Performance Analysis and Design of Tandem Queues with Blocking

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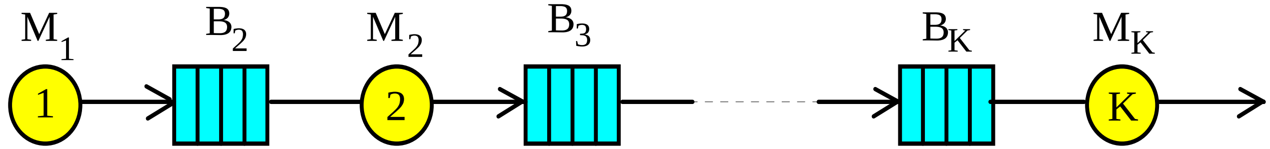
Analysis of Tandem Systems
(Queues in Series with Finite Buffers)



- Random processing times
- Finite-capacity buffers
- Blocking
- Starvation

$X_i =$ processing time at M_i

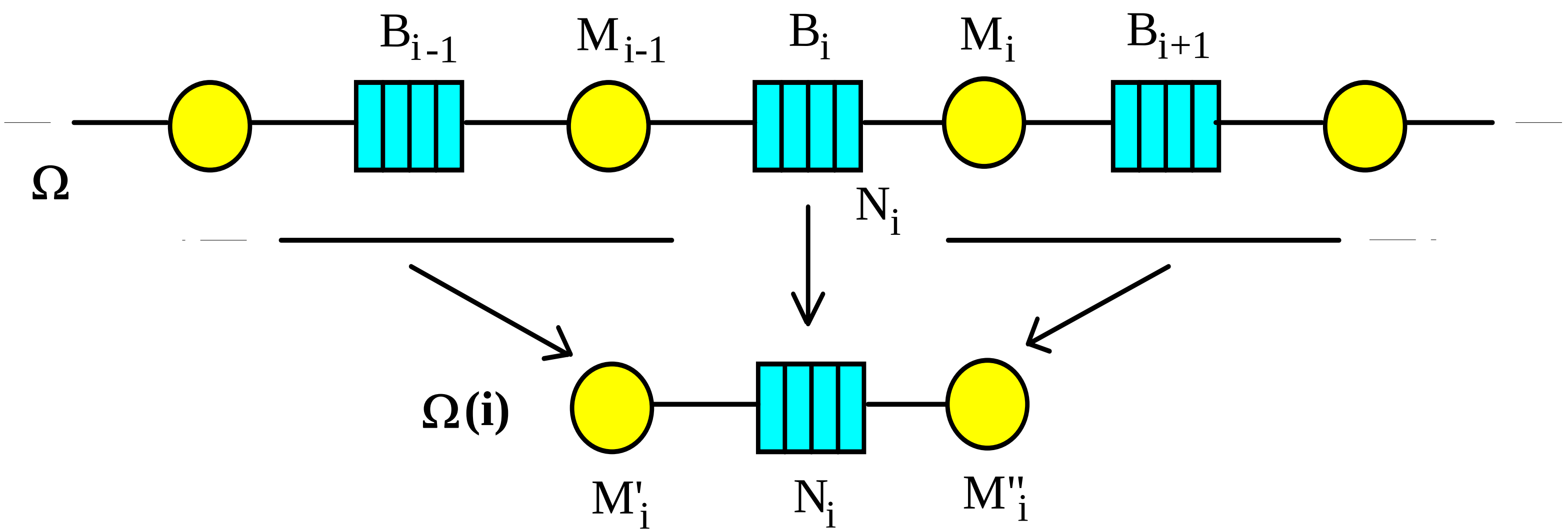
$$\{N_1(t), J_1(t); N_2(t), J_2(t); \ldots; N_K(t), J_K(t),\ t \geq 0\}$$

<u>Metrics of Interest:</u>

$$\begin{cases} \text{Throughput:} & \overline{o}_l \\ \text{Avg. WIP:} & \overline{N}_i \end{cases}$$

$P_i(0),\ P_i(B)$

Concept of Disaggregation



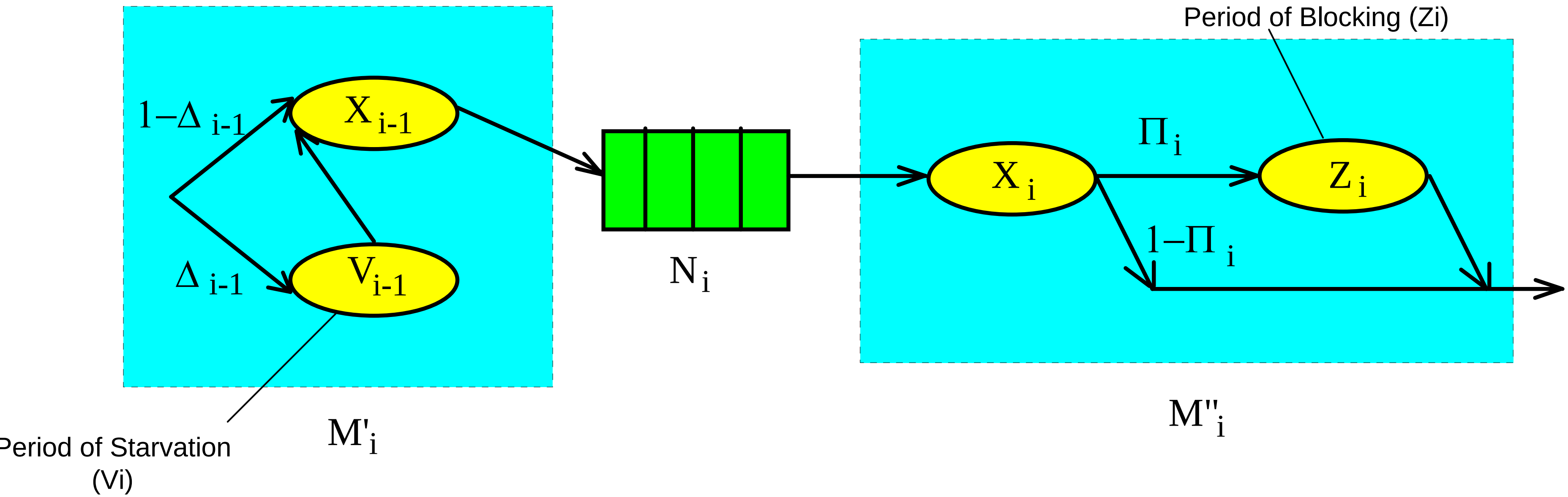
$$\Delta_i = \text{Pr(a departing job leaves } M_i \text{ empty and idle)}$$

$$V_i = \text{Length of the idle period at } M_i$$

$$\Pi_i = \text{Pr(a departing job blocks } M_i)$$

$$Z_i = \text{Length of period } \mathrm{M}_\mathrm{i} \text{ remains blocked}$$

Isolation of Buffers



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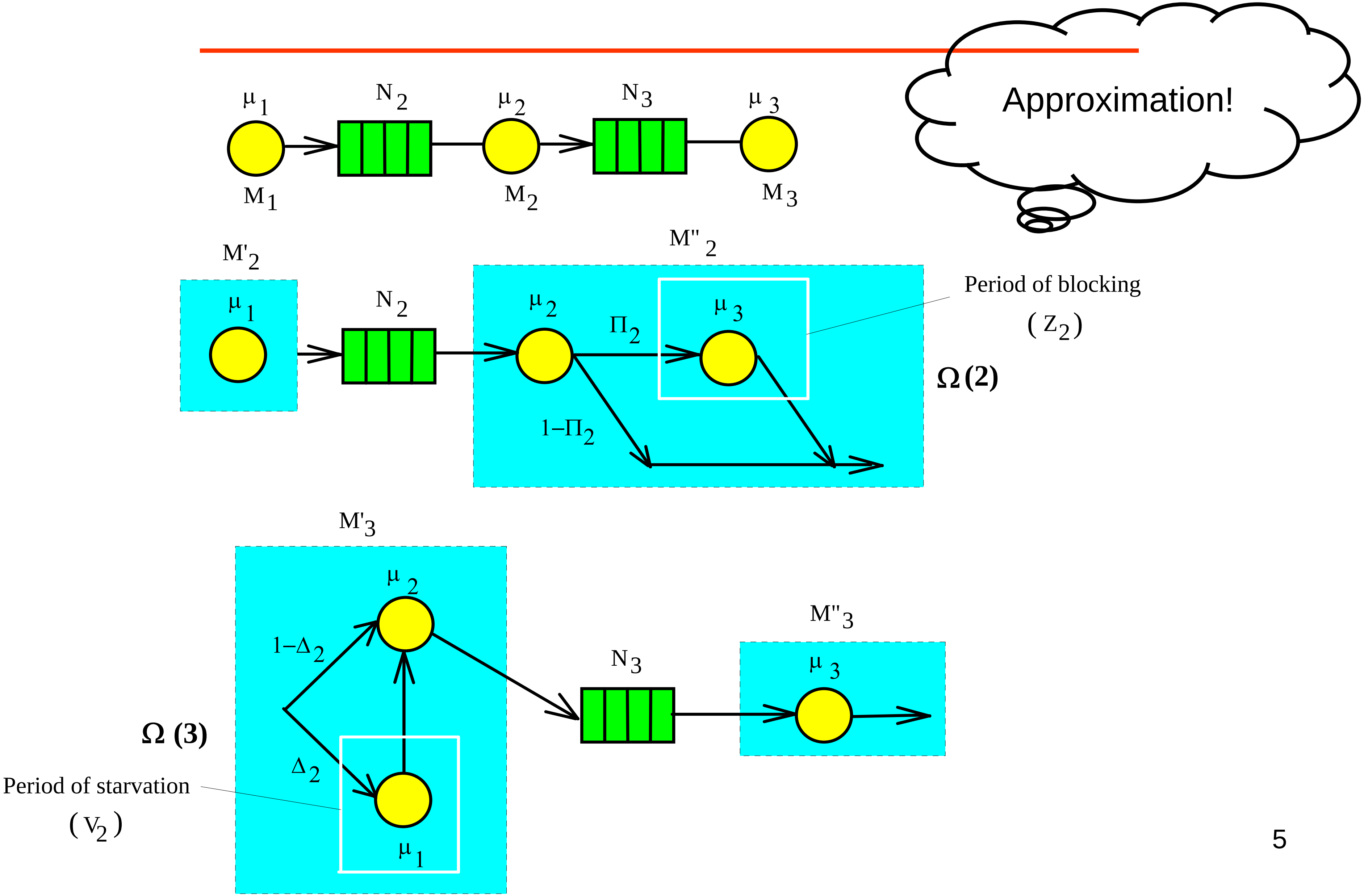
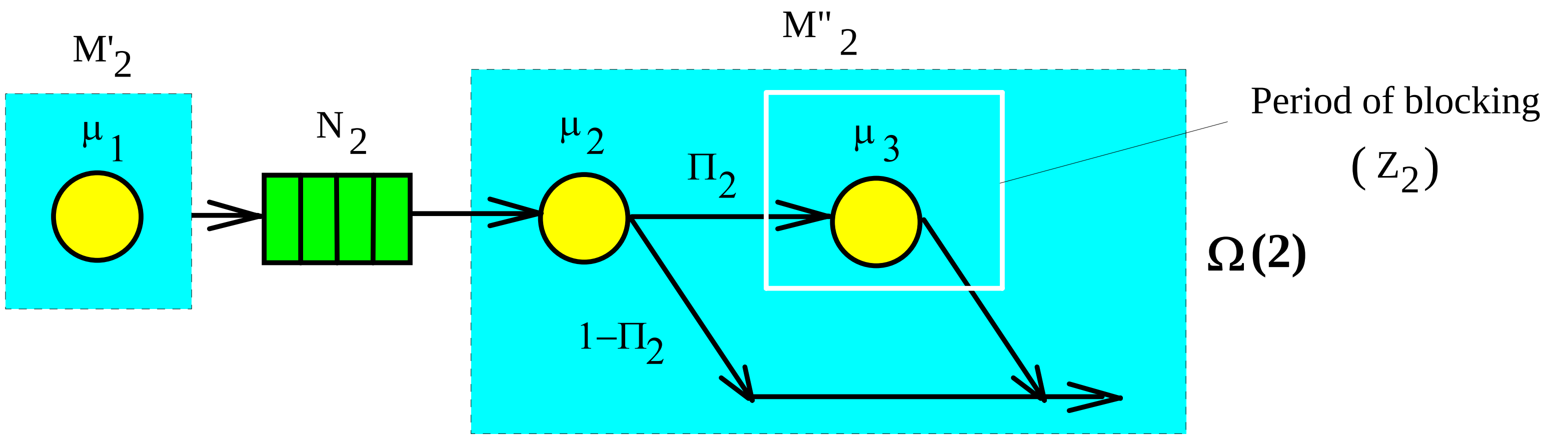
Two Nodes Decomposition: An Example



Ω(2)



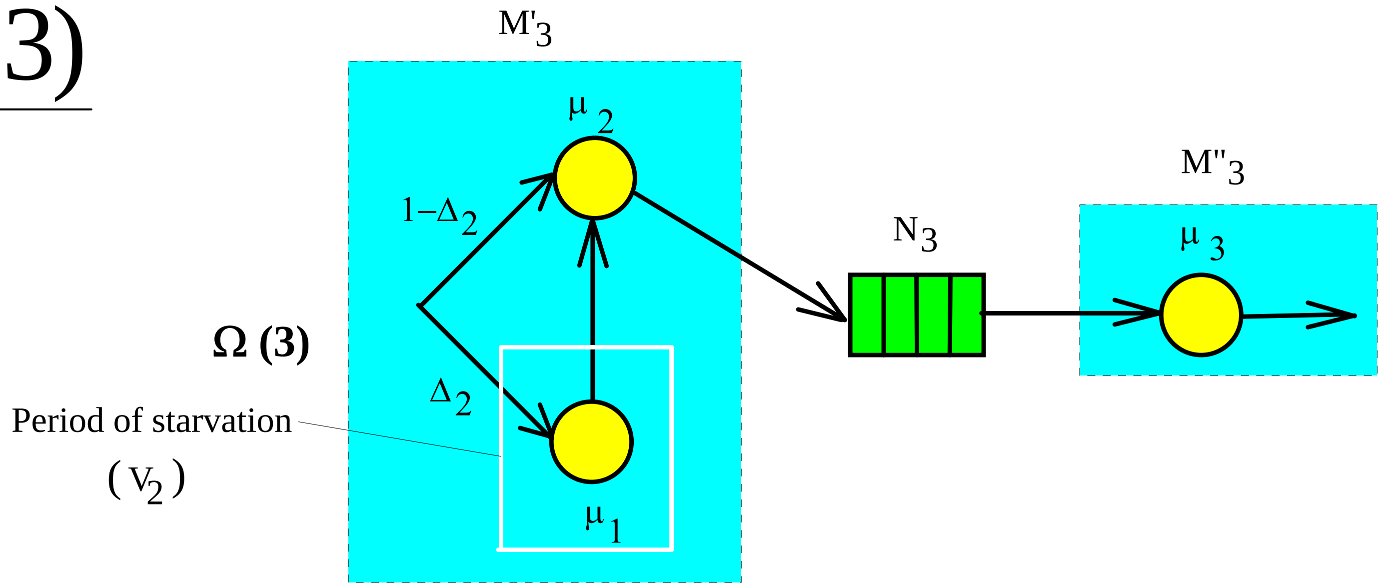
$\{N_2(t), J_2(t), t \geq 0\}$ is a continuous-time MC.

SS probabilities can be solved for using matrix-recursive methods.

$$P_2(n, j),\ n = 0,1,2,..,N_2, B, \quad j = 0,1,2$$

$$P_2(n) = \sum_j P_2(n, j)$$

$\Omega(3)$



$$\{J_2(t), N_3(t), t \geq 0\}$$ is a continuous-time MC.

$$P_3(j,n),\ j = 0,1,2,\ n = 0,1,2,..,N_3,B$$

$$P_3(n) = \sum_j P_3(j,n)$$

Linking $\Omega(2)$ & $\Omega(3)$

- Need to compute Δ_2 and Π_2

<u>Using Little's formula</u>:

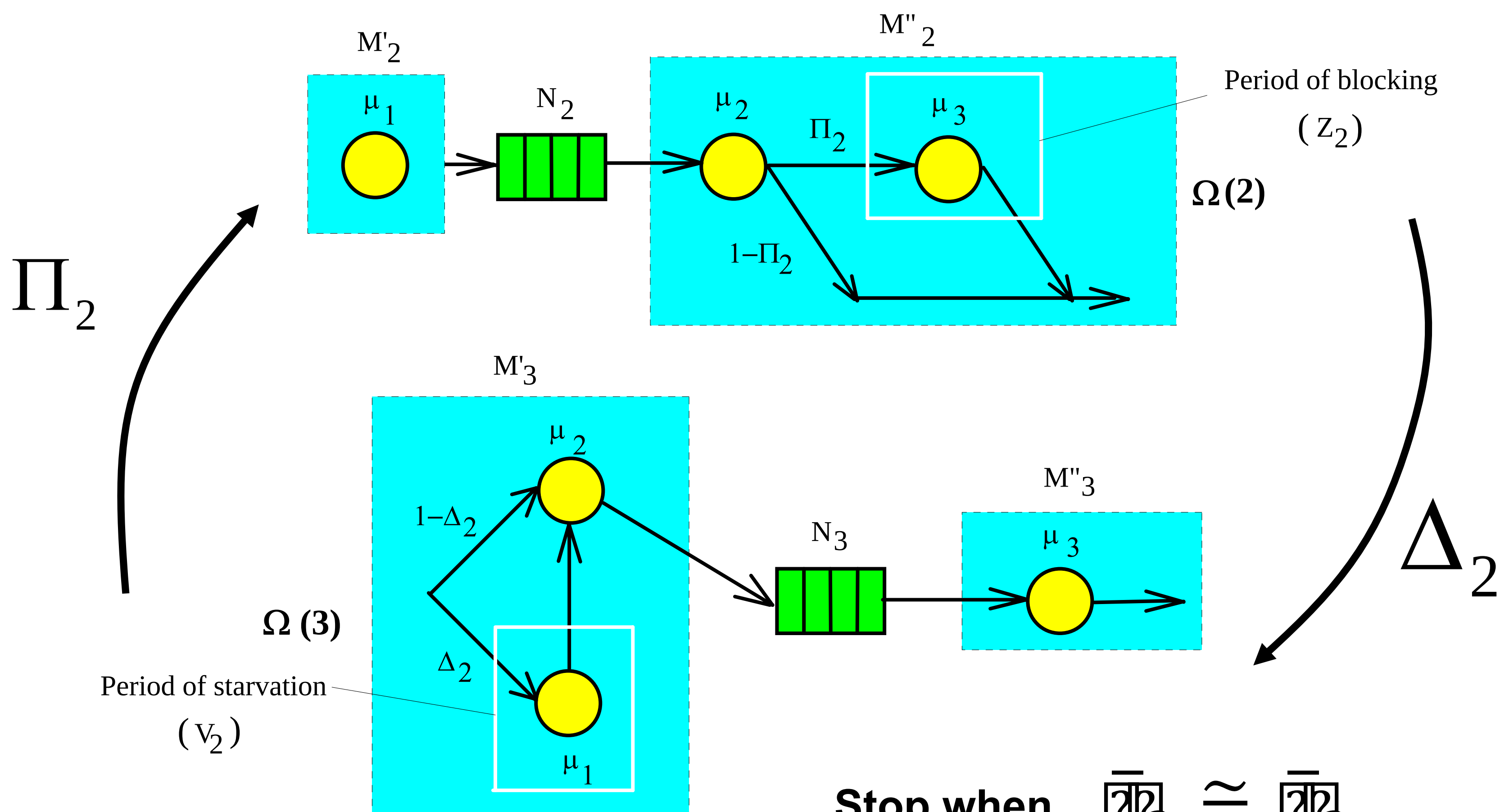
???? = ??

$$\Omega(3) \qquad \bar{o}_{\ell}\Pi_i E[Z_i] = P_i \ (B)$$

$$\Omega(2) \qquad \bar{o}_{\ell}\Delta_i E[V_i] = P_i(0),$$

The Iteration

$$\bar{o}_\ell \Delta_i E[V_i] = P_i(0),$$



Stop when $\bar{?}_2 \cong \bar{?}_3$

$$\bar{o}_\ell \Pi_i E[Z_i] = P_{i+1}(B)$$

Convergence of the Throughput

$K = 3$

$\mu_i = 2$

$N_i = 1$

Iteration No	Δ_2	Π_2	$\bar{o}_i$
0	0.1	0.1	-
1	0.47499	-	1.26984
2	-	0.38125	1.07744
3	0.40469	-	1.11986
4	-	0.39883	1.10895
5	0.40029	-	1.11165
6	-	0.39993	1.11098
7	0.40002	-	1.11115
8	-	0.39999	1.11110
9	0.40000	-	1.11111
10	-	0.39999	1.11111

Exact Throughput: 1.1282 [Hunt (1956)]

Numerical Results

5 station Decomposition:

Buffers	$\overline{N}_{App}$	$\overline{N}_{Sim}$	Rel. Err
			%
1	3.5095	3.5293	-0.56
2	2.2104	2.3098	-4.3
3	7.4080	6.9328	6.85
4	3.7905	3.7151	2.03

$$\underline{\mu} = (2.5, 4, 3, 2, 5)$$

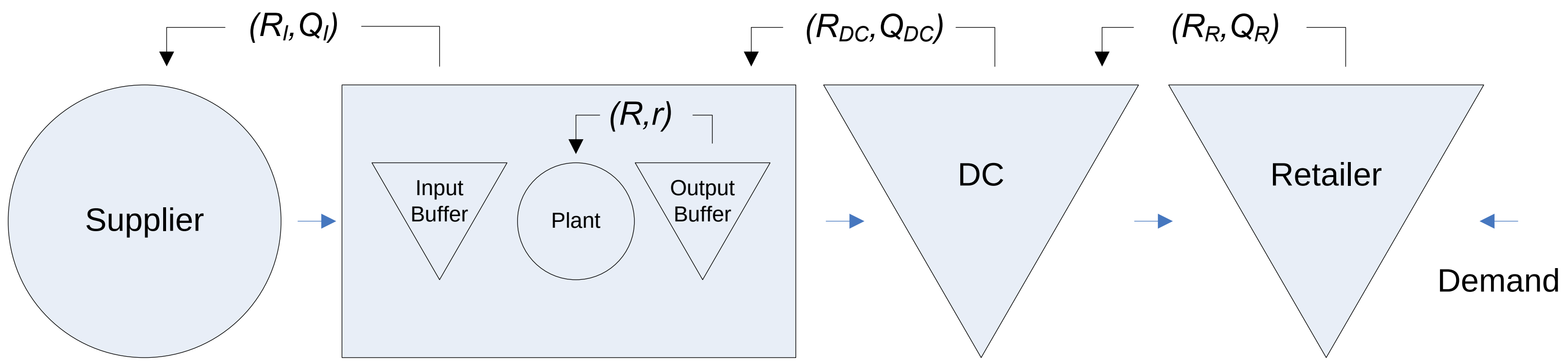
$$\underline{N} = (4, 4, 10, 4)$$

$$Cv^2 = (2.5, 0.25, 3.0, 0.3, 0.75)$$

$\overline{O}_l$

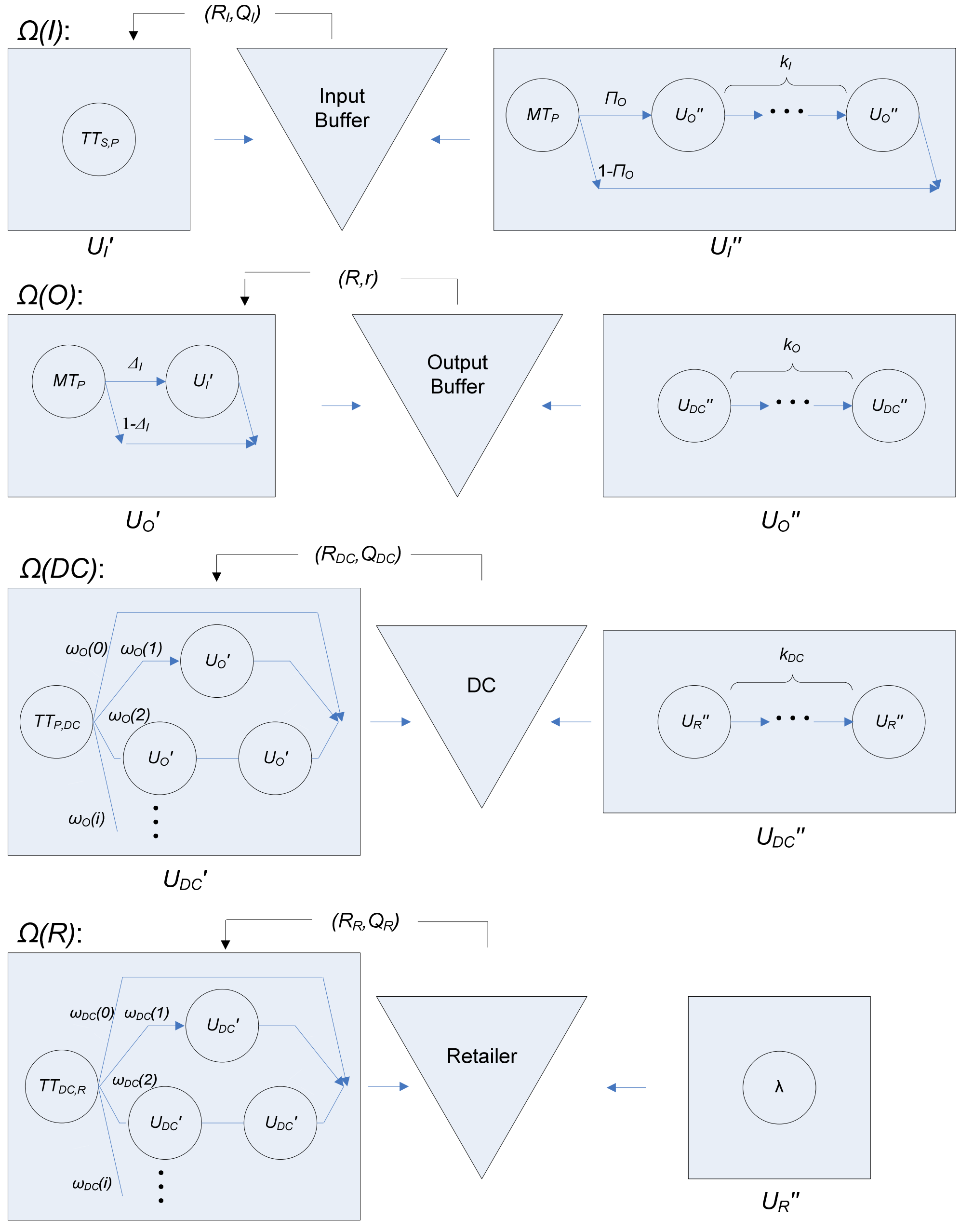
Appx=0.1984
Sim=0.1964

Supply Chain Applications

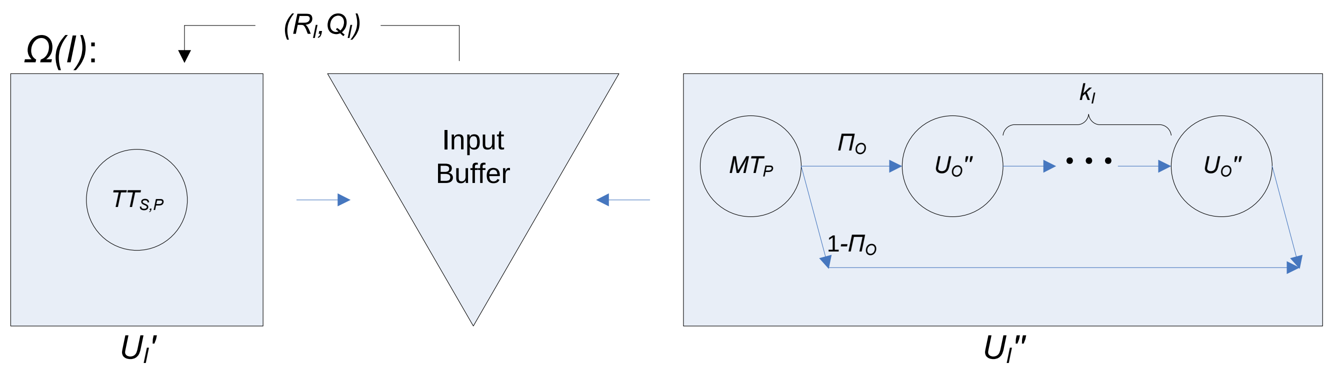


- Single-product
- (R,Q), (R,r) continuous review policies
- Backordering among echelons
- Several outstanding orders
- General transportation times (PH)

Decomposition



Steady-State Analysis of Subsystems



$$I_t = \begin{cases} i, & U_I' \text{ is in phase } i \quad i = 1, 2 \\ B, & U_I' \text{ is blocked} \end{cases}$$

$$J_t = \begin{cases} 0, & U_I'' \text{ is starving} \\ i, & U_I'' \text{ is in phase } i \end{cases}$$

$$N_t = 0, 1, 2, \ldots R_I + Q_I$$

$\{ I_t, J_t, N_t,\ t \geq 0 \}$ Markov chain with a finite number of states.

Computational Accuracy

- Backorder Case

		$R_I = 10$	$R = 30$	$\mu_1 = 2$	$R_{DC} = 10$	$R_R = 5$
Parameters:		$Q_I = 13$	$r = 10$	$\mu_2 = 1$	$Q_{DC} = 20$	$Q_R = 10$

Computational Accuracy

		λ=1.3			λ=1.5		
		Inv. Level	**BO Level**	**Cust. Sat.**	**Inv. Level**	**BO Level**	**Cust. Sat.**
Input Buffer	**Analytic**	14.3861	N/A	99.84%	13.9829	N/A	99.83%
	Simulation	14.3896	N/A	99.84%	13.9857	N/A	99.80%
	Rel. Error	-0.02%	N/A	0.00%	-0.02%	N/A	0.03%
Output Buffer	**Analytic**	20.2831	0.0515	95.14%	14.7011	1.4737	71.67%
	Simulation	20.3260	0.0495	95.30%	14.7971	1.3927	72.17%
	Rel. Error	-0.21%	4.04%	-0.17%	-0.65%	5.82%	-0.69%
DC	**Analytic**	22.2038	0.0004	99.98%	18.7423	0.7061	94.10%
	Simulation	22.2087	0.0035	99.92%	19.5314	0.4381	95.71%
	Rel. Error	-0.02%	-88.57%	0.06%	-4.04%	61.17%	-1.68%
Retailer	**Analytic**	7.9228	0.0528	96.78%	7.0957	0.3687	89.68%
	Simulation	7.9205	0.0537	96.75%	7.3054	0.3470	92.87%
	Rel. Error	0.03%	-1.68%	0.03%	-2.87%	6.25%	-3.43%

Designing Operational Policies

- Complex optimal control policy
  - Restrict the structure of the control policy
  - Use the simple reorder policies at all installations
- Coordinate simple decision rules in the best possible way
  - Minimize steady-state expected total system-wide costs
- Decision variables: (R_I, Q_I), (R, r), (R_{DC}, Q_{DC}), (R_R, Q_R)

Designing Operational Policies

Total Cost: Set-up, holding, backordering, shortage cost

- SC: Set-up cost per order
- h: unit holding cost per unit time
- g: unit backordering cost per unit time
- p: shortage cost per unit short
- λ: mean rate of demand

Min Total Cost (Entire System)

$$TC(\underline{Q},\underline{R}) = \sum_{k} \frac{SC \times \lambda}{Q} + hE[I] + gE[B] + p\lambda \, \mathrm{Pr(Backorder)}$$

s.t. *RI, QI, R, r, RDC, QDC, RR, QR integer*

Optimization

Example

				Input	Output	DC	Retailer
	$\mu_1 = 2$	$\beta_S = 1$	**SC**	30	25	20	15
$\lambda=1.5$	$\mu_2 = 1$	$\beta_P = 1$	**h**	*0.2*	0.4	0.6	0.8
	$a = 0.1$	$\beta_{DC} = 1$	**g**		0.6	0.4	0.2
			p	100	10	50	25

Optimization

Example

Input		Output		DC		Retailer		Cost					
Q_I	R_I	R	r	Q_{DC}	R_{DC}	Q_R	R_R	Input	Output	DC	Retailer	TOTAL	Fill Rate
13	10	30	10	20	10	10	5	9.3077	12.8896	17.4546	11.8722	51.524	89.67%
14	9	31	11	20	11	11	6	8.9591	12.892	15.5895	11.5487	48.9892	
		35	13	22	12				12.9953	16.3616		49.8646	
15	8	35	13	22	12	12	5	8.6927	13.108	16.4506	10.9812	49.2324	
		37	13	24	10				13.3169	17.219		50.2099	
16	7	37	13	24	9	12	5	8.4961	13.4786	16.9707	10.9483	49.8937	
		37	13	24	7				13.4786	15.8918		48.8148	
17	6	38	14	24	6	12	5	8.3861	13.7273	15.7524	11.1456	49.0115	
		38	14	24	4				13.7273	14.688		47.9471	
17	6	38	14	24	3	12	5	8.3821	13.7285	14.159	11.1474	47.417	
		38	14	24	1				13.7285	13.0948		46.3528	
17	6	38	14	24	1	12	5	8.3821	13.7285	13.0948	11.1474	46.3528	
		38	14	24	1				13.7285	13.0948		46.3528	
													Fill Rate
17	6	38	14	24	1	12	5	8.3821	13.7285	13.0948	11.1474	46.3528	93.05%

Any Questions?