

Brascamp-Lieb Inequalities for Non-Commutative Integration

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HAPPY BIRTHDAY ANDRÁS



CLASSICAL YOUNG'S INEQUALITY

For non negative measurable functions f_1, f_2 and f_3 on \mathbb{R} , and $1/p_1 + 1/p_2 + 1/p_3 = 2$,

$$\int_{\mathbb{R}^2} f_1(x_1)f_2(x_1 - x_2)f_3(x_2)d^2x \leq \left(\int_{\mathbb{R}} f_1^{p_1}(t)dt \right)^{1/p_1} \left(\int_{\mathbb{R}} f_2^{p_2}(t)dt \right)^{1/p_2} \left(\int_{\mathbb{R}} f_3^{p_3}(t)dt \right)^{1/p_3} . \quad (1)$$

This is related to the Prékopa-Leindler theorem.

Note: Direct application of Hölder's inequality yields $\int_{\mathbb{R}^2} = \infty$ on right side.

Define the maps $\phi_j : \mathbb{R}^2 \rightarrow \mathbb{R}$, $j = 1, 2, 3$, by

$$\phi_1(x_1, x_2) = x \quad \phi_2(x_1, x_2) = x_1 - x_2 \quad \text{and} \quad \phi_3(x_1, x_2) = x_2 .$$

Then (1) can be rewritten as

$$\int_{\mathbb{R}^2} \left(\prod_{j=1}^3 f_j \circ \phi_j \right)(x_1, x_2) d^2x \leq \prod_{j=1}^3 \left(\int_{\mathbb{R}} f_j^{p_j}(t)dt \right)^{1/p_j} .$$

BRASCAMP-LIEB, GENERALIZED YOUNG

There is now no particular reason to limit ourselves to products of only three functions, or to integrals over \mathbb{R}^2 and \mathbb{R} , or even any space for that matter, e.g., spheres:

DEFINITION. Given measure spaces $(\Omega, \mathcal{S}, \mu)$ and $(M_j, \mathcal{M}_j, \nu_j)$, $j = 1, \dots, N$, not necessarily distinct, together with measurable functions $\phi_j : \Omega \rightarrow M_j$ and numbers p_1, \dots, p_N with $1 \leq p_j \leq \infty$, $1 \leq j \leq N$, we say that **a generalized Young's inequality** holds for $\{\phi_1, \dots, \phi_N\}$ and $\{p_1, \dots, p_N\}$ in case there is a finite constant C such that

$$\int_{\Omega} \prod_{j=1}^N \mathbf{f}_j \circ \phi_j \, d\mu \leq C \prod_{j=1}^N \|\mathbf{f}_j\|_{L^{p_j}(\nu_j)}$$

In particular, B-L proved that for suitable p_1, \dots, p_N , and linear functions $\phi_j : \mathbb{R}^n \rightarrow \mathbb{R}$

$$\int_{\mathbb{R}^n} \prod_{j=1}^N (\mathbf{f}_j \circ \phi_j)(\mathbf{x}_1, \dots, \mathbf{x}_n) \, d\mathbf{x} \leq C \prod_{j=1}^N \|\mathbf{f}_j\|_{L^{p_j}(\mathbb{R})},$$

The sharp constant C is determined by considering only **centered Gaussians**.

Another example, this time on the **unit sphere** $\mathbb{S}^{n-1} \subset \mathbb{R}^n$ instead of on \mathbb{R}^n , was given by Carlen, L, and Loss. Write the coordinates as $x_1^2 + \cdots + x_n^2 = 1$. Then

$$\int_{\mathbb{S}^{n-1}} \prod_{j=1}^n f_j(x_j) \, d\mu \leq \prod_{j=1}^n \|f_j\|_{L^p(\mathbb{S}^{n-1})} \quad \text{for all } p \geq 2.$$

This, in turn, leads to an **entropy subadditivity inequality**:

Let $F(x_1 \dots x_n)$ be a probability density on \mathbb{S}^{n-1} and let $f_j(x_j)$ be its j^{th} marginal.

Then

$$2S(F) \leq \sum_{j=1}^n S(f_j),$$

where $\mathbf{S}(\mathbf{F}) = - \int_{\mathbb{S}^{n-1}} F \ln F \, d\mu$ and $\mathbf{S}(\mathbf{f}_j) = - \int_{\mathbb{S}^{n-1}} f_j \ln f_j \, d\mu$. The interesting point here is the factor of 2, which is best possible, and which gives a subadditivity stronger than the usual one.

A GENERALIZED YOUNG'S INEQUALITY IN THE CONTEXT OF NON COMMUTATIVE INTEGRATION

In non-commutative integration theory, as expounded by I. Segal and J. Dixmier, the basic data is an operator algebra \mathcal{A} equipped with positive linear functional λ .

Basic Example: \mathcal{A} is bounded linear ops on Hilbert space \mathcal{H} and $\lambda(\mathbf{A}) = \text{Trace}(\mathbf{A})$.

In the previous story, the algebra \mathcal{A} corresponded to the algebra of bounded measurable functions, and applying the positive linear functional λ to a positive operator corresponds to taking the integral of a positive function. That is,

$$A \mapsto \lambda(\mathbf{A}) \quad \text{corresponds to} \quad f \mapsto \int_M f d\nu .$$

We replace the measure spaces by non-commutative integration spaces. For $A \in (\mathcal{A}, \lambda)$,

$$\|\mathbf{A}\|_{q,\lambda} := \left(\lambda(A^* A)^{q/2} \right)^{1/q} \quad \text{and the right side becomes} \quad \prod_{j=1}^N \|\mathbf{A}_j\|_{(q_j, \lambda_j)} .$$

What shall we do about the product on the left side, $\prod_{j=1}^N (A_j \circ \phi_j)$?

The A_j are operators and do not commute? But since they are supposed to be positive we can think of the A_j as $\exp(H_j)$ and then $\prod_{j=1}^N A_j \circ \phi_j$ as $\exp(\sum_{j=1}^N \phi_j(H_j))$.

DEFINITION. Given **non commutative integration spaces** (\mathcal{A}, λ) and $(\mathcal{A}_j, \lambda_j)$, $j = 1, \dots, N$, together with C^* algebra endomorphisms $\phi_j : \mathcal{A}_j \rightarrow \mathcal{A}$, $j = 1, \dots, N$, and indices $1 \leq p_j \leq \infty$, $j = 1, \dots, N$, a **generalized Young's inequality** holds for $\{\phi_1, \dots, \phi_N\}$ and $\{p_1, \dots, p_N\}$ if there is a finite constant C so that

$$\lambda \left(\exp \left[\sum_{j=1}^N H_j \circ \phi_j \right] \right) \leq C \prod_{j=1}^N [\lambda_j(e^{p_j H_j})]^{1/p_j}$$

whenever H_j is self adjoint in \mathcal{A}_j , $j = 1, \dots, N$.

We want to find the indices and the best constant C for which such an inequality holds, and shall focus on two cases arising in mathematical physics: The first is operators on tensor products of **Hilbert spaces**, and the second is **Clifford algebras**.

GENERALIZED YOUNG'S INEQUALITY FOR TENSOR PRODUCTS

Let \mathcal{H}_j , $j = 1, \dots, n$ be Hilbert spaces, and let \mathcal{K} be the tensor product

$$\mathcal{K} = \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n .$$

Define \mathcal{A} to be $\mathcal{B}(\mathcal{K})$, the algebra of bounded operators on \mathcal{K} , and define λ to be the trace, Tr , on \mathcal{K} , so that $(\mathcal{A}, \lambda) = (\mathcal{B}(\mathcal{K}), \text{Tr})$.

For any non empty subset J of $\{1, \dots, n\}$, let \mathcal{K}_J denote the tensor product

$$\mathcal{K}_J = \otimes_{j \in J} \mathcal{H}_j .$$

Define \mathcal{A}_J to be $\mathcal{B}(\mathcal{K}_J)$, the algebra of bounded operators on \mathcal{K}_J , and define λ_J to be the trace on \mathcal{K}_J , so that $(\mathcal{A}_J, \lambda_J) = (\mathcal{B}(\mathcal{K}_J), \text{Tr}_J)$.

There are natural endomorphisms ϕ_J embedding the $2^n - 1$ algebras \mathcal{A}_J into \mathcal{A} . For instance, if $J = \{1, 2\}$,

$$\phi_{\{1,2\}}(A_1 \otimes A_2) = A_1 \otimes A_2 \otimes I_{\mathcal{H}_3} \otimes \cdots \otimes I_{\mathcal{H}_N} ,$$

In case $J \cap K = \emptyset$ and $J \cup K = \{1, \dots, n\}$, then for all $H_J \in \mathcal{A}_J$ and $H_K \in \mathcal{A}_K$,

$$\mathrm{Tr}_{\mathcal{K}} \left(e^{\phi_J(H_J) + \phi_K(H_K)} \right) = \mathrm{Tr}_{\mathcal{K}_J} \left(e^{H_J} \right) \mathrm{Tr}_{\mathcal{K}_K} \left(e^{H_K} \right) .$$

However, things are more interesting when $J \cap K \neq \emptyset$.

There is an analogue of Hölder's inequality in the commutative case, but it is not useful: By the **Golden-Thompson inequality** for 2 factors (but not more than 2),

$$\mathrm{Tr}_{\mathcal{K}} \left(e^{\phi_J(H_J) + \phi_K(H_K)} \right) \leq \mathrm{Tr}_{\mathcal{K}} \left(e^{\phi_J(H_J)} e^{\phi_K(H_K)} \right) .$$

. We can then use **Schwarz's inequality**: $(\mathrm{Tr} e^A e^B)^2 \leq (\mathrm{Tr} e^{2A}) (\mathrm{Tr} e^{2B})$, so

$$\mathrm{Tr}_{\mathcal{K}} \left(e^{\phi_J(H_J) + \phi_K(H_K)} \right) \leq \left(\mathrm{Tr}_{\mathcal{K}} e^{2H_J} \right)^{1/2} \left(\mathrm{Tr}_{\mathcal{K}} e^{2H_K} \right)^{1/2}$$

This can be disastrous because the traces on the right carry an unwanted factor $d^{(n-|K|)}$ or $d^{(n-|J|)}$, where d is the dimension of the small Hilbert spaces. Our **non-commutative BL inequality** is much better.

THEOREM. Let J_1, \dots, J_N be N non empty subsets of $\{1, \dots, n\}$. For each $j \in \{1, \dots, n\}$, let $p(j)$ denote the number of the sets J_1, \dots, J_N that contain j , and let $1 \leq q \leq \min_j p(j)$. Then, for self adjoint operators H_j on \mathcal{K}_{J_j} , $j = 1, \dots, N$,

$$\mathrm{Tr}_{\mathcal{K}} \left[\exp \left(\sum_{j=1}^N \phi_{J_j}(\mathbf{H}_j) \right) \right] \leq \prod_{j=1}^N \left(\mathrm{Tr}_{\mathcal{K}_{J_j}} e^{q\mathbf{H}_j} \right)^{1/q}$$

For all $q > \min_j p(j)$, it is possible for the left hand side to be infinite, while the right hand side is finite.

As in the commuting case, the important thing here is that while the left side involves a trace on the big Hilbert space \mathcal{K} , the right side involves only traces on the small Hilbert spaces \mathcal{K}_{J_j} , appropriate for the operator \mathbf{H}_j .

The proof of this theorem is not elementary. It relies on a deep fact in quantum information theory – **the strong subadditivity of entropy**.

EXAMPLE – SPIN-CHAINS

The Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2, \dots, \mathcal{H}_n$ are identical, namely \mathbb{C}^k , and are arranged in a one-dimensional array.

The ‘interactions’ consist of overlapping pairs with a periodic boundary condition, i.e.,

$$J_j = \{j, j+1\} \quad j = 1, \dots, n-1 \quad \text{and} \quad J_n = \{n, 1\} . \quad H_j = H^{j, j+1} .$$

Here, $N = n$, and obviously $p(j) = 2, \forall j$. Therefore,

$$\mathrm{Tr}_{\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \dots \otimes \mathcal{H}_n} \left[\exp \left(\sum_{j=1}^n H^{j, j+1} \right) \right] \leq \prod_{j=1}^n \left(\mathrm{Tr}_{\mathcal{H}_j \otimes \mathcal{H}_{j+1}} e^{2H^{j, j+1}} \right)^{1/2} .$$

The inequality has an interesting statistical mechanical interpretation as an upper bound on the *partition function* of an arbitrarily long chain of interacting spins in terms of a product of partition functions of simple constituent two-spin systems.

Again, the inequality (1) is non trivial owing to the “overlap” in the algebras \mathcal{A}_j .

A GENERALIZED YOUNG'S INEQUALITY IN CLIFFORD ALGEBRAS

Our next class is Clifford algebras, which as Segal emphasized, allow one to represent Fermion Fock space as an L^2 space – albeit a non-commutative L^2 space, but still with many of the advantages of having a Hilbert space represented as a function space, as in the usual Schrödinger representation in quantum mechanics.

In the finite dimensional setting, with n degrees of freedom, one starts with n operators Q_1, \dots, Q_n on some Hilbert space \mathcal{H} that satisfy the *canonical anticommutation relations*

$$Q_i Q_j + Q_j Q_i = 2\delta_{i,j} I .$$

One can concretely construct such operators acting on $\mathcal{H} = (\mathbb{C}^2)^{\otimes n}$, the n -fold tensor product of \mathbb{C}^2 with itself.

The Clifford algebra \mathcal{C} over \mathbb{R}^n is the operator algebra on \mathcal{H} that is generated by Q_1, \dots, Q_n . Its dimension is 2^n . A basis is

$$Q^\alpha = Q_1^{\alpha_1} Q_2^{\alpha_2} \cdots Q_n^{\alpha_n} .$$

where each α_i is 0 or 1.

The positive linear functional λ on \mathcal{C} is defined by

$$\lambda \left(\sum_{\alpha} x_{\alpha} Q^{\alpha} \right) = x_{(0, \dots, 0)} = 2^{-n} \text{Tr}_{\mathcal{H}} \left(\sum_{\alpha} x_{\alpha} Q^{\alpha} \right) ,$$

which picks off the coefficient of the identity in $A = \sum_{\alpha} x_{\alpha} Q^{\alpha}$.

As Segal emphasized, (\mathcal{C}, τ) is in many way a non commutative analog of \mathbb{R}^n with Gaussian measure $(2\pi)^{-n/2} \exp(-|x|^2/2) dx$.

Again we have a **theorem** of the following kind:

$$\tau \left(\exp \left(\sum_{j=1}^N \phi_j(H_j) \right) \right) \leq \prod_{j=1}^N (\tau_j e^{p_j H_j})^{1/p_j}$$

for all self-adjoint operators $H_j \in \mathcal{A}_j =$ (Clifford algebra over V_j , which is a subspace of \mathbb{R}^n) if and only if

$$\sum_{j=1}^N \frac{1}{p_j} P_j \leq I_{\mathbb{R}^n} .$$

where P_j is the orthogonal projection onto V_j in \mathbb{R}^n .

AN AMUSING COROLLARY OF THE CLIFFORD ALGEBRA THEOREM

Consider the special case that all the subspaces V_1, \dots, V_N are one-dimensional, and let u_1, \dots, u_N be unit vectors in these subspaces. Suppose that a weighted sum of the orthogonal projectors onto these vectors is \leq the identity, i.e.,

$$\sum_{j=1}^N \frac{1}{p_j} u_j \otimes u_j \leq I_{\mathbb{R}^n} .$$

Then, our Clifford B-L inequality states that for any collection b_1, \dots, b_n of real numbers,

$$\ln \cosh \left(\left| \sum_{j=1}^N b_j u_j \right| \right) \leq \sum_{j=1}^N \frac{1}{p_j} \ln \cosh(p_j b_j).$$

Notes: 1. After much effort we finally found an independent proof of this inequality, but it is not easy. 2. The Clifford algebra structure is in some ways more ‘non-commutative’ than the simple tensor product structure.

THANKS FOR LISTENING!