Some Observations on Boolean Logic and Optimization

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Outline

- Logic and cutting planes
- Logic of 0-1 inequalities
- Logic and linear programming
- Inference duality
- Constraint programming
- Good logic models

Logic and Cutting Planes

Logic and cutting planes

Theorem (Quine). The resolution method generates all **prime implicates** of a set of logical clauses.

Prime implicates = undominated implications.

Logic and cutting planes

Theorem (Quine). The resolution method generates all **prime implicates** of a set of logical clauses.

Prime implicates = undominated implications.

This means that resolution is a complete inference method for clauses.

Logic and cutting planes

Theorem (Quine). The resolution method generates all **prime implicates** of a set of logical clauses.

$$x_1 \vee \bar{x}_2$$
$$\bar{x}_1 \vee x_3$$
$$x_2 \vee x_3$$

Example

Logic and cutting planes

Theorem (Quine). The resolution method generates all **prime implicates** of a set of logical clauses.

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Resolve on x_1

Logic and cutting planes

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$$x_2 \vee x_3$$
$$\bar{x}_2 \vee x_3$$

Resolve on x_2

Logic and cutting planes

Theorem (Quine). The resolution method generates all **prime implicates** of a set of logical clauses.

$$\begin{array}{l} x_1 \vee \bar{x}_2 \\ \bar{x}_1 \vee x_3 \\ x_2 \vee x_3 \end{array} \qquad \begin{array}{l} x_1 \vee \bar{x}_2 \\ \bar{x}_1 \vee x_3 \\ \boxed{\begin{array}{l} x_2 \vee x_3 \\ \bar{x}_2 \vee x_3 \end{array}} \end{array} \qquad \begin{array}{l} x_1 \vee \bar{x}_2 \\ \bar{x}_1 \vee x_3 \\ x_2 \vee x_3 \\ \bar{x}_2 \vee x_3 \\ x_3 \end{array}$$

Resolve on x_2

Logic and cutting planes

Theorem (Quine). The resolution method generates all **prime implicates** of a set of logical clauses.

$$
\begin{array}{lll}
x_1 \vee \bar{x}_2 & & \\
\bar{x}_1 & & \vee x_3 \\
 & x_2 & \vee x_3
\end{array}
\qquad
\begin{array}{lll}
x_1 \vee \bar{x}_2 & & \\
\bar{x}_1 & & \vee x_3 \\
 & x_2 & \vee x_3 \\
 & \bar{x}_2 & \vee x_3
\end{array}
\qquad
\begin{array}{lll}
x_1 \vee \bar{x}_2 & & \\
\bar{x}_1 & & \vee x_3 \\
 & x_2 & \vee x_3 \\
 & \bar{x}_2 & \vee x_3 \\
 & & x_3
\end{array}
$$

Drop redundant clauses

Logic and cutting planes

Theorem (Quine). The resolution method generates all **prime implicates** of a set of logical clauses.

$x_1 \vee \bar{x}_2$	$x_1 \vee \bar{x}_2$	$x_1 \vee \bar{x}_2$	$x_1 \vee \bar{x}_2$
$\bar{x}_1 \vee x_3$	$\bar{x}_1 \vee x_3$	$\bar{x}_1 \vee x_3$	
$x_2 \vee x_3$	$x_2 \vee x_3$	$x_2 \vee x_3$	
	$\bar{x}_2 \vee x_3$	$\bar{x}_2 \vee x_3$	
		x_3	x_3
		Drop redundant clauses	

Logic and cutting planes

Theorem (Quine). The resolution method generates all **prime implicates** of a set of logical clauses.

$$
\begin{aligned}
&x_1 \vee \bar{x}_2 \\
&\bar{x}_1 \vee x_3 \\
&x_2 \vee x_3
\end{aligned}
$$

$$
\begin{aligned}
&x_1 \vee \bar{x}_2 \\
&\bar{x}_1 \vee x_3 \\
&x_2 \vee x_3 \\
&\bar{x}_2 \vee x_3
\end{aligned}
$$

$$
\begin{aligned}
&x_1 \vee \bar{x}_2 \\
&\bar{x}_1 \vee x_3 \\
&x_2 \vee x_3 \\
&\bar{x}_2 \vee x_3 \\
&x_3
\end{aligned}
$$

$$
\begin{aligned}
&x_1 \vee \bar{x}_2 \\
&x_3
\end{aligned}
$$

Prime implicates remain

Logic and cutting planes

Theorem (Chvátal). Every cutting plane for a 0-1 system $Ax \geq b$ can be generated by repeatedly taking nonnegative linear combinations and rounding up.

This might be regarded as the fundamental theorem of cutting plane theory.

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A key step of the proof uses the **resolution method**.

Logic and cutting planes

Theorem (Chvátal). Every cutting plane for a 0-1 system $Ax \geq b$ can be generated by repeatedly taking nonnegative linear combinations and rounding up.

This might be regarded as the fundamental theorem of cutting plane theory.

A key step of the proof uses the **resolution method**.

This suggests there are deep connections between resolution and cutting planes.

Logic and cutting planes

A resolution step generates a **rank 1 cut** (i.e., a cut generated by one step of Chvátal's method).

$$\begin{array}{lll} x_1 \vee \bar{x}_2 & & \\ \bar{x}_1 & \vee x_3 \end{array} \qquad \begin{array}{rrrl} x_1 & + (1-\bar{x}_2) & & \geq 1 \\ (1-x_1) & & + x_3 & \geq 1 \end{array}$$

Convert to 0-1 inequalities

Logic and cutting planes

A resolution step generates a **rank 1 cut** (i.e., a cut generated by one step of Chvátal's method).

$$x_1 \vee \bar{x}_2$$
$$\bar{x}_1 \vee x_3$$

$$\begin{array}{rl}
x_1 + (1-\bar{x}_2) \geq 1 & (1/2) \\
(1-x_1) + x_3 \geq 1 & (1/2) \\
(1-x_2) \geq 0 & (1/2) \\
x_3 \geq 0 & (1/2) \\
\hline
(1-\bar{x}_2) + x_3 \geq 1/2 &
\end{array}$$

Take linear combination

Logic and cutting planes

A resolution step generates a **rank 1 cut** (i.e., a cut generated by one step of Chvátal's method).

$$x_1 \vee \bar{x}_2$$
$$\bar{x}_1 \vee x_3$$

$$\begin{array}{rl} x_1 + (1-\bar{x}_2) \ge 1 & (1/2) \\ (1-x_1) + x_3 \ge 1 & (1/2) \\ (1-x_2) \ge 0 & (1/2) \\ x_3 \ge 0 & (1/2) \\ \hline (1-\bar{x}_2) + x_3 \ge 1/2 & \\ (1-\bar{x}_2) + x_3 \ge 1 & \text{Round up} \end{array}$$

$$\bar{x}_2 \vee x_3$$

Resolvent

Logic and cutting planes

Theorem (JNH). **Input resolution** generates precisely those clauses that are rank 1 cuts.

$$
\begin{array}{lcr}
x_1 \vee \bar{x}_2 & & \\
\bar{x}_1 & & \vee\, x_3 \\
 & x_2 & \vee\, x_3 \\
\hline
 & & x_3
\end{array}
$$

Result of input resolution

Input resolution = use at least one of the original clauses to obtain each resolvent

Logic and cutting planes

Theorem (JNH). **Input resolution** generates precisely those clauses that are rank 1 cuts.

$$
\begin{array}{l}
x_1 \vee \bar{x}_2 \\
\bar{x}_1 \vee x_3 \\
x_2 \vee x_3 \\
\hline
x_3
\end{array}
$$

Result of input resolution

$$
\begin{array}{lr}
x_1 + (1 - x_2) \geq 1 & (1/4) \\
(1 - x_1) + x_3 \geq 1 & (1/4) \\
x_2 + x_3 \geq 1 & (1/2) \\
(1 - x_2) \geq 0 & (1/4) \\
x_3 \geq 0 & (1/4) \\
\hline
x_3 \geq 1/4 &
\end{array}
$$

Take linear combination

Logic and cutting planes

Theorem (JNH). **Input resolution** generates precisely those clauses that are rank 1 cuts.

$$x_1 \vee \bar{x}_2$$
$$\bar{x}_1 \vee x_3$$
$$x_2 \vee x_3$$
$$\overline{}$$
$$x_3$$

Result of input resolution

$$
\begin{array}{lr}
x_1 + (1 - x_2) \ge 1 & (1/4) \\
(1 - x_1) + x_3 \ge 1 & (1/4) \\
x_2 + x_3 \ge 1 & (1/2) \\
(1 - x_2) \ge 0 & (1/4) \\
x_3 \ge 0 & (1/4) \\
\hline
x_3 \ge 1/4 & \\
x_3 \ge 1 & \text{Round up}
\end{array}
$$

Logic and cutting planes

By generating enough Chvátal cuts, we obtain the **convex hull** of the 0-1 solutions.

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Can we obtain the convex hull by generating **resolvents**?

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That is, do the **prime implicates** define the convex hull?

Logic and cutting planes

By generating enough Chvatal cuts, we obtain the **convex hull** of the 0-1 solutions.

Can we obtain the convex hull by generating **resolvents**?

That is, do the **prime implicates** define the convex hull?

Not in general.

They do, if and only if the underlying **set covering problems** define convex hulls.

Logic and cutting planes

Theorem (JNH). The prime implicates of a clause set define an integral polytope if and only if all maximal **monotone** subsets of the prime implicates define an integral polytope.

monotone = every variable has the same sign in all occurrences.

A monotone subset of clauses is a set covering problem (after complementing negated variables).

Logic and cutting planes

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$$x_1 \vee x_2$$
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Prime implicates

Logic and cutting planes

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Prime implicates

$$x_1 \vee x_2$$
$$x_1 \vee x_3$$

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Maximal monotone subsets

Logic and cutting planes

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Prime implicates

$$x_1 \vee x_2$$
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Maximal monotone subsets

$$x_1 + x_2 \geq 1$$
$$x_1 + x_3 \geq 1$$

$$x_1 + x_3 \geq 1$$
$$(1 - x_2) + x_3 \geq 1$$

These systems define integral polytopes

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Prime implicates

$$
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\end{aligned}
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Maximal monotone subsets

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x_1 + x_2 &\geq 1 \\
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These systems define integral polytopes

$$
\begin{aligned}
x_1 + x_2 &\geq 1 \\
x_1 + x_3 &\geq 1 \\
(1 - x_2) + x_3 &\geq 1
\end{aligned}
$$

Therefore this system defines an integral polytope

Logic and cutting planes

Theorem (JNH). The prime implicates of a clause set define an integral polytope if and only if all maximal **monotone** subsets of the prime implicates define an integral polytope.

Generalized by Guenin, and by Nobili & Sassano.

Logic of 0-1 Inequalities

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0-1 inequalities can be viewed as **logical propositions**.

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Can the **resolution algorithm** be generalized to 0-1 inequalities?

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Yes. This results in a **logical analog** of Chvátal's theorem.

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Can the **resolution algorithm** be generalized to 0-1 inequalities?

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Theorem (JNH). Classical resolution + **diagonal summation** generates all 0-1 prime implicates (up to logical equivalence).

Logic of 0-1 inequalities

Diagonal summation:

$$\begin{array}{r} x_1 + 5x_2 + 3x_3 + x_4 \geq 4 \\ 2x_1 + 4x_2 + 3x_3 + x_4 \geq 4 \\ 2x_1 + 5x_2 + 2x_3 + x_4 \geq 4 \\ 2x_1 + 5x_2 + 3x_3 \geq 4 \\ \hline 2x_1 + 5x_2 + 3x_3 + x_4 \geq 5 \end{array}$$

Each inequality is implied by an inequality in the set to which 0-1 resolution is implied.

Diagonal sum

Logic of 0-1 inequalities

Diagonal summation:

$$x_1 + 5x_2 + 3x_3 + x_4 \geq 4$$
$$2x_1 + 4x_2 + 3x_3 + x_4 \geq 4$$
$$2x_1 + 5x_2 + 2x_3 + x_4 \geq 4$$
$$2x_1 + 5x_2 + 3x_3 + 0x_4 \geq 4$$
$$\overline{2x_1 + 5x_2 + 3x_3 + x_4 \geq 5}$$

Each inequality is implied by an inequality in the set to which 0-1 resolution is implied.

Diagonal sum

Logic and Linear Programming

Logic and linear programming

Theorem: A **renamable Horn** set of clauses is satisfiable if and only if it has a **unit refutation**.

Horn = at most one positive literal per clause

Renamable Horn = Horn after complementing some variables.

Unit refutation = resolution proof of unsatisfiability in which at least one parent of each resolvent is a unit clause.

Logic and linear programming

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$$
\begin{array}{l}
x_1 \\
\bar{x}_1 \vee \bar{x}_2 \\
\bar{x}_1 \qquad\quad \vee\, x_3 \\
\bar{x}_1 \vee x_2 \vee \bar{x}_3
\end{array}
$$

Horn set

Logic and linear programming

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Horn set	Unit resolution
x_1	x_1
$\bar{x}_1 \vee \bar{x}_2$	$\bar{x}_1 \vee \bar{x}_2$
$\bar{x}_1 \vee x_3$	$\bar{x}_1 \vee x_3$
$\bar{x}_1 \vee x_2 \vee \bar{x}_3$	$\bar{x}_1 \vee x_2 \vee \bar{x}_3$

Logic and linear programming

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Horn set	Unit resolution	Unit resolution
x_1	x_1	x_1
$\bar{x}_1 \vee \bar{x}_2$	$\bar{x}_1 \vee \bar{x}_2$	$\bar{x}_1 \vee \bar{x}_2$
$\bar{x}_1 \vee x_3$	$\bar{x}_1 \vee x_3$	$\bar{x}_1 \vee x_3$
$\bar{x}_1 \vee x_2 \vee \bar{x}_3$	$\bar{x}_1 \vee x_2 \vee \bar{x}_3$	$\bar{x}_1 \vee x_2 \vee \bar{x}_3$

Logic and linear programming

We don't know a **necessary and sufficient condition** for solubility by unit refutation.

But we can identify sufficient conditions by **generalizing Horn sets.**

For example, to **extended Horn sets**, which rely on a rounding property of linear programming.

Logic and linear programming

Theorem: A satisfiable **Horn** set can be solved by rounding down a solution of the linear programming relaxation.

Logic and linear programming

Theorem: A satisfiable **Horn** set can be solved by rounding down a solution of the linear programming relaxation.

$$x_1$$
$$\bar{x}_1 \vee \bar{x}_2 \vee x_3$$
$$\bar{x}_2 \vee \bar{x}_3$$
$$\bar{x}_1 \vee x_2 \vee \bar{x}_3$$

Horn set

Logic and linear programming

Theorem: A satisfiable **Horn** set can be solved by rounding down a solution of the linear programming relaxation.

$$\begin{array}{l} x_1 \\ \bar{x}_1 \vee \bar{x}_2 \vee x_3 \\ \qquad \bar{x}_2 \vee \bar{x}_3 \\ \bar{x}_1 \vee x_2 \vee \bar{x}_3 \end{array}$$

Horn set

$$\begin{array}{l} x_1 \geq 1 \\ (1-x_1)+(1-x_2)+x_3 \geq 1 \\ (1-x_2)+(1-x_3) \geq 1 \\ (1-x_1)+x_2+(1-x_3) \geq 1 \\ 0 \leq x_j \leq 1 \end{array}$$

LP relaxation

Logic and linear programming

Theorem: A satisfiable **Horn** set can be solved by rounding down a solution of the linear programming relaxation.

$$x_1$$
$$\bar{x}_1 \vee \bar{x}_2 \vee x_3$$
$$\bar{x}_2 \vee \bar{x}_3$$
$$\bar{x}_1 \vee x_2 \vee \bar{x}_3$$

Horn set

$$\begin{aligned} x_1 &\geq 1 \\ (1-x_1)+(1-x_2)+x_3 &\geq 1 \\ (1-x_2)+(1-x_3) &\geq 1 \\ (1-x_1)+x_2+(1-x_3) &\geq 1 \\ 0 \leq x_j \leq 1 & \end{aligned}$$

LP relaxation

Solution: $(x_1,x_2,x_3) = (1,1/2,1/2)$

Logic and linear programming

Theorem: A satisfiable **Horn** set can be solved by rounding down a solution of the linear programming relaxation.

$$\begin{aligned} & x_1 \\ & \bar{x}_1 \vee \bar{x}_2 \vee x_3 \\ & \bar{x}_2 \vee \bar{x}_3 \\ & \bar{x}_1 \vee x_2 \vee \bar{x}_3 \end{aligned}$$

Horn set

$$\begin{aligned} x_1 &\geq 1 \\ (1-x_1) + (1-x_2) + x_3 &\geq 1 \\ (1-x_2) + (1-x_3) &\geq 1 \\ (1-x_1) + x_2 + (1-x_3) &\geq 1 \\ 0 \leq x_j &\leq 1 \end{aligned}$$

LP relaxation

Solution: $(x_1,x_2,x_3) = (1,1/2,1/2)$

Round down: $(x_1,x_2,x_3) = (1,0,0)$

Logic and linear programming

To generalize this, we use the following:

Theorem (Chandrasekaran): If $Ax \geq b$ has integral components and T is nonsingular such that:

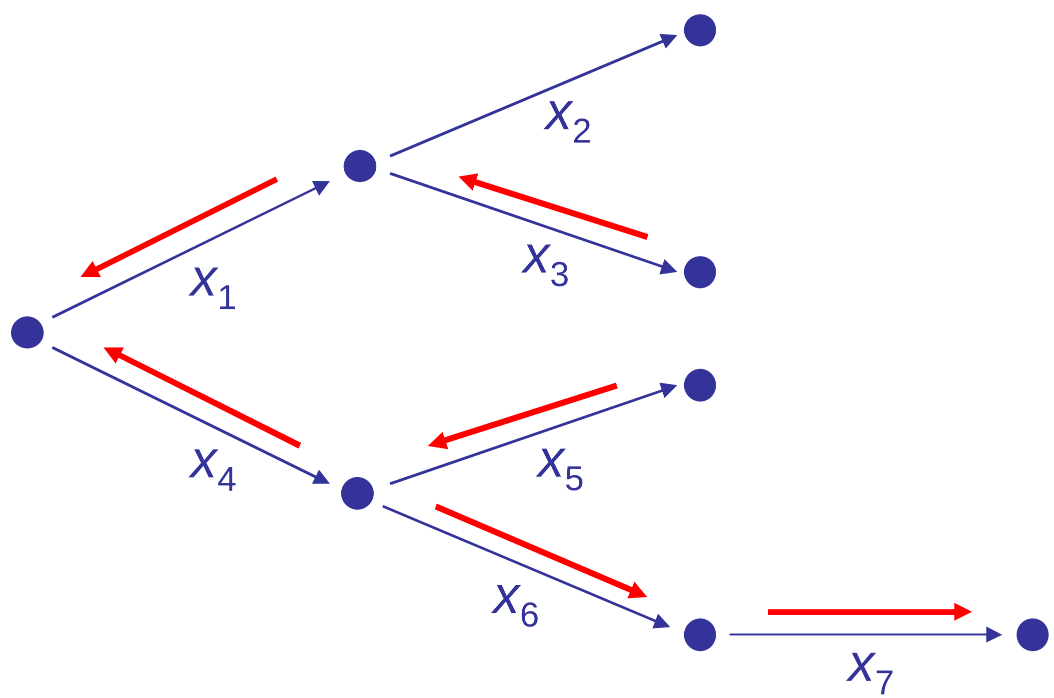
- T and T^{-1} are integral
- Each row of T^{-1} contains at most one negative entry, namely –1
- Each row of AT^{-1} contains at most one negative entry, namely –1

Then if x solves $Ax \geq b$, so does $T^{-1}\lceil Tx \rceil$

Logic and linear programming

A clause has the **extended star-chain property** if it corresponds to a set of edge-disjoint flows into the root of an arborescence and a flow on one additional chain.

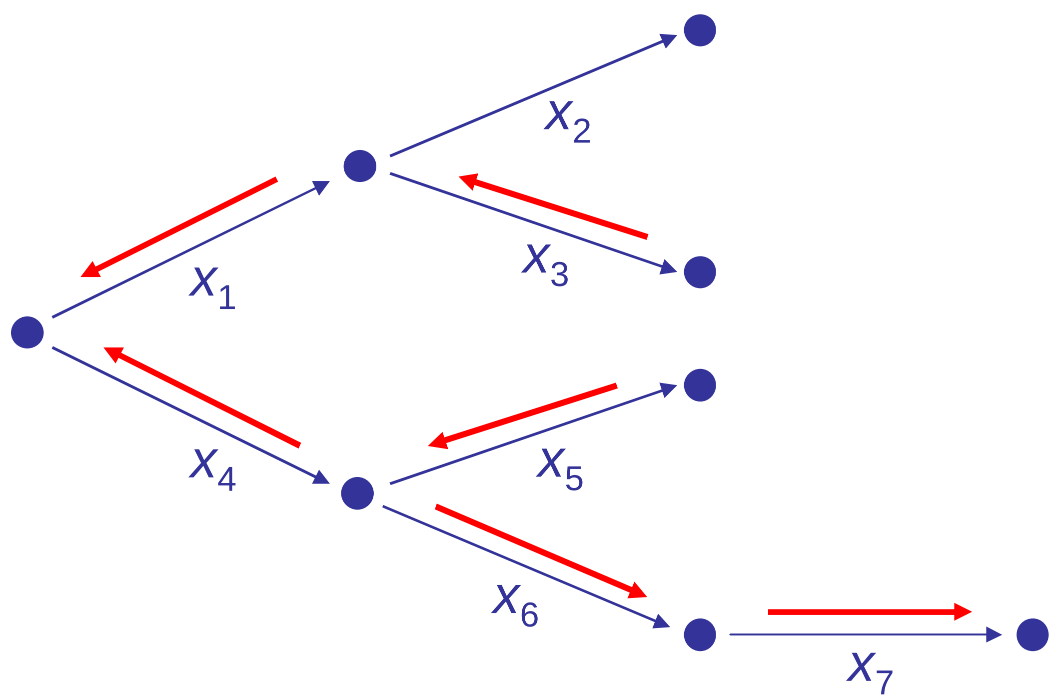
$$\bar{x}_1 \vee \bar{x}_3 \vee \bar{x}_4 \vee \bar{x}_5 \vee x_6 \vee x_7$$



Logic and linear programming

A clause set is **extended Horn** if there is an arborescence for which every clause in the set has the extended star-chain property.

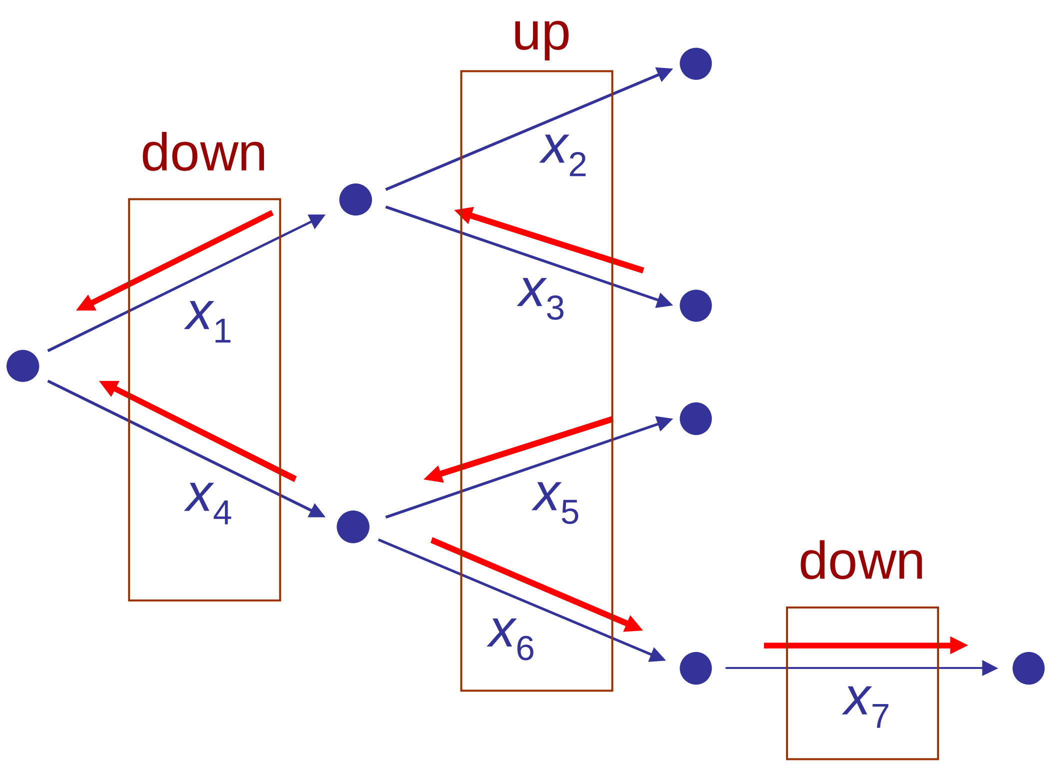
$$\bar{x}_1 \vee \bar{x}_3 \vee \bar{x}_4 \vee \bar{x}_5 \vee x_6 \vee x_7$$



Logic and linear programming

Theorem (Chandru and JNH). A satisfiable extended Horn clause set can be solved by rounding a solution of the LP relaxation as shown:

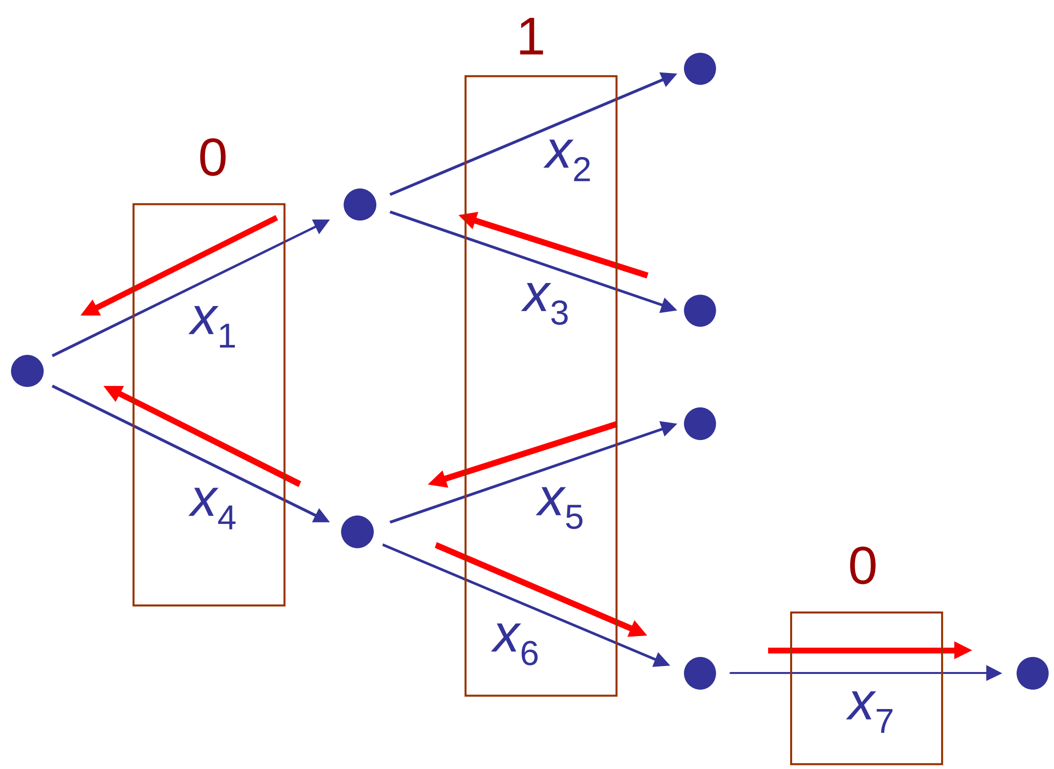
$$\bar{x}_1 \vee \bar{x}_3 \vee \bar{x}_4 \vee \bar{x}_5 \vee x_6 \vee x_7$$



Logic and linear programming

Corollary. A satisfiable extended Horn clause set can be solved by assigning values as shown:

$$\bar{x}_1 \vee \bar{x}_3 \vee \bar{x}_4 \vee \bar{x}_5 \vee x_6 \vee x_7$$



Logic and linear programming

Theorem (Chandru and JNH). A **renamable** extended Horn clause is satisfiable if and only if it has no unit refutation.

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Theorem (Schlipf, Annexstein, Franco & Swaminathan). These results hold when then incoming chains are not edge disjoint.

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Logic and linear programming

Theorem (Chandru and JNH). A **renamable** extended Horn clause is satisfiable if and only if it has no unit refutation.

Theorem (Schlipf, Annexstein, Franco & Swaminathan). These results hold when then incoming chains are not edge disjoint.

Corollary (Schlipf, Annexstein, Franco & Swaminathan). A one-step lookahead algorithm solves a satisfiable extended Horn problem without knowledge of the arborescence.

Inference duality

Inference duality

Consider an optimization problem:

$$\min f(x)$$
$$S$$
$$x \in D$$

S ← Constraint set

$x \in D$ ← Variable domain

Inference duality

Consider an optimization problem:

$$\begin{aligned} &\min f(x) \\ &\mathcal{S} \\ &x \in D \end{aligned}$$

$\mathcal{S}$ ← Constraint set

$x \in D$ ← Variable domain

An inference dual is:

$$\begin{aligned} &\max v \\ &\mathcal{S} \overset{P}{\Rightarrow} \left(f(x) \geq v \right) \\ &v \in \mathbb{R}, \quad P \in \mathscr{P} \end{aligned}$$

There is a proof P of $f(x) \geq v$ from premises in $\mathcal{S}$

Family of admissible proofs

Inference duality

Linear programming:

$$\begin{aligned} &\min cx \\ &Ax \geq b \\ &x \geq 0 \end{aligned}$$

Let $Ax \geq b \Rightarrow cx \geq v$ when $uAx \geq ub$ **dominates** $cx \geq v$ for some $u \geq 0$.

dominates = $uA \leq c$ and $ub \geq v$

Inference dual is:

$$\begin{aligned} &\max v \\ &(Ax \geq b) \overset{P}{\Rightarrow} (cx \geq v) \\ &v \in \mathbb{R}, \quad P \in \mathscr{P} \end{aligned}$$

Inference duality

Linear programming:

$$\min cx$$
$$Ax \geq b$$
$$x \geq 0$$

Let $Ax \geq b \Rightarrow cx \geq v$ when $uAx \geq ub$ **dominates** $cx \geq v$ for some $u \geq 0$.

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Inference dual is:

$$\max v$$
$$(Ax \geq b) \overset{P}{\Rightarrow} (cx \geq v)$$
$$v \in \mathbb{R},\quad P \in \mathscr{P}$$

This becomes the **classical LP dual.**

Inference duality

Linear programming:

$$\min cx$$
$$Ax \geq b$$
$$x \geq 0$$

Let $Ax \geq b \Rightarrow cx \geq v$ when $uAx \geq ub$ **dominates** $cx \geq v$ for some $u \geq 0$.

dominates = $uA \leq c$ and $ub \geq v$

Inference dual is:

$$\max v$$
$$(Ax \geq b) \overset{P}{\Rightarrow} (cx \geq v)$$
$$v \in \mathbb{R},\quad P \in \mathscr{P}$$

This becomes the **classical LP dual.**

This is a **strong dual** because the inference method is **complete** (Farkas Lemma).

Inference duality

General inequality constraints:

$$\begin{aligned}&\min f(x)\\&g(x) \geq 0\\&x \in S\end{aligned}$$

Let $g(x) \geq 0 \Rightarrow f(x) \geq v$ when $ug(x) \geq 0$ **implies** $f(x) \geq v$ for some $u \geq 0$.

implies = all $x \in S$ satisfying $ug(x) \geq 0$ satisfy $f(x) \geq v$.

Inference dual is:

$$\begin{aligned}&\max v\\&\left(g(x) \geq 0\right) \stackrel{P}{\Rightarrow} \left(f(x) \geq v\right)\\&v \in \mathbb{R},\quad P \in \mathscr{P}\end{aligned}$$

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This becomes the **surrogate dual**.

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This becomes the **Lagrangean dual**

Inference duality

Integer linear programming:

$$\min cx$$
$$Ax \geq b$$
$$x \in S$$

Let $Ax \geq$ b $\Rightarrow cx \geq v$ when $h(Ax) \geq h(b)$ dominates $cx \geq v$ for some **subadditive** and homogeneous function *h*.

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This becomes the **subadditive dual**.

This is a **strong dual** because the inference method is complete, due to Chvátal's theorem.

Appropriate Chvátal function is subadditive and can found by Gomory's cutting plane method.

Inference duality

Inference duality permits a generalization of **Benders decomposition.**

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For general optimization, a Benders cut does the same, but the proof of optimality is a solution of the general **inference dual**.

This has led to orders-of-magnitude speedups in solution of scheduling and other problems by **logic-based Benders decomposition**.

Constraint Programming

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Inference takes the form of **consistency maintenance**.

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A constraint set S containing variables $x_1, \ldots, x_n$ is ***k*-consistent** if
- for any subject of variables $x_1, \ldots, x_j, x_{j+1}$
- and any partial assignment $(x_1, \ldots, x_j) = (v_1, \ldots, v_j)$ that violates no constraint in S,

there is a v_{j+1} such that $(x_1, \ldots, x_{j+1}) = (v_1, \ldots, v_{j+1})$ violates no constraint in S.

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S is **strongly *k*-consistent** if it is j-consistent for $j = 1, \ldots, k$.

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Theorem (Freuder). If constraint set S is strongly k-consistent, and its **dependency graph** has width less than k (with respect to the branching order), then S can be solved without backtracking.

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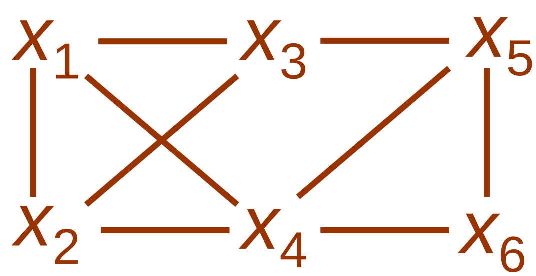
$$x_1 \vee \bar{x}_2 \vee x_4$$

$$x_3 \vee x_5$$

$$x_4 \vee \bar{x}_5 \vee x_6$$

Dependency graph



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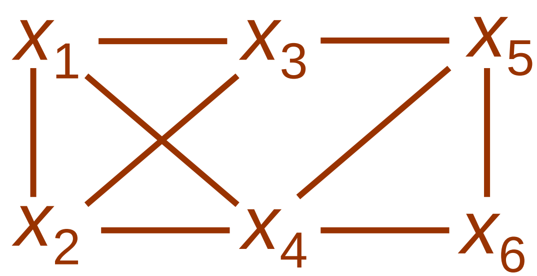

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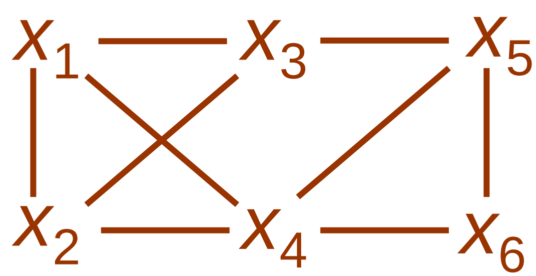

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We will show that this is strongly 3-consistent.

We can therefore solve it without backtracking

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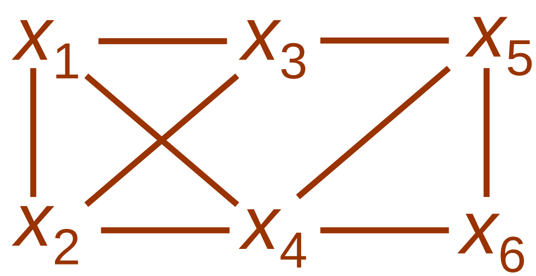

Width = max in-degree = 2

x_1	x_2	x_3	x_4	x_5	x_6
0					

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$$
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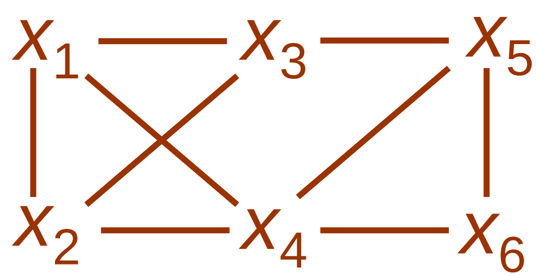
&x_4 \vee \bar{x}_5 \vee x_6
\end{aligned}
$$

Dependency graph



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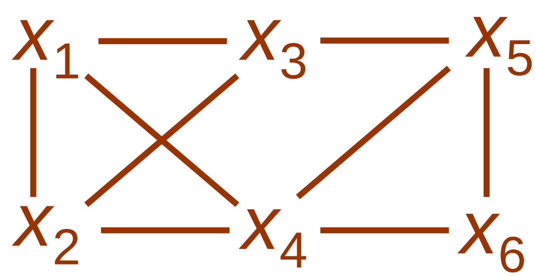

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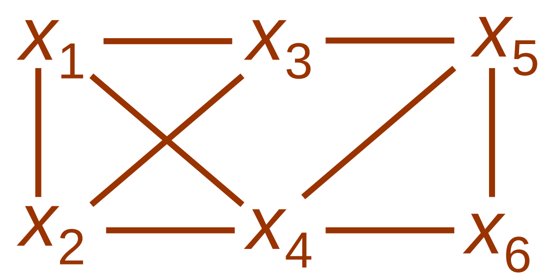

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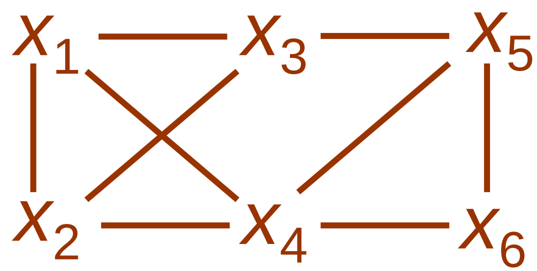

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x_1	x_2	x_3	x_4	x_5	x_6
0	0	0			

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x_5

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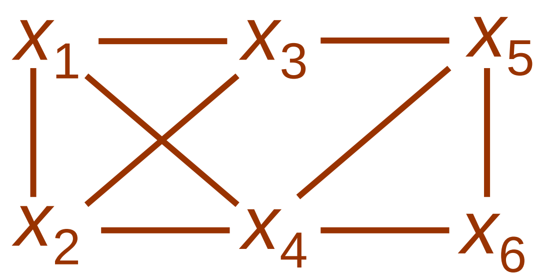

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x_1	x_2	x_3	x_4	x_5	x_6
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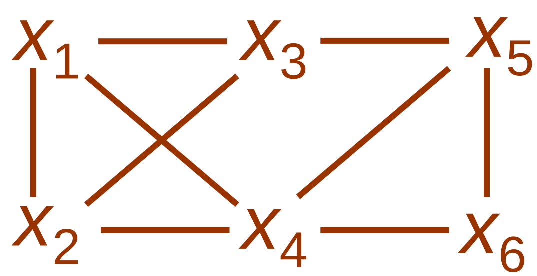

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0	0	0	0		

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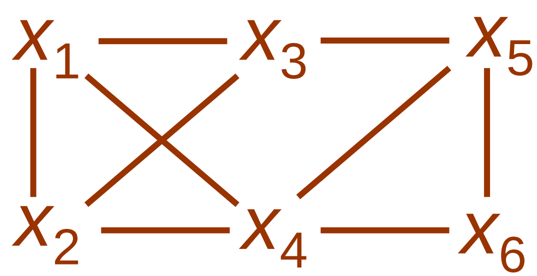

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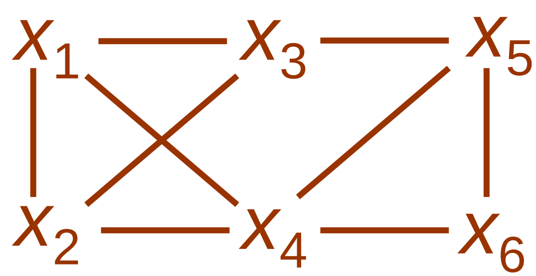

Width = max in-degree = 2

x_1	x_2	x_3	x_4	x_5	x_6
0	0	0	0	1	

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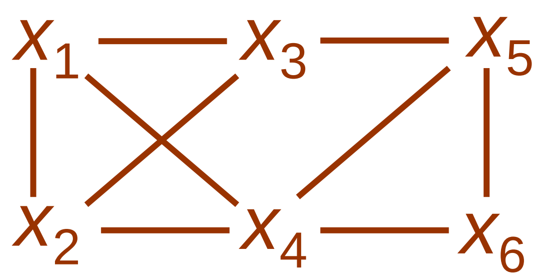
x_6

Dependency graph



Width = max in-degree = 2

x_1	x_2	x_3	x_4	x_5	x_6
0	0	0	0	1	

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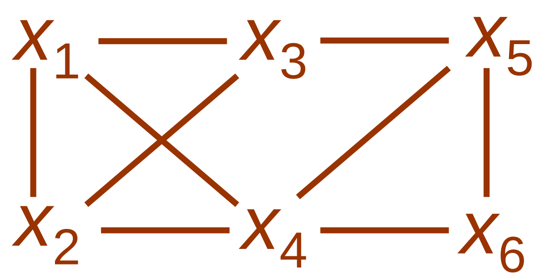

Width = max in-degree = 2

x_1	x_2	x_3	x_4	x_5	x_6
0	0	0	0	1	1

Constraint programming

Theorem. Application of ***k*-resolution** makes a clause set strongly *k*-consistent.

k-resolution = generate only resolventss with fewer than k literals.

$$\bar{x}_1 \vee x_2 \vee \bar{x}_3$$
$$x_1 \vee \bar{x}_2 \vee x_4$$
$$x_3 \vee x_5$$
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All resolvents have 3 or more literals.

Clause set is therefore strongly 3-consistent, as claimed.

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Filtering (**= logical inference**) is the workhorse of constraint programming, as solving relaxations is the workhorse of integer programming.

Good Logic Models

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- A high degree of **consistency** (in the constraint programming sense)
  - We talked about **resolution** as a means of achieving consistency for boolean models.
- A tight **linear relaxation**.
  - We talked about logic and **cutting planes**.
  - Logic constraints can also be given **convex hull formulations**…

Good logic models

Example: **cardinality rules**

We have 3 possible sites for factories and 3 possible products.

Rule 1: If at least 2 plants are built, then at least 2 products should be made.

Rule 2. Only 1 product should be made, unless plants are built at both sites 1 and 2.

$$(x_1 + x_2 + x_3 \geq 2) \Rightarrow (y_1 + y_2 + y_3 \geq 2)$$

$$(y_1 + y_2 + y_3 \geq 2) \Rightarrow (x_1 + x_2 \geq 2)$$

Good logic models

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Inequality form:

$$-2(x_1 + x_2 + x_3) + 2(y_1 + y_2 + y_3) \geq -2$$
$$-2(x_1 + x_2) + y_1 + y_2 + y_3 \geq -2$$
$$-2(x_1 + x_3) + y_1 + y_2 + y_3 \geq -2$$
$$-2(x_2 + x_3) + y_1 + y_2 + y_3 \geq -2$$
$$-x_1 - x_2 - x_3 + 2(y_1 + y_2) \geq -1$$
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$$(y_1 + y_2 + y_3 \geq 2) \Rightarrow (x_1 + x_2 \geq 2)$$

Inequality form:

$$-2(y_1 + y_2 + y_3) + x_1 \geq -3$$
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$$-2(y_1 + y_3) + x_1 \geq -1$$
$$-2(y_2 + y_3) + x_1 \geq -1$$

Good logic models

Theorem (Yan and JNH): These describe the convex hull of the feasible set.

Generalized by Balas, Bockmayr, Pisaruk & Wolsey.