Linear Reformulation of Probabilistically Constrained Optimization Problems Using Combinatorial Patterns

Miguel A. Lejeune*

George Washington University

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Problem Formulation

Probabilistically constrained programming problem

$$\begin{aligned} \min\ & c^T x \\ \text{subject to } & Ax \geq b \\ & \mathcal{P}\left(T_j x \geq \xi_j, j \in J\right) \geq p \\ & x \in \mathcal{R}_+ \end{aligned}$$

with ξ having a multivariate probability distribution with finite support

Approach: Combinatorial pattern approach

Rationale: Capturing impact of single variables and “interactions” between variables on constraint satisfiability

Output: Linear reformulations

Example

$$\min x_1 + 2x_2$$

$$\text{subject to } \mathcal{P}\left\{ \begin{array}{l} 8 - x_1 - 2x_2 \geq \xi_1 \\ 8x_1 + 6x_2 \geq \xi_2 \end{array} \right\} \geq 0.7$$

$$x_1, x_2 \geq 0$$

k	ω_1^k	ω_2^k	$F(\omega^k)$
1	6	3	0.2
2	2	3	0.1
3	1	4	0.1
4	4	5	0.3
5	3	6	0.3
6	4	8	0.5
7	6	8	0.7
8	1	9	0.2
9	4	9	0.7
10	5	10	0.8

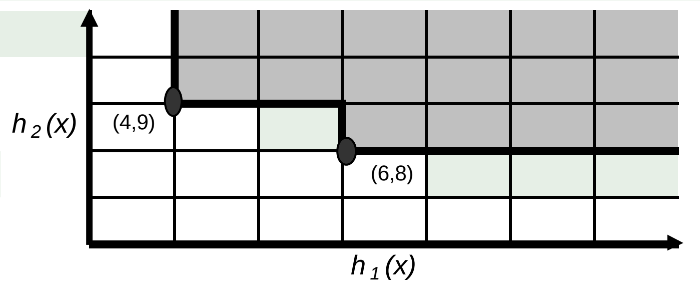
with $p_k = 0.1, k = 1, \ldots, 10 \in \Omega$.

Example

Feasibility set is the union of the two following polyhedra:

- $S_1 = \{(x_1, x_2) \in \mathcal{R}^2_+ : 8 - x_1 - 2x_2 \geq 6,\ 8x_1 + 6x_2 \geq 8\}$,
- $S_2 = \{(x_1, x_2) \in \mathcal{R}^2_+ : 8 - x_1 - 2x_2 \geq 4,\ 8x_1 + 6x_2 \geq 9\}$,

and is non-convex:



Could also be disconnected (Henrion, 2002).

Applications:

- Power management (Prékopa et al., 1980)
- Finance: cash matching (Henrion, 2001; Dentcheva et al., 2004), market risk (Pirvu, 2009)
- Reservoir management (Prékopa, Szántai, 1978)
- Water management (Takyi, Lence, 1999)
- Chemical distillation process (Henrion et al., 2001)
- Wireless communication (Chalise et al., 2007)
- Pollution (Gren, 2008)
- Production-distribution problems (LE, Ruszczyński, 2007)
- Munitions prepositioning (Avital, 2005)
- Military supply chain operations (Kress et al., 2007)
- Medical service design (Noyan, 2009)
- Vaccination strategies (Tanner et al., 2008)
- etc.

Solution Methods

- *p*-efficiency concept (Prékopa, 1990): disjunctive problem:
  - Identification of finite, unknown number of *p*-efficient points
  - Enumerative algorithm (Prékopa, 1990; Prékopa, 1995; Prékopa et al., 1997; Beraldi, Ruszczyński, 2002; LE, 2008) or optimization-based generation (LE, Noyan, 2009)
  - Convexification (Dentcheva et al., 2001)
  - Column generation algorithm (LE, Ruszczyński, 2007)
- MIP approach
  - List possible realizations of multivariate random vector
  - Associate a binary variable with each scenario
  - MIP formulation with cover constraint
  - Use of structural properties (Ruszczyński, 2002; Cheon et al., 2006; Luedtke et al., 2009; Küçükyavuz, 2009)
- Robust approach
  - Derivation of convex approximations (Calafiore, Campi, 2005; Nemirovski, Shapiro, 2005, 2006)

Structure

- Combinatorial patterns defining sufficient conditions for

$$\mathcal{P}\left(T_j x \geq \xi_j, j \in J\right) \geq p$$

  - Binarization of probability distribution F
  - Generation of set of relevant realizations
  - Representation of combination (F, p) of probability distribution F and probability level p as a partially defined Boolean function (pdBf)
  - Extension as a disjunctive normal form - collection of patterns
  - **Linear** programming generation of combinatorial patterns

- Derivation of:
  - **Linear programming tight** inner approximation
  - Linear deterministic equivalent

- Numerical results

- Extendibility and conclusion

Partitioning of Extended Set of Realizations

Definition (p-Sufficient Realization)

A realization k is p-sufficient if $\mathcal{P}(\xi \leq \omega^k) = F(\omega^k) \geq p$ and is p-insufficient if $F(\omega^k) < p$.

$$k : \omega_j^k \geq F_j^{-1}(p), \; j \in J$$

Let:

$$Z_j = \{\omega_j^k : F_j(\omega_j^k) \geq p, k \in \Omega\}$$

$$Z = Z_1 \times \ldots \times Z_j \times \ldots \times Z_{|J|}$$

Extended set of points:

$$\Omega^E = \Omega \bigcup Z$$

Example ($F_1^{-1}(0.7) = 4; F_2^{-1}(0.7) = 8$)

	k	ω_1^k	ω_2^k	$\mathcal{I}^k$
Extended set of p-insufficient points	1	6	3	0
	2	2	3	0
	3	1	4	0
	4	4	5	0
	5	3	6	0
	6	4	8	0
	8	1	9	0
	11	**5**	**8**	**0**
Extended set of p-sufficient points	7	6	8	1
	9	4	9	1
	10	5	10	1
	12	**4**	**10**	**1**
	13	**5**	**9**	**1**
	14	**6**	**9**	**1**
	15	**6**	**10**	**1**

Binarization of Probability Distribution

- Mapping of numerical vector ω^k into a binary one β^k:

$$\omega^k \quad \rightarrow \quad \beta^k = \left[\beta^k_{11}, \beta^k_{21}, \ldots, \beta^k_{ij}, \ldots\right], \; k \in \Omega^E$$

- Set C of cut points c_{ij}

$$\beta^k_{ij} = \begin{cases} 1 & \text{if } \omega^k_j \geq c_{ij} \\ 0 & \text{otherwise} \end{cases} \quad , \; i = 1, \ldots, n_j, j \in J$$

- Consistent set C of cut points: preserves the disjointedness between Ω^{E-} and Ω^{E+}

Definition

A sufficient-equivalent set of cut points C^e

$$C^e = \{c_{ij} : c_{ij} = \omega^k_j, F_j(\omega^k_j) \geq p, \; i = 1, \ldots, n_j, j \in J, k \in \Omega^{E+}\}$$

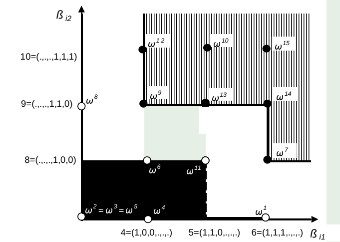
is consistent

Representation of (F, p) as a pdBf

Example ($C^e = \{c_{11} = 4;\ c_{21} = 5;\ c_{31} = 6;\ c_{12} = 8;\ c_{22} = 9;\ c_{32} = 10\}$)

	Numerical		Binarized Images $\bar{\Omega}_B^-$ and $\bar{\Omega}_B^+$						
k	ω_1^k	ω_2^k	β_{11}^k	β_{21}^k	β_{31}^k	β_{12}^k	β_{22}^k	β_{32}^k	$\mathcal{I}^k$
1	6	3	1	1	1	0	0	0	0
2	2	3	0	0	0	0	0	0	0
3	1	4	0	0	0	0	0	0	0
4	4	5	1	0	0	0	0	0	0
5	3	6	0	0	0	0	0	0	0
6	4	8	1	0	0	1	0	0	0
7	6	8	1	1	1	1	0	0	1
8	1	9	0	0	0	1	1	0	0
9	4	9	1	0	0	1	1	0	1
10	5	10	1	1	0	1	1	1	1
11	5	8	1	1	0	1	0	0	0
12	4	10	1	0	0	1	1	1	1
13	5	9	1	1	0	1	1	0	1
14	6	9	1	1	1	1	1	0	1
15	6	10	1	1	1	1	1	1	1

Example



	Numerical		Binarized Images $\bar{\Omega}_B^-$ and $\bar{\Omega}_B^+$						
k	ω_1^k	ω_2^k	β_{11}^k	β_{21}^k	β_{31}^k	β_{12}^k	β_{22}^k	β_{32}^k	$\mathcal{I}^k$
6	4	8	1	0	0	1	0	0	0
7	6	8	1	1	1	1	0	0	1
9	4	9	1	0	0	1	1	0	1
10	5	10	1	1	0	1	1	1	1
11	5	8	1	1	0	1	0	0	0
12	4	10	1	0	0	1	1	1	1
13	5	9	1	1	0	1	1	0	1
14	6	9	1	1	1	1	1	0	1
15	6	10	1	1	1	1	1	1	1

Extension

- Binarization process: pdBf $g(\bar{\Omega}_B^+, \bar{\Omega}_B^-)$ representing (F, p)
- Objective: Simple and compact *extension* f of $g(\bar{\Omega}_B^+, \bar{\Omega}_B^-)$

Definition (Boros et al., 1997)

Let $\mathcal{B} = \{0, 1\}$ and pdBf g defined by $(\bar{\Omega}_B^+, \bar{\Omega}_B^-)$: $\bar{\Omega}_B^+, \bar{\Omega}_B^- \subseteq \mathcal{B}^n$.

A function $f : \mathcal{B}^n \to \mathcal{B}$ is called an extension of the pdBf $g(\bar{\Omega}_B^+, \bar{\Omega}_B^-)$ if:

$$\bar{\Omega}_B^+ \subseteq \bar{\Omega}_B^+(f) \text{ and } \bar{\Omega}_B^- \subseteq \bar{\Omega}_B^+(f)\ .$$

- Any Boolean function can be represented by a *disjunctive normal form* (DNF)

DNF Extension for $g(\bar{\Omega}_B^+, \bar{\Omega}_B^-)$

- DNF: $f = \bigvee_{v \in V} t_v$
- *Term*: $t = \bigwedge_{ij \in P_t} \beta_{ij} \bigwedge_{ij \in N_t} \bar{\beta}_{ij}$, with β_{ij}, $\bar{\beta}_{ij}$ called *literals*
- Binary mapping of realization: $\omega^k \rightarrow \beta^k = \left[\beta_{11}^k, \ldots, \beta_{ij}^k, \ldots\right]$
- Term *covers* a realization k: $t(k) = 1 \Leftrightarrow \bigwedge_{ij \in P} \beta_{ij}^k \bigwedge_{ij \in N} \bar{\beta}_{ij}^k = 1$
- *Degree* of a term: number of literals: $d = |P| + |N|$

Theorem (LE, 2009)

The DNF $f = \bigvee_{v \in V} t_v$*, such that*

$$f(k) = 1,\ k \in \bar{\Omega}_B^+ \ \text{and}\ f(k) = 0,\ k \in \bar{\Omega}_B^-\ ,$$

is an extension of $g(\bar{\Omega}_B^+, \bar{\Omega}_B^-)$*. It provides a deterministic equivalent reformulation of the* $\mathcal{P}\left(T_j x \geq \xi_j, j \in J\right) \geq p$.

Pattern Properties

DNF as a disjunction of p-patterns

Definition

A p-pattern is a term t such that: $\bigvee_{k \in \bar{\Omega}_B^+} t(k) \geq 1$ and $\bigwedge_{k \in \bar{\Omega}_B^-} t(k) = 0$.

Corollary

Consider a sufficient-equivalent set C^e of cut points and the set of relevant realizations $\bar{\Omega}_B$:

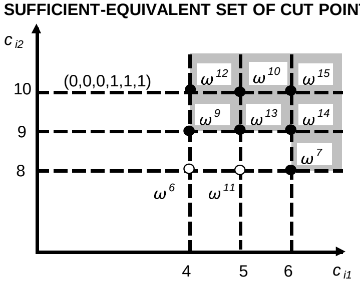
$$t(k) = 0, k \in \bar{\Omega}_B^- \; \Rightarrow \; t(k) = 1 \textit{ for at least one } k \in \bar{\Omega}_B^+ \, .$$

Prime p-patterns are of (large) degree $|J|$

$\Rightarrow$ Rationale for optimization-based generation

Approach

- Derive the "optimal" *p*-pattern: defines minimal conditions for $\mathcal{P}\left(T_j x \geq \xi_j, j \in J\right) \geq p$: $\rightarrow$ deterministic equivalent
- No particular/stylized form
- Derive a *p*-pattern: defines sufficient conditions for $\mathcal{P}\left(T_j x \geq \xi_j, j \in J\right) \geq p \rightarrow$ *tight* inner approximation
- Coverage criterion



Decision variables:

y^k: defines whether $k \in \bar{\Omega}_B^+$ is covered by t

u_{ij}: defines whether literal β_{ij} is included in t

Integer Programming Formulation `IP1` for *p*-Patterns

Theorem

Any feasible solution $(\mathbf{u}, \mathbf{y})$ *of* `IP1`

$$
\begin{aligned}
z = \min \sum_{k \in \bar{\Omega}_B^+} y^k & \\
\text{subject to} \quad \sum_{j \in J} \sum_{i=1}^{n_j} \beta_{ij}^k u_{ij} + |J| y^k \geq |J|, & \quad k \in \bar{\Omega}_B^+ \\
\sum_{j \in J} \sum_{i=1}^{n_j} \beta_{ij}^k u_{ij} \leq |J| - 1, & \quad k \in \bar{\Omega}_B^- \\
\sum_{i=1}^{n_j} u_{ij} = 1, & \quad j \in J \\
y^k, u_{ij} \in \{0, 1\}, & \quad j \in J, i = 1, \ldots, n_j, k \in \bar{\Omega}_B^+
\end{aligned}
$$

defines a prime p-pattern $\quad t = \bigwedge_{\substack{\mathbf{u}_{ij}=1 \\ j \in J, i=1,\ldots,n_j}} \beta_{ij}$

of degree $|J|$ *and* $(\mathbf{u}^*, \mathbf{y}^*)$ *defines the p-pattern with maximal coverage.*

Integer Programming Formulation `IP2` for *p*-Patterns

Theorem

Any feasible solution $(\mathbf{u},\mathbf{y})$ *of* `IP2`

$$
\begin{aligned}
z = \min & \sum_{k\in\bar\Omega_B^+} y^k \\
\textit{subject to} & \sum_{j\in J}\sum_{i=1}^{n_j} \beta_{ij}^k u_{ij} + y^k = |J|, & k \in \bar\Omega_B^+ \\
& \sum_{j\in J}\sum_{i=1}^{n_j} \beta_{ij}^k u_{ij} \le |J| - 1, & k \in \bar\Omega_B^- \\
& \sum_{i=1}^{n_j} u_{ij} = 1, & j \in J \\
& u_{ij} \in \{0,1\}, & j \in J, i = 1,\dots,n_j \\
& 0 \le y^k \le |J|, & k \in \bar\Omega_B^+
\end{aligned}
$$

defines a prime p-pattern with degree $|J|$: $\quad t = \bigwedge_{\substack{\mathbf{u}_{ij}=1 \\ j\in J, i=1,\dots,n_j}} \beta_{ij}$.

Example

With (both) IP1 and IP2, we obtain: $\mathbf{u}_{11}^* = \mathbf{u}_{22}^* = 1\ ,\ \mathbf{z}^* = 1$. Thus,

$$t = \beta_{11}\ \beta_{22}$$

	Numerical		Binarized Images $\bar{\Omega}_B^-$ and $\bar{\Omega}_B^+$							Coverage
k	ω_1^k	ω_2^k	β_{11}^k	β_{21}^k	β_{31}^k	β_{12}^k	β_{22}^k	β_{32}^k	$\mathcal{I}^k$	$\mathbf{y}^{\mathbf{k}^*}$
6	4	8	1	0	0	1	0	0	0	–
11	5	8	1	1	0	1	0	0	0	–
7	6	8	1	1	1	1	0	0	1	1
9	4	9	1	0	0	1	1	0	1	0
10	5	10	1	1	0	1	1	1	1	0
12	4	10	1	0	0	1	1	1	1	0
13	5	9	1	1	0	1	1	0	1	0
14	6	9	1	1	1	1	1	0	1	0
15	6	10	1	1	1	1	1	1	1	0

Linear Programming Formulation LP1 for p-Patterns

Theorem

Any feasible solution $(\mathbf{u}, \mathbf{y})$ *of* LP1

$$z = \min \sum_{k \in \bar{\Omega}_B^+} y^k$$

$$\begin{aligned}
\textit{subject to} \quad & \sum_{j \in J} \sum_{i=1}^{n_j} \beta_{ij}^k u_{ij} + |J| y^k \geq |J|, && k \in \bar{\Omega}_B^+ \\
& \sum_{j \in J} \sum_{i=1}^{n_j} \beta_{ij}^k u_{ij} \leq |J| - 1, && k \in \bar{\Omega}_B^- \\
& \sum_{i=1}^{n_j} u_{ij} = 1, && j \in J \\
& 0 \leq y^k, u_{ij} \leq 1, && j \in J, i = 1, \ldots, n_j, k \in \bar{\Omega}_B^+
\end{aligned}$$

defines a p-pattern

$$t = \bigwedge_{\substack{\mathbf{u}_{ij} > 0 \\ j \in J, i = 1, \ldots, n_j}} \beta_{ij} \ .$$

Prime p-Pattern with Linear Programming Formulation

- Objective function: relaxed number of non-covered p-sufficient realizations
- p-pattern not necessarily prime

Example

$(\mathbf{u_{11} = 0, u_{21} = 0.5, u_{31} = 0.5, u_{12} = 0.5, u_{22} = 0.5, u_{32} = 0})$
is feasible for `LP1` and results in the pattern $t = \beta_{21}\ \beta_{31}\ \beta_{12}\ \beta_{22}$

Corollary

A prime p-pattern

$$t = \bigwedge_{\substack{\bar{u}_{i_j j}=1 \\ j \in J, i=1,\dots,n_j}} \beta_{ij}, \text{ with}$$

$$\bar{i}_j = \operatorname*{argmax}_i \mathbf{u}_{ij} > 0, j \in J \quad , \quad \bar{u}_{ij} \begin{cases} 1 & \text{if} \quad i = \bar{i}_j \\ 0 & \text{otherwise} \end{cases}$$

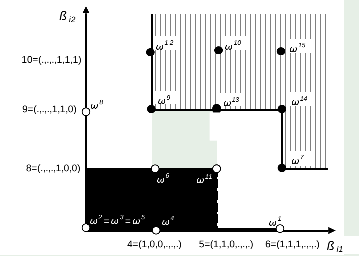
can immediately be derived from any feasible solution $(\mathbf{u}, \mathbf{y})$ of the linear programming problem.

Pricing Literals

- Pricing literals so that t defines (close to) minimal conditions
- Define parameters b_{ij} associated with literals β_{ij}
  - intra-component j pricing: include in t the literals imposing the least demanding conditions. If possible, preferable to include $\beta_{i'j}$ than β_{ij} if $i > i'$. Thus:

$$b_{ij} > b_{i'j}, \quad i > i', \qquad j \in J$$

Example



  - inter-component pricing: value of $b_{ij}, i = 1, \ldots n_j$, associated with j, is an increasing function of the cost associated with component j.

Linear Programming Formulation LP2 for p-Patterns

b_{ij}: price of including literal β_{ij} in t

Theorem

Any feasible solution $(\mathbf{u})$ *of* LP2

$$z = \min \sum_{j \in J} \sum_{i=1}^{n_j} b_{ij}\, u_{ij}$$

$$\begin{aligned} \textit{subject to} \quad & \sum_{j \in J} \sum_{i=1}^{n_j} \beta_{ij}^k u_{ij} \le |J| - 1, && k \in \bar{\Omega}_B^- \\ & \sum_{i=1}^{n_j} u_{ij} = 1, && j \in J \\ & 0 \le u_{ij} \le 1, && j \in J, i = 1, \ldots, n_j \end{aligned}$$

defines a p-pattern

$$t = \bigwedge_{\substack{\mathbf{u}_{ij} > 0 \\ j \in J, i = 1, \ldots, n_j}} \beta_{ij} \, .$$

Only includes n continuous variables and $n + |\Omega_B^-|$ constraints

Optimal solution of $(\mathbf{u})^*$ of LP2 defining “least costly” p-pattern

LP Inner Approximation with p-Pattern

Consider a p-pattern $t = \bigwedge_{ij \in P} \beta_{ij}$, with P the set of literals included in t.

$$t(k) = 1 \Leftrightarrow \prod_{ij \in P} \beta_{ij}^k = 1 \Leftrightarrow \omega_j^k \geq c_{ij}, ij \in P$$

Theorem

The LP problem

$$\begin{aligned} \min\ & c^T x \\ \textit{subject to } & Ax \geq b \\ & T_j x \geq c_{ij},\ ij \in P \\ & x \in \mathcal{R}_+ \end{aligned}$$

is an inner approximation of the probabilistic problem.

Evaluate tightness

Test Set

Supply chain-distribution problem

$$\begin{aligned} \min \quad & \sum_{k\in K}\sum_{j\in J} c_{kj}x_{kj} & \\ \text{subject to} \quad & \sum_{j\in J} x_{kj} \le M_k \;, & k\in K \\ & x_{kj} \le V_{kj} \;, & k\in K, j\in J \\ & \mathcal{P}\Big(\sum_{k\in K} x_{kj} \ge \xi_j, j\in J\Big) \ge p & \\ & x \ge 0 & \end{aligned}$$

32 types of instances differentiating along tuple $(|J|, |\Omega|, p)$:

- $|J|$=10, 20
- $|\Omega|$=5000,10000,20000,50000
- p=0.875, 0.9, 0.925, 0.95

8 instances per type

MATLAB - AMPL - CPLEX 11.1

Inner Approximation: Speed and Tightness I

Instance Type			IP1		IP2		LP1		LP2	
$\lvert J \rvert$	$\lvert \Omega \rvert$	p	Time	Gap	Time	Gap	Time	Gap	Time	Gap
10	5000	0.875	1.59	1.316%	0.41	1.449%	0.03	1.698%	0.03	0.802%
10	5000	0.9	0.11	0.458%	0.06	0.253%	0.02	0.485%	0.02	0.000%
10	5000	0.925	0.03	0.734%	0.03	0.971%	0.03	0.856%	0.03	0.560%
10	5000	0.95	0.03	0.325%	0.03	0.346%	0.01	0.457%	0.02	0.116%
10	10000	0.875	1.62	0.986%	0.39	0.960%	0.02	1.442%	0.02	1.292%
10	10000	0.9	0.30	0.396%	0.12	0.432%	0.03	0.421%	0.03	0.868%
10	10000	0.925	0.03	0.778%	0.02	0.939%	0.04	0.967%	0.03	0.000%
10	10000	0.95	0.03	0.001%	0.02	0.001%	0.04	0.001%	0.03	0.001%
10	20000	0.875	1.60	0.292%	0.41	0.487%	0.01	0.541%	0.02	1.631%
10	20000	0.9	0.28	1.532%	0.09	1.509%	0.03	2.360%	0.03	1.341%
10	20000	0.925	0.03	0.352%	0.03	0.369%	0.01	0.796%	0.02	0.004%
10	20000	0.95	0.03	0.775%	0.03	1.031%	0.03	1.029%	0.03	0.002%
10	50000	0.875	1.58	0.231%	0.41	0.472%	0.02	0.237%	0.02	0.000%
10	50000	0.9	0.39	1.241%	0.14	1.525%	0.03	2.254%	0.03	1.033%
10	50000	0.925	0.03	0.381%	0.03	0.815%	0.01	1.259%	0.02	0.137%
10	50000	0.95	0.03	0.918%	0.03	0.846%	0.03	2.638%	0.03	0.853%

Inner Approximation: Speed and Tightness II

Instance Type			IP1		IP2		LP1		LP2	
$\|J\|$	$\|\Omega\|$	p	Time	Gap	Time	Gap	Time	Gap	Time	Gap
20	5000	0.875	14.56	0.796%	2.43	0.801%	0.22	1.234%	0.24	0.800%
20	5000	0.9	3.94	0.610%	1.31	0.608%	0.03	0.896%	0.03	0.459%
20	5000	0.925	0.05	0.260%	0.05	0.257%	0.04	0.364%	0.03	0.000%
20	5000	0.95	0.02	0.001%	0.03	0.000%	0.02	0.011%	0.02	0.126%
20	10000	0.875	15.70	0.924%	4.70	0.922%	0.26	1.264%	0.30	0.925%
20	10000	0.9	11.61	0.401%	3.69	0.406%	0.05	0.856%	0.03	0.001%
20	10000	0.925	0.05	0.205%	0.05	0.207%	0.02	0.745%	0.02	0.069%
20	10000	0.95	0.03	0.332%	0.02	0.340%	0.06	0.867%	0.03	0.068%
20	20000	0.875	26.54	0.904%	4.27	0.896%	0.13	1.762%	0.13	0.896%
20	20000	0.9	12.70	0.404%	2.28	0.408%	0.04	2.697%	0.05	0.878%
20	20000	0.925	0.02	0.175%	0.00	0.175%	0.01	1.671%	0.02	0.000%
20	20000	0.95	0.03	0.142%	0.02	0.137%	0.03	2.012%	0.03	0.137%
20	50000	0.875	28.96	0.862%	4.98	0.862%	1.78	3.002%	1.62	0.862%
20	50000	0.9	21.17	0.666%	3.58	0.678%	1.36	3.216%	1.44	0.956%
20	50000	0.925	0.05	0.459%	0.05	0.479%	0.01	2.891%	0.02	0.275%
20	50000	0.95	0.03	0.181%	0.05	0.189%	0.01	2.421%	0.02	0.252%

1. LP formulations the fastest ($<$ 2 sec)
2. IP formulations also fast ($<$28 sec): `IP2` faster
3. Very tight approximations with `IP1`, `IP2`, `LP2`
4. Tightest most often with `LP2`: literal pricing approach

	IP1	IP2	LP1	LP2
Tightest	8	8	1	21

Concurrent Pattern Generation and Solution of Deterministic Equivalent Problem

Theorem

The MIP problem DEIP

$$
\begin{aligned}
& \min c^T x \\
\textit{subject to } & Ax \geq b \\
& T_j x \geq \sum_{i=1}^{n_j} c_{ij}\, u_{ij}, && j \in J \\
& \sum_{j \in J} \sum_{i=1}^{n_j} \beta_{ij}^k\, u_{ij} \leq |J| - 1, && k \in \bar{\Omega}_B^- \\
& \sum_{i=1}^{n_j} u_{ij} = 1, && j \in J \\
& u_{ij} \in \{0, 1\}, && j \in J, i = 1, \ldots, n_j \\
& x \in \mathcal{R}_+
\end{aligned}
$$

is a deterministic equivalent of the probabilistically constrained problem.

- $(\mathbf{u}^*, \mathbf{x}^*)$ defines the p-pattern $t = \bigwedge_{\mathbf{u}_{\mathbf{ij}}^* = \mathbf{1}} \beta_{ij}$ imposing the minimal conditions for the satisfiability of the probabilistic constraint
- Number of integer variables = number of cut points

Computational Results for Deterministic Equivalent DEIP

Optimality reached in at most 3 sec: 32 families (256 instances)

$\|J\|$	$\|\Omega\|$	p	Time	$\|J\|$	$\|\Omega\|$	p	Time
10	5000	0.875	2.212	20	5000	0.875	1.036
10	5000	0.9	0.063	20	5000	0.9	0.500
10	5000	0.925	0.016	20	5000	0.925	0.032
10	5000	0.95	0.005	20	5000	0.95	0.015
10	10000	0.875	0.141	20	10000	0.875	1.874
10	10000	0.9	0.078	20	10000	0.9	1.391
10	10000	0.925	0.016	20	10000	0.925	0.031
10	10000	0.95	0.016	20	10000	0.95	0.016
10	20000	0.875	0.109	20	20000	0.875	2.891
10	20000	0.9	0.078	20	20000	0.9	2.297
10	20000	0.925	0.031	20	20000	0.925	0.031
10	20000	0.95	0.031	20	20000	0.95	0.031
10	50000	0.875	0.141	20	50000	0.875	2.652
10	50000	0.9	0.094	20	50000	0.9	1.891
10	50000	0.925	0.016	20	50000	0.925	0.031
10	50000	0.95	0.008	20	50000	0.95	0.016

Conclusions and Extensions

- Novel methodology for probabilistically constrained problems
- Combinatorial patterns take into account "interactions" between components ξ_j on satisfiability of joint probabilistic constraint
- Commonalities with Logical Analysis of Data (Hammer, 1986; Boros et al., 1997, 2000)
- Derivation of combinatorial patterns defining sufficient conditions for attainment of prescribed probability level
  - Binarization of probability distribution
  - Representation of (F, p) as pdBf and extended as a DNF
  - **Linear Programming** modeling of patterns
  - Derivation of tight LP inner approximation
  - Concurrent pattern generation and solution
- Computational efficiency
- Handles very **fine** discretizations of random events