Brascamp-Lieb Inequalities for Non-Commutative Integration

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HAPPY BIRTHDAY ANDRÁS

Classical Young's Inequality

For non negative measurable functions f_1, f_2 and f_3 on $\mathbb{R}$, and $1/p_1 + 1/p_2 + 1/p_3 = 2$,

$$\int_{\mathbb{R}^2} f_1(x_1) f_2(x_1 - x_2) f_3(x_2) \mathrm{d}^2 x \leq \left(\int_{\mathbb{R}} f_1^{p_1}(t) \mathrm{d}t \right)^{1/p_1} \left(\int_{\mathbb{R}} f_2^{p_2}(t) \mathrm{d}t \right)^{1/p_2} \left(\int_{\mathbb{R}} f_3^{p_3}(t) \mathrm{d}t \right)^{1/p_3} . \qquad (1)$$

This is related to the Prékopa-Leindler theorem.

Note: Direct application of Hölder's inequality yields $\int_{\mathbb{R}^2} = \infty$ on right side.

Define the maps $\phi_j : \mathbb{R}^2 \to \mathbb{R}$, $j = 1, 2, 3$, by

$$\phi_1(x_1, x_2) = x \qquad \phi_2(x_1, x_2) = x_1 - x_2 \qquad \text{and} \qquad \phi_3(x_1, x_2) = x_2 \ .$$

Then (1) can be rewritten as

$$\boxed{\int_{\mathbb{R}^2} \left(\prod_{j=1}^{3} f_j \circ \phi_j \right) (x_1,\ x_2)\, \mathrm{d}^2 x \leq \prod_{j=1}^{3} \left(\int_{\mathbb{R}} f_j^{p_j}(t) \mathrm{d}t \right)^{1/p_j} \ .}$$

Brascamp-Lieb, Generalized Young

There is now no particular reason to limit ourselves to products of only three functions, or to integrals over $\mathbb{R}^2$ and $\mathbb{R}$, or even any space for that matter, e.g., spheres:

DEFINITION. *Given measure spaces $(\Omega, \mathcal{S}, \mu)$ and $(M_j, \mathcal{M}_j, \nu_j)$, $j = 1, \dots, N$, not necessarily distinct, together with measurable functions $\phi_j : \Omega \to M_j$ and numbers $p_1, \dots, p_N$ with $1 \le p_j \le \infty$, $1 \le j \le N$, we say that* ***a generalized Young's inequality*** *holds for $\{\phi_1, \dots, \phi_N\}$ and $\{p_1, \dots, p_N\}$ in case there is a finite constant C such that*

$$\int_\Omega \Pi_{j=1}^N \mathbf{f_j} \circ \phi_\mathbf{j} \, \mathrm{d}\mu \le C \, \Pi_{j=1}^N \|\mathbf{f_j}\|_{L^{p_j}(\nu_j)}$$

In particular, B-L proved that for suitable $p_1, ..., p_N$, and linear functions $\phi_j : \mathbb{R}^n \to \mathbb{R}$

$$\boxed{\int_{\mathbb{R}^\mathbf{n}} \mathbf{\Pi_{j=1}^N} \, (\mathbf{f_j} \circ \phi_\mathbf{j})(\mathbf{x_1}, ..., \mathbf{x_n}) \mathrm{d}\mathbf{x} \le \mathbf{C} \, \mathbf{\Pi_{j=1}^N} \, \|\mathbf{f_j}\|_{\mathbf{L^{p_j}}(\mathbb{R})},}$$

The sharp constant C is determined by considering only **centered Gaussians.**

Another example, this time on the **unit sphere** $\mathbb{S}^{n-1} \subset \mathbb{R}^n$ instead of on $\mathbb{R}^n$, was given by Carlen, L, and Loss. Write the coordinates as $x_1^2 + \cdots + x_n^2 = 1$. Then

$$\boxed{\int_{\mathbb{S}^{n-1}} \Pi_{j=1}^n f_j(x_j)\, \mathrm{d}\mu \le \Pi_{j=1}^n \|f_j\|_{L^p(\mathbb{S}^{n-1})} \qquad \text{for all } p \ge 2.}$$

This, in turn, leads to an **entropy subadditivity inequality**:

Let $F(x_1 \dots x_n)$ be a probability density on $\mathbb{S}^{n-1}$ and let $f_j(x_j)$ be its j^{th} marginal. Then

$$\boxed{2S(F) \le \sum_{j=1}^n S(f_j),}$$

where $\mathbf{S(F)} = -\int_{\mathbb{S}^{n-1}} F \ln F\, \mathrm{d}\mu$ and $\mathbf{S(f_j)} = -\int_{\mathbb{S}^{n-1}} f_j \ln f_j\, \mathrm{d}\mu$. The interesting point here is the factor of 2, which is best possible, and which gives a subadditivity stronger than the usual one.

A generalized Young's inequality in the context of non commutative integration

In non-commutative integration theory, as expounded by I. Segal and J. Dixmier, the basic data is an operator algebra $\mathcal{A}$ equipped with positive linear functional λ.

Basic Example: $\mathcal{A}$ is bounded linear ops on Hilbert space $\mathcal{H}$ and $\boxed{\lambda(\mathbf{A}) = \mathrm{Trace}\,(\mathbf{A}).}$

In the previous story, the algebra $\mathcal{A}$ corresponded to the algebra of bounded measurable functions, and applying the positive linear functional λ to a positive operator corresponds to taking the integral of a positive function. That is,

$$A \mapsto \lambda(\mathbf{A}) \qquad \text{corresponds to} \qquad f \mapsto \int_M f \mathrm{d}\nu \ .$$

We replace the measure spaces by non-commutative integration spaces. For $A \in (\mathcal{A}, \lambda)$,

$$\|\mathbf{A}\|_{q,\lambda} := \left(\lambda(A^* A)^{q/2}\right)^{1/q} \quad \text{and the right side becomes} \quad \Pi_{j=1}^N \|\mathbf{A_j}\|_{(q_j,\lambda_j)} \ .$$

What shall we do about the product on the left side, $\Pi_{j=1}^{N}(A_j \circ \phi_j)$?
The A_j are operators and do not commute? But since they are supposed to be positive we can think of the A_j as $\exp(H_j)$ and then $\Pi_{j=1}^{N} A_j \circ \phi_j$ as $\exp(\sum_{j=1}^{N} \phi_j(H_j))$.

DEFINITION. *Given* **non commutative integration spaces** $(\mathcal{A}, \lambda)$ *and* $(\mathcal{A}_j, \lambda_j)$, $j = 1, \ldots, N$, *together with* C^* *algebra endomorphisms* $\phi_j : \mathcal{A}_j \to \mathcal{A}$, $j = 1, \ldots, N$, *and indices* $1 \leq p_j \leq \infty$, $j = 1, \ldots, N$, *a* ***generalized Young's inequality*** holds for $\{\phi_1, \ldots, \phi_N\}$ and $\{p_1, \ldots, p_N\}$ *if there is a finite constant* C *so that*

$$\boxed{\lambda\left(\exp\left[\sum_{j=1}^{N} H_j \circ \phi_j\right]\right) \leq C \prod_{j=1}^{N} \left[\lambda_j(e^{p_j H_j})\right]^{1/p_j}}$$

whenever H_j *is self adjoint in* $\mathcal{A}_j$, $j = 1, \ldots, N$.

We want to find the indices and the best constant C for which such an inequality holds, and shall focus on two cases arising in mathematical physics: The first is operators on tensor products of **Hilbert spaces**, and the second is **Clifford algebras**.

GENERALIZED YOUNG'S INEQUALITY FOR TENSOR PRODUCTS

Let $\mathcal{H}_j$, $j = 1, \ldots, n$ be Hilbert spaces, and let Let $\mathcal{K}$ be the tensor product

$$\mathcal{K} = \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n \ .$$

Define $\mathcal{A}$ to be $\mathcal{B}(\mathcal{K})$, the algebra of bounded operators on $\mathcal{K}$, and define λ to be the trace, Tr, on $\mathcal{K}$, so that $(\mathcal{A}, \lambda) = (\mathcal{B}(\mathcal{K}), \mathrm{Tr})$.

For any non empty subset J of $\{1, \ldots, n\}$, let $\mathcal{K}_J$ denote the tensor product

$$\mathcal{K}_J = \otimes_{j \in J} \mathcal{H}_j \ .$$

Define $\mathcal{A}_J$ to be $\mathcal{B}(\mathcal{K}_J)$, the algebra of bounded operators on $\mathcal{K}_J$, and define λ_J to be the trace on $\mathcal{K}_J$, so that $(\mathcal{A}_\mathbf{J}, \lambda_\mathbf{J}) = (\mathcal{B}(\mathcal{K}_\mathbf{J}), \mathrm{Tr}_\mathbf{J})$.

There are natural endomorphisms ϕ_J embedding the $2^n - 1$ algebras $\mathcal{A}_J$ into $\mathcal{A}$. For instance, if $J = \{1, 2\}$,

$$\phi_{\{\mathbf{1},\mathbf{2}\}}(A_1 \otimes A_2) = A_1 \otimes A_2 \otimes I_{\mathcal{H}_3} \otimes \cdots \otimes I_{\mathcal{H}_N} \ ,$$

In case $J \cap K = \emptyset$ and $J \cup K = \{1, \ldots, n\}$, then for all $H_J \in \mathcal{A}_J$ and $H_K \in \mathcal{A}_K$,

$$\mathrm{Tr}_{\mathcal{K}} \left(e^{\phi_J(H_J) + \phi_K(H_K)} \right) = \mathrm{Tr}_{\mathcal{K}_J} \left(e^{H_J} \right) \mathrm{Tr}_{\mathcal{K}_K} \left(e^{H_K} \right) \ .$$

However, things are more interesting when $J \cap K \neq \emptyset$.

There is an analogue of Hölder's inequality in the commutative case, but it is not useful: By the **Golden-Thompson inequality** for 2 factors (but not more than 2),

$$\mathrm{Tr}_{\mathcal{K}} \left(e^{\phi_J(H_J) + \phi_K(H_K)} \right) \leq \mathrm{Tr}_{\mathcal{K}} \left(e^{\phi_J(H_J)} \, e^{\phi_K(H_K)} \right) .$$

. We can then use **Schwarz's inequality**: $\left(\mathrm{Tr}\, e^A e^B \right)^2 \leq \left(\mathrm{Tr}\, e^{2A} \right) \left(\mathrm{Tr}\, e^{2B} \right)$, so

$$\boxed{\mathrm{Tr}_{\mathcal{K}} \left(e^{\phi_J(H_J) + \phi_K(H_K)} \right) \leq \left(\mathrm{Tr}_{\mathcal{K}} \ e^{2H_J} \right)^{1/2} \ \left(\mathrm{Tr}_{\mathcal{K}} \ e^{2H_K} \right)^{1/2}}$$

This can be disastrous because the traces on the right carry an unwanted factor $d^{(n-|K|)}$ or $d^{(n-|J|)}$, where d is the dimension of the small Hilbert spaces. Our **non-commutative BL inequality** is much better.

THEOREM. *Let* $J_1, \ldots, J_N$ *be* N *non empty subsets of* $\{1, \ldots, n\}$ *For each* $j \in \{1, \ldots, n\}$*, let* $p(j)$ *denote the number of the sets* $J_1, \ldots, J_N$ *that contain* j*, and let* $1 \leq q \leq \min_j p(j)$*. Then, for self adjoint operators* H_j *on* $\mathcal{K}_{J_j}$*,* $j = 1, \ldots, N$*,*

$$\boxed{\mathrm{Tr}_{\mathcal{K}}\left[\exp\left(\sum_{\mathbf{j}=\mathbf{1}}^{\mathbf{N}} \phi_{\mathbf{J_j}}(\mathbf{H_j})\right)\right] \leq \prod_{\mathbf{j}=\mathbf{1}}^{\mathbf{N}} \left(\mathrm{Tr}_{\mathcal{K}_{\mathbf{J_j}}}\, \mathbf{e}^{\mathbf{qH_j}}\right)^{\mathbf{1/q}}}$$

For all $q > \min_j p(j)$*, it is possible for the left hand side to be infinite, while the right hand side is finite.*

As in the commuting case, the important thing here is that while the left side involves a trace on the big Hilbert space $\mathcal{K}$, the right side involves only traces on the small Hilbert spaces $\mathcal{K}_{\mathcal{K}_{\mathbf{J_j}}}$, appropriate for the operator $\mathbf{H_j}$.

The proof of this theorem is not elementary. It relies on a deep fact in quantum information theory – **the strong subadditivity of entropy**.

Example – spin-chains

The Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2, ..., \mathcal{H}_n$ are identical, namely $\mathbb{C}^k$, and are arranged in a one-dimensional array.
The 'interactions' consist of overlapping pairs with a periodic boundary condition, i.e.,

$$J_j = \{j, j+1\} \qquad j = 1, \dots, n-1 \qquad \text{and} \quad J_n = \{n, 1\}\ . \quad H_j = H^{j,j+1}.$$

Here, $N = n$, and obviously $p(j) = 2$, $\forall j$. Therefore,

$$\mathrm{Tr}_{\,\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \ldots \otimes \mathcal{H}_n} \left[\exp \left(\sum_{j=1}^{n} H^{j,j+1} \right) \right] \leq \prod_{j=1}^{n} \left(\mathrm{Tr}_{\,\mathcal{H}_j \otimes \mathcal{H}_{j+1}} e^{2H^{j,j+1}} \right)^{1/2} .$$

The inequality has an interesting statistical mechanical interpretation as an upper bound on the ***partition function*** of an arbitrarily long chain of interacting spins in terms of a product of partition functions of simple constituent two–spin systems.

Again, the inequality (1) is non trivial owing to the "overlap" in the algebras $\mathcal{A}_j$.

A Generalized Young's Inequality in Clifford Algebras

Our next class is Clifford algebras, which as Segal emphasized, allow one to represent Fermion Fock space as an L^2 space – albeit a non-commutative L^2 space, but still with many of the advantages of having a Hilbert space represented as a function space, as in the usual Schrödinger representation in quantum mechanics.

In the finite dimensional setting, with n degrees of freedom, one starts with n operators $Q_1, \ldots, Q_n$ on some Hilbert space $\mathcal{H}$ that satisfy the *canonical anticommutation relations*

$$Q_i Q_j + Q_J Q_i = 2\delta_{i,j} I \ .$$

One can concretely construct such operators acting on $\mathcal{H} = (\mathbb{C}^2)^{\otimes n}$, the n–fold tensor product of $\mathbb{C}^2$ with itself.

The Clifford algebra $\mathcal{C}$ over $\mathbb{R}^n$ is the operator algebra on $\mathcal{H}$ that is generated by $Q_1, \ldots, Q_n$. Its dimension is 2^n. A basis is

$$Q^\alpha = Q_1^{\alpha_1} Q_2^{\alpha_2} \cdots Q_n^{\alpha_n} \ .$$

where each α_i is 0 or 1.

The positive linear functional λ on $\mathcal{C}$ is defined by

$$\lambda\left(\sum_\alpha x_\alpha Q^\alpha\right) = x_{(0,\ldots,0)} = 2^{-n}\mathrm{Tr}_{\mathcal{H}}\left(\sum_\alpha x_\alpha Q^\alpha\right) ,$$

which picks off the coefficient of the identity in $A = \sum_\alpha x_\alpha Q^\alpha$.
As Segal emphasized, $(\mathcal{C}, \tau)$ is in many way a non commutative analog of $\mathbb{R}^n$ with Gaussian measure $(2\pi)^{-n/2}\exp(-|x|^2/2)\mathrm{d}x$.
Again we have a **theorem** of the following kind:

$$\tau\left(\exp\left(\sum_{j=1}^N \phi_j(H_j)\right)\right) \le \prod_{j=1}^N \left(\tau_j\ e^{p_j H_j}\right)^{1/p_j}$$

for all self-adjoint operators $H_j \in \mathcal{A}_j =$ (Clifford algebra over V_j, which is a subspace of $\mathbb{R}^n$) if and only if

$$\sum_{j=1}^N \frac{1}{p_j} P_j \le I_{\mathbb{R}^n} .$$

where P_j is the orthogonal projection onto V_j in $\mathbb{R}^n$.

An Amusing Corollary of the Clifford Algebra Theorem

Consider the special case that all the subspaces $V_1, \ldots, V_N$ are one-dimensional, and let $u_1, \ldots, u_N$ be unit vectors in these subspaces. Suppose that a weighted sum of the orthogonal projectors onto these vectors is $\leq$ the identity, i.e.,

$$\boxed{\sum_{j=1}^{N} \frac{1}{p_j} u_j \otimes u_j \leq I_{\mathbb{R}^n} \ .}$$

Then, our Clifford B-L inequality states that for any collection $b_1, \ldots, b_n$ of real numbers,

$$\boxed{\ln\cosh\left(\left|\sum_{j=1}^{N} b_j u_j\right|\right) \leq \sum_{j=1}^{N} \frac{1}{p_j} \ln\cosh(p_j b_j).}$$

Notes: 1. After much effort we finally found an independent proof of this inequality, but it is not easy. 2. The Clifford algebra structure is in some ways more 'non-commutative' than the simple tensor product structure.

THANKS FOR LISTENING!