number of bookstores)} \\
& 500\,(x_{1,1} + 2x_{1,2} + 3x_{1,3}) + \cdots + 900\,(x_{5,1} + 2x_{5,2} + 3x_{5,3}) \leq 10000 \qquad \text{(space)} \\
& x_{i,j} \text{ is binary} \qquad i = 1, 2, \ldots, 5; j = 1, 2, 3.
\end{aligned}
$$

3. (Easy Sudoku.) Decision variables:

$$
x_{i,j,k} = \begin{cases} 1 & \text{if in row } i \text{ and column } j \text{ we put } k \\ 0 & \text{otherwise} \end{cases}.
$$

Here i, j, and k are all in $1, \ldots, 9$.

The constraints: in every row and in every column and in every 3 by 3 "box" there is exactly one from each number. Moreover, each cell contains exactly one of the nine numbers. See the algebra below. There is no objective function, since there is nothing to optimize. If this is frustrating, we can optimize anything, for example minimize $x_{1,1,1}$, or whatever, subject to the constraints:.

$$
\begin{aligned}
\text{s.t.}\quad & x_{i,j,1} + \cdots + x_{i,j,9} = 1 && i = 1,\ldots,9; j = 1,\ldots,9 && \text{(cells)}\\
& x_{i,1,k} + \cdots + x_{i,9,k} = 1 && i = 1,\ldots,9; k = 1,\ldots,9 && \text{(rows)}\\
& x_{1,j,k} + \cdots + x_{9,j,k} = 1 && j = 1,\ldots,9; k = 1,\ldots,9 && \text{(columns)}\\
& x_{1,1,k} + \cdots + x_{3,3,k} = 1 && k = 1,\ldots,9; && \text{(upper left box)}\\
& \vdots \\
& x_{7,7,k} + \cdots + x_{9,9,k} = 1 && k = 1,\ldots,9; && \text{(lower right box)}\\
& x_{i,j,k} \text{ is binary} && i = 1,\ldots,9; j = 1,\ldots,9; k = 1,\ldots,9.
\end{aligned}
$$

Finally, of course, we need to add the constraints that some numbers are given as clues. For example, if there is a number 5 written in the second column of the fourth row, then $x_{4,2,5} = 1$.

4. (Harder Sudoku.) We will use the same constraints as above, but now there is an objective function. Given the initial solution that we started from, we can try to *maximize the number of those cells in a solution that contain a different number from the number that occupies the same cell in the initial solution.* If this number is zero, it means there is no other solution than the one we already knew about. If this number is positive, then our initial solution is not unique. This is clearly a linear objective function, as it is simply the sum of some of the variables. Algebraically (I'm using the sum notation here is to make things a lot easier):

$$\max \sum_i \sum_j \sum_{k:\, x_{i,j,k}=0} x_{i,j,k}$$

subject to the constraints of the previous problem.

An open problem: somebody asked me after class, what is the minimum number of clues we must give to make a Sudoku problem uniquely solvable. It seems that nobody knows the answer, but the conjecture is 17. (There are thousands of different problems with 17 clues only, and none with 16.) Unfortunately we cannot easily answer this problem by simply solving a simple integer programming problem. While it is tempting to write this question as "minimize the number of clues, subject to the puzzle being uniquely solvable", we cannot formulate the constraint "the puzzle is uniquely solvable" easily! This is quite surprising; compare it to the solution we just gave to problem 4, and see why that idea doesn't work to solve this open problem!