Polynomially computable sharp probability bounds

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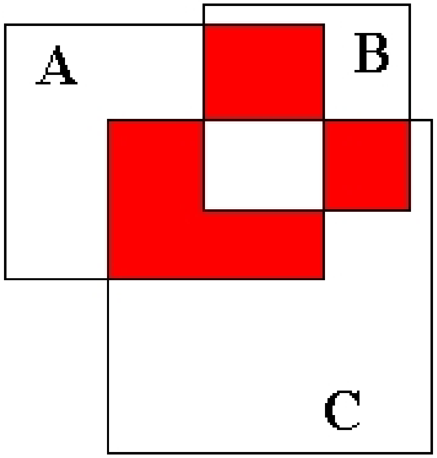
Many More Happy Birthdays Andras!

Joint work with **Andrea Scozzari**, **Fabio Tardella**, and **Pierangela Veneziani**.

Boole's Problem



$$\mathbf{E_1} = (\mathbf{A} \cap \mathbf{B} \cap \overline{\mathbf{C}}) \cup (\mathbf{A} \cap \overline{\mathbf{B}} \cap \mathbf{C}) \cup (\overline{\mathbf{A}} \cap \mathbf{B} \cap \mathbf{C}) \qquad Prob(\mathbf{E_1}) = \frac{1}{3}$$

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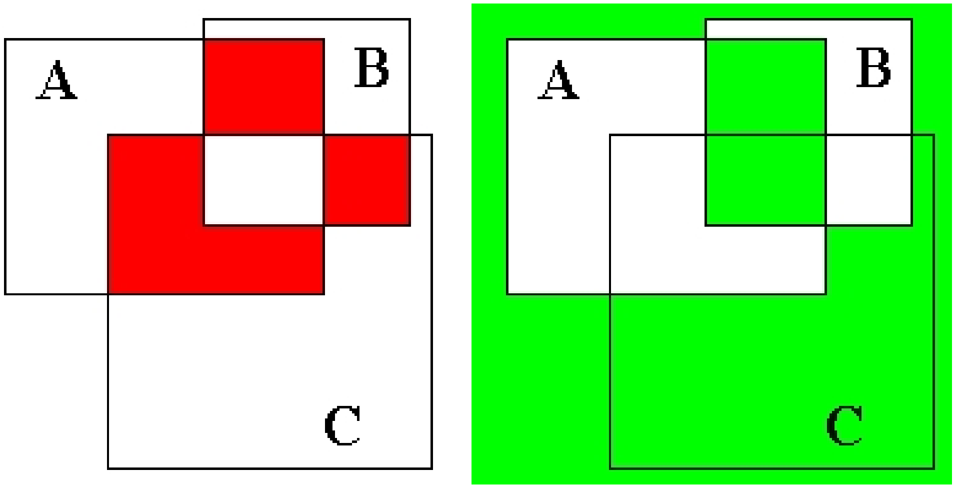

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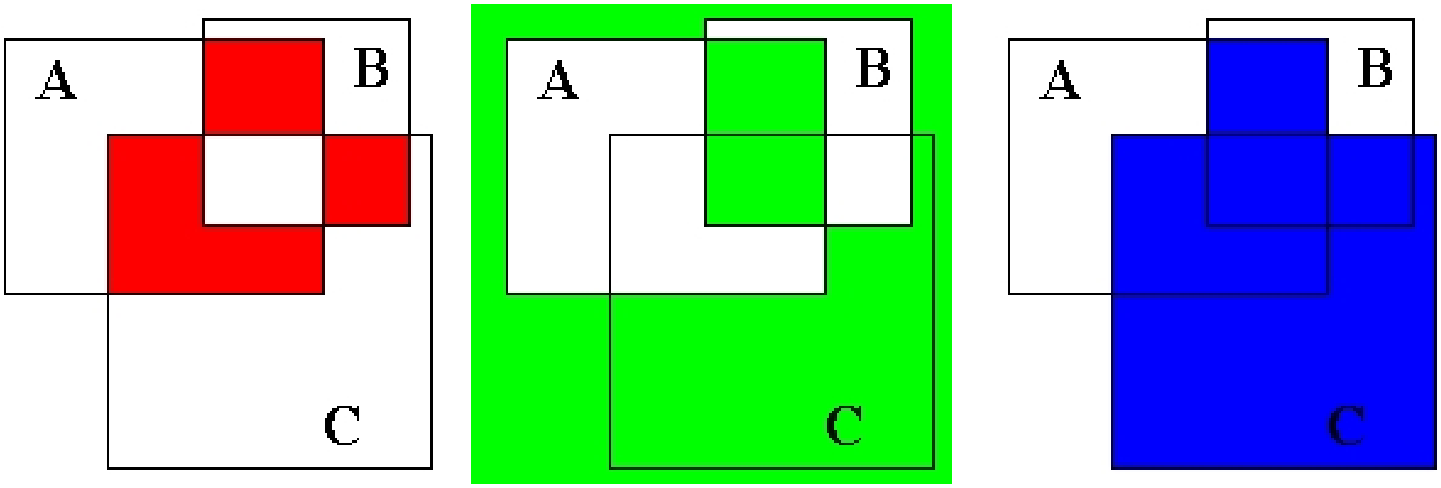

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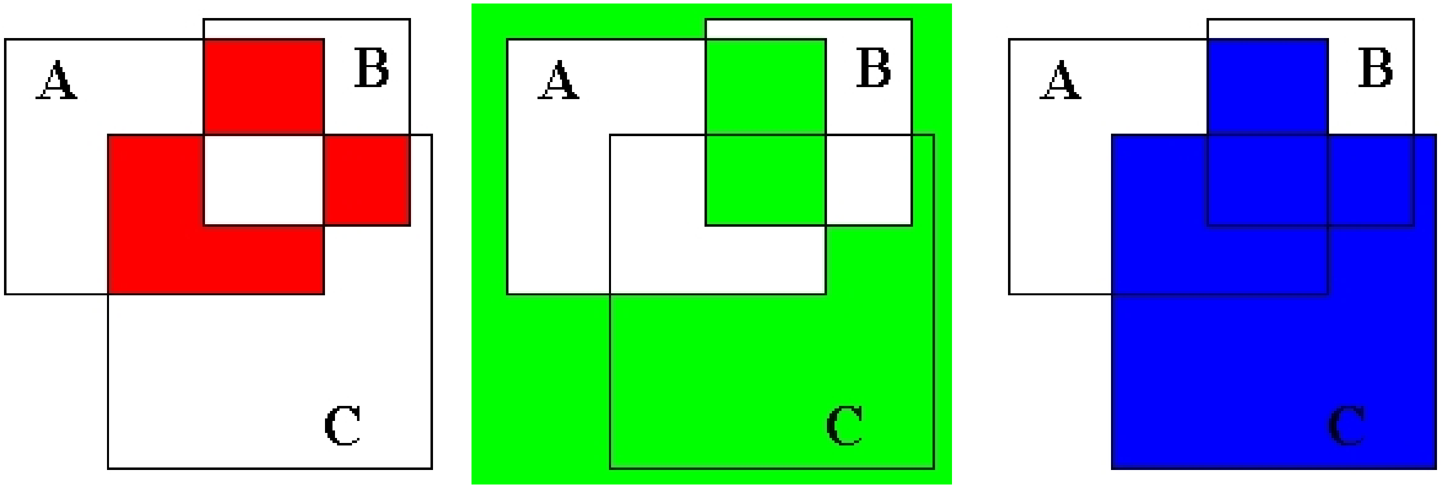

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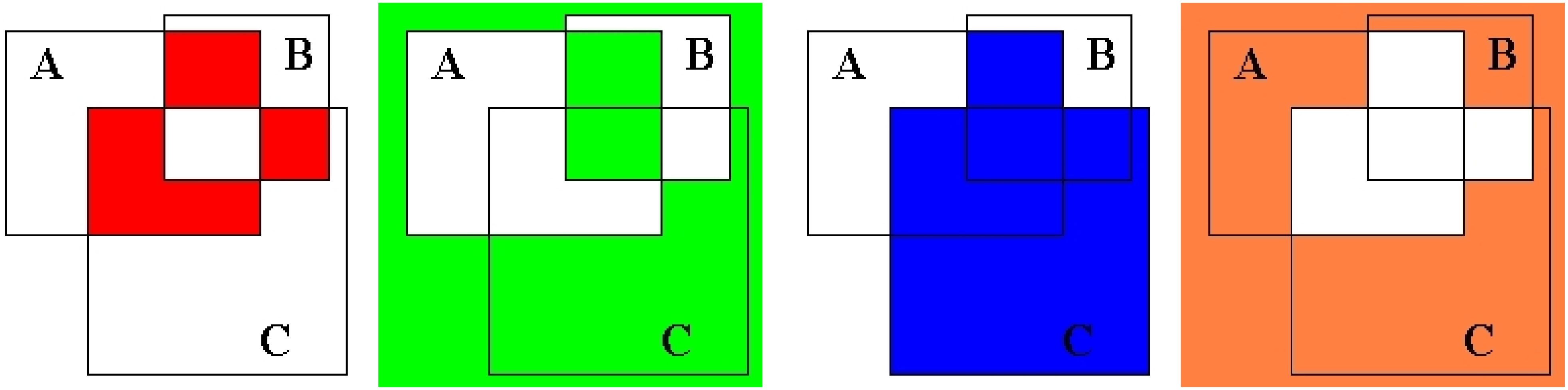
Is this possible?

Boole's Problem



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$\mathbf{E}_4 = (\overline{\mathbf{A}} \cap \overline{\mathbf{B}}) \cup (\overline{\mathbf{A}} \cap \overline{\mathbf{C}}) \cup (\overline{\mathbf{B}} \cap \overline{\mathbf{C}})$ $\qquad Prob(\mathbf{E}_4) = ?$

How large (small) can $Prob(\mathbf{E}_4)$ be?

Brief History

- Boole's Problem (Boole 1854, 1868 (1850))

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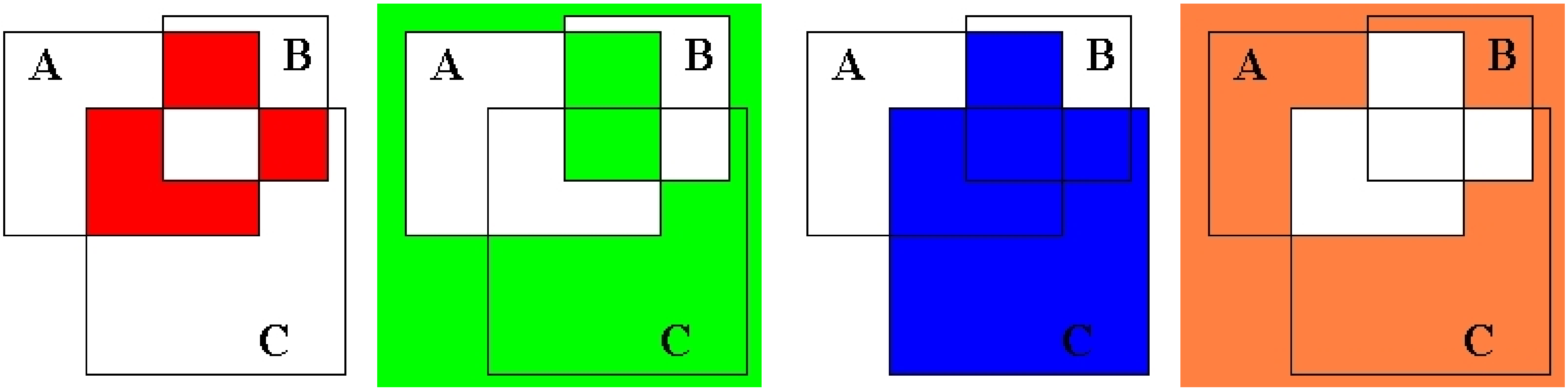
- Linear programming formulation (Hailperin 1965)

Linear Programming Formulation



$$\mathbf{x}_0 = Prob(\overline{\mathbf{A}} \cap \overline{\mathbf{B}} \cap \overline{\mathbf{C}}), \qquad \mathbf{x}_1 = Prob(\mathbf{A} \cap \overline{\mathbf{B}} \cap \overline{\mathbf{C}}), \qquad \cdots, \qquad \mathbf{x}_7 = Prob(\mathbf{A} \cap \mathbf{B} \cap \mathbf{C})$$

$\emptyset$	$\{\mathbf{A}\}$	$\{\mathbf{B}\}$	$\{\mathbf{C}\}$	$\{\mathbf{A},\mathbf{B}\}$	$\{\mathbf{A},\mathbf{C}\}$	$\{\mathbf{B},\mathbf{C}\}$	$\{\mathbf{A},\mathbf{B},\mathbf{C}\}$		
$\mathbf{x}_0$	$+\mathbf{x}_1$	$+\mathbf{x}_2$	$+\mathbf{x}_3$					$\longrightarrow$	$\left\{\begin{matrix}\text{max}\\ \text{min}\end{matrix}\right\}$
$\mathbf{x}_0$	$+\mathbf{x}_1$	$+\mathbf{x}_2$	$+\mathbf{x}_3$	$+\mathbf{x}_4$	$+\mathbf{x}_5$	$+\mathbf{x}_6$	$+\mathbf{x}_7$	$=$	1
				$+\mathbf{x}_4$	$+\mathbf{x}_5$	$\mathbf{x}_6$		$=$	$\frac{1}{3}$
$\mathbf{x}_0$			$+\mathbf{x}_3$	$+\mathbf{x}_4$			$+\mathbf{x}_7$	$=$	$\frac{1}{2}$
			$+\mathbf{x}_3$	$+\mathbf{x}_4$	$+\mathbf{x}_5$	$+\mathbf{x}_6$	$+\mathbf{x}_7$	$=$	$\frac{5}{6}$
							$\mathbf{x}_j$	$\geq$	$0, \ j = 0, ..., 8$

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- Boole's Problem (Boole 1854, 1868 (1850))
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  Potentially exponential number of variables!
- Probabilistic logic (Nilsson 1986)
- Probabilistic satisfiability (PSAT)
  (Georgakopoulos, Kavvadias and Papadimitriou 1988)
  Feasibility is **NP-hard!**
  What about optimization with feasible input?

A Special Case: Union of Events

Events: $\mathbf{A}_i \subseteq \Omega$, $i \in \mathbf{V} = \{1, 2, ..., n\}$

Input: $p_I = Prob\left(\bigcap_{i \in I} \mathbf{A}_i\right)$ for $I \subseteq \mathbf{V}$, $|I| \leq m$, $(p_\emptyset = 1)$

Problem: Find lower and upper bounds for the probability of the union of these n events:

$$LB\left(p_I \mid I \subseteq \mathbf{V}, |I| \leq m\right) \leq Prob\left(\bigcup_{i \in \mathbf{V}} \mathbf{A}_i\right) \leq UB\left(p_I \mid I \subseteq \mathbf{V}, |I| \leq m\right)$$

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Remark [Fréchet 1935]: For $m = 1$ the bounds

$$\max_{1\leq i\leq n} Prob(\mathbf{A}_i) \quad \leq Prob\left(\bigcup_{i\in\mathbf{V}} \mathbf{A}_i\right) \leq \quad \min\left\{1, \sum_{i=1}^{n} Prob(\mathbf{A}_i)\right\}$$

are **sharp**.

Linear Programming Formulation

Events: $\mathbf{A}_i \subseteq \Omega$, $i \in \mathbf{V} = \{1, 2, ..., n\}$

Input: $p_I = Prob\,(\bigcap_{i \in I} \mathbf{A}_i)$ for $I \subseteq \mathbf{V}$, $|I| \leq m$, $(p_\emptyset = 1)$

Variables: $\mathbf{x}_J = Prob\left(\left(\bigcap_{i \in J} \mathbf{A}_i\right) \cap \left(\bigcap_{i \notin J} \overline{\mathbf{A}}_i\right)\right)$ for $J \subseteq \mathbf{V}$

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With this notation we have

$$Prob\left(\bigcup_{i \in \mathbf{V}} \mathbf{A}_i\right) = \sum_{\emptyset \neq J \subseteq \mathbf{V}} \mathbf{x}_J$$

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With this notation we have

$$Prob\left(\bigcup_{i \in \mathbf{V}} \mathbf{A}_i\right) = \sum_{\emptyset \neq J \subseteq \mathbf{V}} \mathbf{x}_J$$

and

$$p_I = \sum_{\mathbf{V} \supseteq J \supseteq I} \mathbf{x}_J \qquad \text{for all } I \subseteq \mathbf{V}, |I| \leq m$$

Linear Programming Formulation

Let us define

$$LB_m^* = \min \sum_{\emptyset \neq J \subseteq \mathbf{V}} \mathbf{x}_J \quad \text{and} \quad UB_m^* = \max \sum_{\emptyset \neq J \subseteq \mathbf{V}} \mathbf{x}_J$$

subject to the constraints

$$\sum_{\mathbf{V} \supseteq J \supseteq I} \mathbf{x}_J = p_I \quad \text{for all } I \subseteq \mathbf{V}, |I| \leq m$$

$$\mathbf{x}_J \geq 0 \quad \text{for all } J \subseteq \mathbf{V}$$

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Claim: Bounds for the probability of any other event defined in terms of $\mathbf{A}_1$, $\mathbf{A}_2$, ..., $\mathbf{A}_n$ can be computed from a similar LP formulation, in which only the objective function will be different.

E.g. "at least r out of these n events occur", "at most q out of these n events occur", etc.

Linear Programming Formulation

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Claim [Hailperin 1965]: The bounds $LB_m^* = LB\,(p_I \mid I \subseteq \mathbf{V}, |I| \leq m)$ and $UB_m^* = UB\,(p_I \mid I \subseteq \mathbf{V}, |I| \leq m)$ are **sharp**.

From the optimal solutions of these linear programs one can construct examples for which $Prob(\bigcup_{i \in \mathbf{V}} \mathbf{A}_i)$ attains these bounds. $(\sum_{J \subseteq \mathbf{V}} \mathbf{x}_J = p_\emptyset = 1)$

Linear Programming Formulation

Claim: Computing LB_m^* and UB_m^* **maybe hard**!

- Feasibility is **NP-hard**.
  (Georgakopoulos, Kavvadias and Papadimitriou 1988)

- Exponentially many variables in LP formulation!

- Column generation (row generation in dual LP) is an **NP-hard** sub-problem (even for $m = 2$).
  (Jaumard, Hansen and Poggi de Aragão 1991)

Relaxation I: Aggregation

Summing up equations for $I \subseteq \mathbf{V}$, $|I| = k$ (for $k = 0, 1, ..., m$), and introducing new variables $y_j = \sum_{J \subseteq \mathbf{V},\ |J|=j} \mathbf{x}_J$ (for $j = 0, 1, ..., n$) yields a **relaxation**, the so called Binomial Moment Problem (Prékopa 1988):

$$\widetilde{LB}_m = \min \sum_{j \geq 1} y_j \quad \text{and} \quad \widetilde{UB}_m = \max \sum_{j \geq 1} y_j$$

subject to the constraints

$$\sum_{j=0}^{n} \binom{j}{k} y_j = \mathbf{S}_k \qquad \text{for } k = 0, 1, ..., m$$

$$y_j \geq 0 \qquad \text{for } j = 0, 1, ..., n$$

$\mathbf{S}_k = \sum_{\substack{I \subseteq \mathbf{V} \\ |I|=k}} p_I$ is called the k-th binomial moment of $\mathbf{A}_1$, $\mathbf{A}_2$, ..., $\mathbf{A}_n$.

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If ξ is the random variable denoting the number of events occurring from $\{\mathbf{A}_1, \mathbf{A}_2, ..., \mathbf{A}_n\}$, then

$$\mathbf{S}_k = Exp\left[\binom{\xi}{k}\right].$$

Binomial Moment Problem

- $\widetilde{LB}_m \leq LB^*_m$ and $\widetilde{UB}_m \geq UB^*_m$ are polynomially computable sharp bounds (sharp in terms of $\{\mathbf{S}_k \mid k \leq m\}$, but may not be sharp in terms of $\{p_I \mid I \subseteq \mathbf{V},\ |I| \leq m\}$).

- Dual feasible basic solutions are characterized (Prékopa 1988) $\rightsquigarrow$ closed form optimal solutions for $m \leq 4$ (Prékopa 1988; B and Prékopa 1989)

- Several closed form bounds in the literature are of this type, or generalized by these type of bounds.

Binomial Moments Based Bounds

- $\mathbf{S}_1 - \mathbf{S}_2 + \cdots - \mathbf{S}_{2s} \leq \widetilde{LB}_{2s}$ and $\widetilde{UB}_{2s+1} \leq \mathbf{S}_1 - \mathbf{S}_2 + \cdots + \mathbf{S}_{2s+1}$

  (Bonferroni 1937)

- $\widetilde{LB}_2 \quad \geq \quad \frac{\mathbf{S}_1^2}{\mathbf{S}_1 + 2\mathbf{S}_2}$ (Chung and Erdös 1952)

- $\widetilde{LB}_2 = \frac{2}{i+1}\mathbf{S}_1 - \frac{2}{i(i+1)}\mathbf{S}_2, \qquad i = 1 + \lfloor \frac{2\mathbf{S}_2}{\mathbf{S}_1} \rfloor$

  (Dawson and Sankoff 1967; Kwerel 1975; Galambos 1977)

- $\widetilde{UB}_2 = \mathbf{S}_1 - \frac{2}{n}\mathbf{S}_2$

  (Kwerel 1975; Sathe, Pradhan and Shah 1980; Platz 1985)

Binomial Moments Based Bounds

- $\widetilde{LB}_3 = \frac{i+2n-1}{(i+1)n}\mathbf{S}_1 - \frac{2(2i+n-2)}{i(i+1)n}\mathbf{S}_2 + \frac{6}{i(i+1)n}\mathbf{S}_3$,

  where $i = 1 + \lfloor \frac{2(n-2)\mathbf{S}_2 - 6\mathbf{S}_3}{(n-1)\mathbf{S}_1 - 2\mathbf{S}_2} \rfloor$, and

- $\widetilde{UB}_3 = \mathbf{S}_1 - \frac{2(2i-1)}{i(i+1)}\mathbf{S}_2 + \frac{6}{i(i+1)}\mathbf{S}_3$,

  where $i = 1 + \lfloor \frac{3\mathbf{S}_3}{\mathbf{S}_2} \rfloor$

(Kwerel 1975; B and Prékopa 1989)

- $\widetilde{UB}_4 = \mathbf{S}_1 - \frac{2((i-1)(i-2)+(2i-1)n)}{i(i+1)n}\mathbf{S}_2 + \frac{6(2i+n-4)}{i(i+1)n}\mathbf{S}_3 - \frac{24}{i(i+1)n}\mathbf{S}_4$,

  where $i = 1 + \lfloor \frac{2(n-2)\mathbf{S}_2 + 3(n-5)\mathbf{S}_3 - 12\mathbf{S}_4}{(n-2)\mathbf{S}_2 - 3\mathbf{S}_3} \rfloor$

(B and Prékopa 1989)

Stronger Lower Bounds

- $LB^*_{m=2} \geq \sum_{i=1}^{n} \alpha_i p_i$, where $p_i = P(\mathbf{A}_i)$, $p_{i,j} = P(\mathbf{A}_i \cap \mathbf{A}_j)$

  and $\sum_{j \neq i} \alpha_j p_{i,j} = (1 - \alpha_i) p_i$ for $i = 1, ..., n$.

  (Gallot 1966; Kounias 1968)

- $LB^*_2 \geq \sum_{i \in \mathbf{V}} \frac{p_i^2}{p_i + \sum_{j \neq i} p_{i,j}} \geq \widetilde{LB}_2$, (de Caen 1997)

- $LB^*_2 \geq \sum_{i \in \mathbf{V}} \left(\frac{\theta_i p_i^2}{(2 - \theta_i) p_i + \sum_{j \neq i} p_{i,j}} + \frac{(1 - \theta_i) p_i^2}{(1 - \theta_i) p_i + \sum_{j \neq i} p_{i,j}} \right) \geq \widetilde{LB}_2$,

  where $\theta_i = \frac{\sum_{j \neq i} p_{i,j}}{p_i} - \lfloor \frac{\sum_{j \neq i} p_{i,j}}{p_i} \rfloor$

  (Kuai, Alajai and Takahara 2000)

Stronger Bounds by Graph Structures

- $UB_2^* \le \mathbf{S}_1 - \sum_{i \neq k} p_{i,k}$ (k is fixed) (Kounias 1968)

- $UB_2^* \le \mathbf{S}_1 - \sum_{(i,j) \in T} p_{i,j} \le \widetilde{UB}_2$,

  where T is a spanning tree (Hunter 1976; Worsley 1982)

- $UB_3^* \le \mathbf{S}_1 - \sum_{(i,j) \in \mathcal{E}} p_{i,j} + \sum_{(i,j,k) \in \mathcal{C}} p_{\{i,j,k\}} \le \widetilde{UB}_3$,

  where $(\mathcal{E}, \mathcal{C})$ is a cherry tree (Bukszár and Prékopa 2001)

Aggregations and Graph Structures

- Several stronger lower and upper bounds, generalizing the previous ones, were derived recently via **partial aggregation**: considering linear combinations instead of the original equations, and introducing new variables, which are linear functions of the original variables in order to obtain a **polynomially sized relaxation**.

  (Prékopa, Vizvári, Regős and Gao 2001; Prékopa and Gao 2001)

- Improved Bonferroni inequalities via binomially bounded functions.

  (Dohmen and Tittmann, 2007)

- Chordal graph bound ($m = 3$, (Veneziani, 2002)) and chordal graph sieve ($m = \chi(G)$, (Dohmen, 2002)).

- Upper bounds for $m = 3$ via graph structures (positive $p_{i,j}$ effect)

  (Veneziani, 2002, 2008)

Tightening the Dual

- **Relaxing** of an LP has the same effect on its optimum value as **tightening** of its dual.

Tightening the Dual

- **Relaxing** of an LP has the same effect on its optimum value as **tightening** of its dual.

- Replace **polynomial resizing** by **efficient tightening** of the dual:

  Try to tighten the dual so that row generation (separation) becomes polynomially solvable. (Size of the formulation may not decrease!)

Simplify LP

Eliminate $x_\emptyset$ and the normalization $\sum_{j \subseteq \mathbf{V}} x_J = 1$, and then dualize.

$$\begin{aligned} \sum_{\emptyset \neq J \subseteq \mathbf{V}} \mathbf{x}_J &\rightarrow \left\{ \begin{array}{c} \text{max} \\ \text{min} \end{array} \right\} \\ \sum_{J \supseteq I} \mathbf{x}_J &= p_I, \;\; I \subseteq \mathbf{V}, |I| \leq m \\ \mathbf{x}_J &\geq 0, \;\; J \subseteq \mathbf{V} \end{aligned} \quad \equiv \quad \begin{aligned} \sum_{\emptyset \neq J \subseteq \mathbf{V}} \mathbf{x}_J &\rightarrow \left\{ \begin{array}{c} \text{max} \\ \text{min} \end{array} \right\} \\ \sum_{\emptyset \neq J \supseteq I} \mathbf{x}_J &= p_I, \;\; \emptyset \neq I \subseteq \mathbf{V}, |I| \leq m \\ \mathbf{x}_J &\geq 0, \;\; \emptyset \neq J \subseteq \mathbf{V} \end{aligned}$$

Simplify LP

Eliminate $x_\emptyset$ and the normalization $\sum_{j\subseteq \mathbf{V}} x_J = 1$, and then dualize.

$$
\begin{array}{lll}
\displaystyle\sum_{\emptyset\neq J\subseteq \mathbf{V}} \mathbf{x}_J & \to \left\{\begin{array}{c}\text{max}\\ \text{min}\end{array}\right\} & \\
\displaystyle\sum_{J\supseteq I} \mathbf{x}_J & = p_I, & I\subseteq \mathbf{V}, |I|\le m \\
\mathbf{x}_J & \ge 0, & J\subseteq \mathbf{V}
\end{array}
\quad \equiv \quad
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\quad \equiv \quad
\begin{array}{lll}
\left\{\begin{array}{c}\text{min}\\ \text{max}\end{array}\right\} & \leftarrow & \displaystyle\sum_{\substack{I\subseteq \mathbf{V}\\ 1\le|I|\le m}} p_I w_I \\
w(S) & \left\{\begin{array}{c}\ge\\ \le\end{array}\right\} & 1, \ \emptyset\neq S\subseteq \mathbf{V}
\end{array}
$$

Here $w(S)=\sum_{I\subseteq S} w_I$ and $w=(w_I \mid 1\le|I|\le m)$.

Tighten-up the Dual

Recall that $w = (w_I \mid 1 \le |I| \le m)$ and $w(S) = \sum_{I \subseteq S} w_I$ for all subsets $S \subseteq \mathbf{V}$.

$$\left\{ \begin{array}{c} \text{min} \\ \text{max} \end{array} \right\} \sum_{\substack{I \subseteq \mathbf{V} \\ 1 \le |I| \le m}} p_I w_I \quad = \quad \left\{ \begin{array}{c} UB_m(\mathcal{F}) \\ LB_m(\mathcal{F}) \end{array} \right\}$$

$$w(S) \left\{ \begin{array}{c} \ge \\ \le \end{array} \right\} 1 \quad \emptyset \ne S \subseteq \mathbf{V}$$

$$w \quad \in \quad \mathcal{F},$$

where $\mathcal{F}$ is a polyhedral set.

Tighten-up the Dual

$$\left\{\begin{array}{c}\text{min}\\\text{max}\end{array}\right\} \sum_{\substack{I\subseteq \mathbf{V}\\ 1\leq |I|\leq m}} p_I w_I \quad = \quad \left\{\begin{array}{c}UB_m(\mathcal{F})\\LB_m(\mathcal{F})\end{array}\right\}$$

$$w(S) \left\{\begin{array}{c}\geq\\\leq\end{array}\right\} 1 \quad \emptyset\neq S\subseteq \mathbf{V}$$

$$w \quad \in \quad \mathcal{F}.$$

Observation 1. If membership in $\mathcal{F}$ can be checked and for all $w\in\mathcal{F}$ the setfunction $w(S)$ can be minimized (resp. maximized) over $S\subseteq\mathbf{V}$ in polynomial time, then $UB_m(\mathcal{F})$ (resp. $LB_m(\mathcal{F})$) can be computed in polynomial time.

Tighten-up the Dual

$$\left\{ \begin{array}{c} \min \\ \max \end{array} \right\} \sum_{\substack{I \subseteq \mathbf{V} \\ 1 \leq |I| \leq m}} p_I w_I \quad = \quad \left\{ \begin{array}{c} UB_m(\mathcal{F}) \\ LB_m(\mathcal{F}) \end{array} \right\}$$

$$w(S) \quad \left\{ \begin{array}{c} \geq \\ \leq \end{array} \right\} \quad 1 \quad \emptyset \neq S \subseteq \mathbf{V}$$

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Observation 2. $LB_m(\mathcal{F}) \leq LB_m^* \leq Prob\left(\bigcup_{i=1}^{n} A_i\right) \leq UB_m^* \leq UB_m(\mathcal{F})$

Submodular Bounds

Set $M = \sum_{i=1}^{m} \binom{n}{i}$, and

$$\mathcal{F}_{sub} \;=\; \{w \in \mathbb{R}^M \mid w(S) \text{ is submodular } \}$$

$$w(S) \;=\; w(X^S) \;=\; \sum_{1 \leq |I| \leq m} w_I \prod_{j \in I} X_j$$

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Recognition is **polynomial** for $m \le 3$ (Billionet and Minoux, 1985)
NP-hard for $m > 3$ (Gallo and Simeone, 1989)

Corollary. For $m \le 3$ the upper bound $UB_m(\mathcal{F}_{sub})$ can be computed in polynomial time (by network flow models).

Nonpositive Bounds

$$\mathcal{F}_N \; = \; \{w \in \mathbb{R}^M \mid w_I \leq 0 \text{ for all } I \text{ with } 1 < |I| \leq m\}$$

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Observation 1. $UB_m(\mathcal{F}_N) = \max Prob(\mathbf{A_1} \cup \mathbf{A_2} \cup \cdots \cup \mathbf{A_n})$ for events s.t.

$$Prob(\mathbf{A_i}) \;=\; p_{\{i\}} \text{ for } i \in \mathbf{V}$$

$$Prob\left(\bigcap_{i \in I} \mathbf{A_i}\right) \;\geq\; p_I \quad \text{for } I \subseteq \mathbf{V},\; 1 < |I| \leq m.$$

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Observation 2. $\mathcal{F}_N \subseteq \mathcal{F}_{sub}$ for all $m \geq 1$ (and $\mathcal{F}_N = \mathcal{F}_{sub}$ for $m = 2$).

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Observation 3. Membership in $\mathcal{F}_N$ is linear time testable.

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Corollary. $UB_m(\mathcal{F}_N)$ can be computed in polynomial time for all $m \ge 1$.

$UB_2(\mathcal{F}_N) = UB_2(\mathcal{F}_{sub})$

$$\sum_{i=1}^{n} p_i w_i^1 + \sum_{1 \le i < j \le n} p_{ij} w_{ij}^2 \;\to\; \text{min} \;=\; UB_2(\mathcal{F}_N) = UB_2(\mathcal{F}_{sub})$$

$$w^1(S) + w^2(S) \;\ge\; 1 \qquad \text{for all } S \subseteq \mathbf{V},\; S \neq \emptyset,$$

$$w^2 \;\le\; 0$$

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Theorem. The vertices of the feasible region are of the form $(w^1, w^2) = (\mathbf{1}, X^T)$ where T is a spanning tree of the complete graph on $\mathbf{V}$.

$UB_2(\mathcal{F}_N) = UB_2(\mathcal{F}_{sub})$

$$\begin{aligned}
\sum_{i=1}^{n} p_i w_i^1 + \sum_{1\le i<j\le n} p_{ij} w_{ij}^2 \;\rightarrow\; \text{min} \quad &= \quad UB_2(\mathcal{F}_N) = UB_2(\mathcal{F}_{sub}) \\
w^1(S) + w^2(S) \quad &\ge \quad 1 \qquad \text{for all } S \subseteq \mathbf{V},\; S \ne \emptyset, \\
w^2 \quad &\le \quad 0
\end{aligned}$$

Theorem. The vertices of the feasible region are of the form $(w^1, w^2) = (\mathbf{1}, X^T)$ where T is a spanning tree of the complete graph on $\mathbf{V}$.

Corollary. $UB_2(\mathcal{F}_N) = UB_2(\mathcal{F}_{sub}) = UB_2(HW)$, where $UB_2(HW)$ is the **best** Hunter-Worsley bound.

These bounds are sharper than any other known upper bound for $m = 2$.

Upper Bounds

- $\widetilde{UB}_{2s+1} \leq \mathbf{S}_1 - \mathbf{S}_2 + \cdots + \mathbf{S}_{2s+1}$ (Bonferroni 1937)

- $UB_2^* \leq \sum_{i=1}^{n} p_i - \sum_{i \neq k} p_{i,k}$ (k is fixed) (Kounias 1968)

- $\widetilde{UB}_2 = \mathbf{S}_1 - \frac{2}{n}\mathbf{S}_2$ (Kwerel 1975; Sathe, Pradhan and Shah 1980; Platz 1985)

- $\widetilde{UB}_3 = \mathbf{S}_1 - \frac{2(2i-1)}{i(i+1)}\mathbf{S}_2 + \frac{6}{i(i+1)}\mathbf{S}_3, \quad \left(i = 1 + \lfloor \frac{3\mathbf{S}_3}{\mathbf{S}_2} \rfloor\right)$ (Kwerel 1975; B and Prékopa 1989)

- $UB_2^* \leq \mathbf{S}_1 - \sum_{(i,j)\in T} p_{i,j} \leq \widetilde{UB}_2,$ where T is a spanning tree

  (Hunter 1976; Worsley 1982)

- $\widetilde{UB}_4 = \mathbf{S}_1 - \frac{2i^2 - i(6-4n)+4-2n}{i(i+1)n}\mathbf{S}_2 + \frac{6(2i+n-4)}{i(i+1)n}\mathbf{S}_3 - \frac{24}{i(i+1)n}\mathbf{S}_4, \quad \left(i = 1 + \lfloor \frac{2(n-2)\mathbf{S}_2 + 3(n-5)\mathbf{S}_3 - 12\mathbf{S}_4}{(n-2)\mathbf{S}_2 - 3\mathbf{S}_3} \rfloor\right)$ (B and Prékopa 1989)

- $UB_3^* \leq \mathbf{S}_1 - \sum_{(i,j)\in\mathcal{E}} p_{i,j} + \sum_{(i,j,k)\in\mathcal{C}} p_{\{i,j,k\}} \leq \widetilde{UB}_3,$ where $(\mathcal{E},\mathcal{C})$ is a cherry tree

  (Bukszár and Prékopa 2001)

$UB_2(\mathcal{F}_N) = UB_2(\mathcal{F}_{sub})$

Observation. $UB_2(\mathcal{F}_N) \neq UB_2^*$

There are infinitely many examples where in the optimum we have $w^2_{i,j} > 0$ for some i, j.

Decomposition Bounds

$$\mathcal{F}_{dec} = \left\{ w \in \mathbb{R}^M \;\middle|\; \begin{array}{c} w_I = \sum_{i \in I} u_i^{|I|} \quad \forall\ I \subseteq \mathbf{V},\ 1 \leq |I| \leq m \\ \text{for some } u^1, \ldots, u^m \in \mathbb{R}^n \end{array} \right\}$$

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Observation. Membership in $\mathcal{F}_{dec}$ can be tested in polynomial time.

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Theorem. For $w \in \mathcal{F}_{dec}$ the setfunction $w(S)$ can be minimized (or maximized) over $S \subseteq \mathbf{V}$ in $O(n^2 \log n)$ time.

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Theorem. For $w \in \mathcal{F}_{dec}$ the setfunction $w(S)$ can be minimized (or maximized) over $S \subseteq \mathbf{V}$ in $O(n^2 \log n)$ time.

Corollary. For any $m \geq 1$ the bounds $LB_m(\mathcal{F}_{dec})$ and $UB_m(\mathcal{F}_{dec})$ can be computed in polynomial time.

Decomposition Bounds

$$\mathcal{F}_{dec} = \left\{ w \in \mathbb{R}^M \;\middle|\; \begin{array}{c} w_I = \sum_{i \in I} u_i^{|I|} \quad \forall\ I \subseteq \mathbf{V},\ 1 \leq |I| \leq m \\ \text{for some } u^1, \ldots, u^m \in \mathbb{R}^n \end{array} \right\}$$

$$w(S) = \sum_{k=1}^{m} \binom{|S|-1}{k-1} \sum_{i \in S} u_i^k$$

Decomposition Bounds $- UB_2(\mathcal{F}_{dec})$

$$w(S) \;=\; u^1(S) + (|S|-1)u^2(S)$$

$$\sum_{i \in \mathbf{V}} \left(p_i u_i^1 + u_i^2 \sum_{j \neq i} p_{ij} \right) \;\to\; \min \;= UB_2(\mathcal{F}_{dec})$$

$$u^1(S) + (|S|-1)u^2(S) \;\geq\; 1 \qquad \forall\; S \subseteq \mathbf{V},\; S \neq \emptyset$$

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Theorem. The vertices of the feasible region are the vectors $(u^1, u^2) = (\mathbf{1}, -\mathbf{e}_i)$ for $i = 1, ..., n$, where $\mathbf{e}_i$ is the i^{th} unit vector.

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Theorem. The vertices of the feasible region are the vectors $(u^1, u^2) = (\mathbf{1}, -\mathbf{e}_i)$ for $i = 1, ..., n$, where $\mathbf{e}_i$ is the i^{th} unit vector.

Corollary. $UB_2(\mathcal{F}_{dec}) \;=\; \sum_{i\in\mathbf{V}} p_i - \max_{i\in\mathbf{V}} \sum_{j\neq i} p_{ij} \;=\; UB_2(Ko)$, where $UB_2(Ko)$ is the **best** bound of the type introduced by Kounias (1968).

Summary

- $LB_m(\mathcal{F}_{dec})$ dominates all known lower bounds.

- $UB_2(\mathcal{F}_N) = UB_2(HW)$ dominates all known upper bounds for $m = 2$.

- $UB_m(\mathcal{F}_{dec})$ and $UB_m(\mathcal{F}_{sub})$ are incomparable, and dominate all known polynomially computable upper bounds.

  - $UB_3(\mathcal{F}_{sub})$, $UB_3(\mathcal{F}_{dec})$ and $UB_3(V)$ are incomparable.

  - $UB_3(V)$ of Veneziani (2002, 2008) dominates $UB_3(BP)$ of Bukszár and Prékopa (2001), and $UB_3(D)$ of Dohmen (2002); none of these are known to be polynomially computable.