DIMACS – RUTCOR Workshop
January 19-22, 2009

ON VARIOUS RELAXATIONS BASED ON REFORMULATION-LINEARIZATION FOR 0-1 MIPs

and specialization to Pseudo-Boolean Optimization

in memory of Peter L. Hammer

M. MINOUX
University Paris 6 – France

SUMMARY

1. INTRODUCTION
2. CONNECTION BETWEEN RANK-1 L&P AND RANK-1 RLT
3. THE RANK-d SRL* RELAXATION FOR ARBITRARY d AND CONNECTIONS WITH P^d_{RLT} AND $P^d_{L\&P}$
4. THE CASE OF LINEARY CONSTRAINED PSEUDOBOLEAN FUNCTION OPTIMIZATION
5. PRELIMINARY COMPUTATIONAL EXPERIMENTS ON LINEARLY CONSTRAINED PBO
6. CONCLUSIONS

1. INTRODUCTION

We consider here the mixed 0-1 integer set P corresponding to the set of feasible solutions to :

$$
(\mathrm{I})\quad \begin{cases} \sum_{j=1}^{n+m} a^j x_j \le b & (1) \\ x_j \le 1 \quad \forall\, j \in E = \{1, 2, \ldots, n\} & (2) \\ -x_j \le 0 \quad \forall\, j \in N = \{1, 2, \ldots, n+m\} & (3) \\ x_j \in \{0,1\} \ \forall\, j \in E & (4) \end{cases}
$$

Where : $a^j \in \mathbb{R}^c$ ($\forall$ $j \in$ N), $b \in \mathbb{R}^c$ (c = number of constraints (1))

In the above, there are n + m variables, n 0-1 integer variables and m real variables subject to non negativity conditions.

The linear relaxation, denoted $\overline{P}$, is the polyhedron defined by (1)-(3).

We focus here on two well-known hierarchies of relaxations for P namely :

- the LIFT-AND-PROJECT (or disjunctive) hierarchy ;
- the RLT hierarchy (Sherali & Adams 1990)

and we investigate connections between these two hierarchies.

As an interesting outcome of this investigation, it will be seen that a *new hierarchy* arises in a natural way : the so-called SRL* hierarchy. Some potentially interesting features of SRL* (in particular w.r.t. computational issues) will be pointed out.

d-factors :

For any integer d such that $1 \le d \le n$, we call d-factor associated with the d-element subset $J^d \subseteq E$ any degree-d polynomial $F_d(J, J^d \setminus J)$ of the form

$$F_d(J, J^d \setminus J) = \prod_{j \in J} x_j \prod_{j \in J^d \setminus J} (1 - x_j)$$

With $J \subseteq J^d$.

2. CONNECTION BETWEEN RANK-1 LIFT-AND-PROJECT AND RANK-1 RLT RELAXATIONS

(See BALAS, CERIA & CORNUEJOLS 1993, BONAMI & MX 2005)

2.1. THE RANK-1 LIFT-AND-PROJECT RELAXATION

The linear representation of $P^1_{L\&P}$ is derived from(1)-(3) as follows.

Each constraint out of the system (1)-(3) gives rise to 2 n (nonlinear) constraints :

- one for each 0-1 variable x_i ($i \in E$), obtained by multiplying both handsides by the factor $F_1(\{i\}, \emptyset) = x_i$
- one for each 0-1 variable x_i ($i \in E$) obtained by multiplying both handsides by the factor $F_1(\emptyset, \{i\}) = 1 - x_i$.

The result of this reformulation is a nonlinear system (II) composed of a set of quadratic inequalities.

The nonlinear system (II) is then ***linearized*** by introducing the 2n(n+m+1) variables

$Z_0^{\{i\},\{i\}}$, $Z_j^{\{i\},\{i\}}$, $Z_0^{\emptyset,\{i\}}$, $Z_j^{\emptyset,\{i\}}$ (i = 1, …, n ; j = 1, … n+ m) where :

$Z_0^{\{i\},\{i\}}$ is a substitute for F_1 ({i}, Ø)

$Z_j^{\{i\},\{i\}}$ is a substitute for x_j F_1 ({i}, Ø)

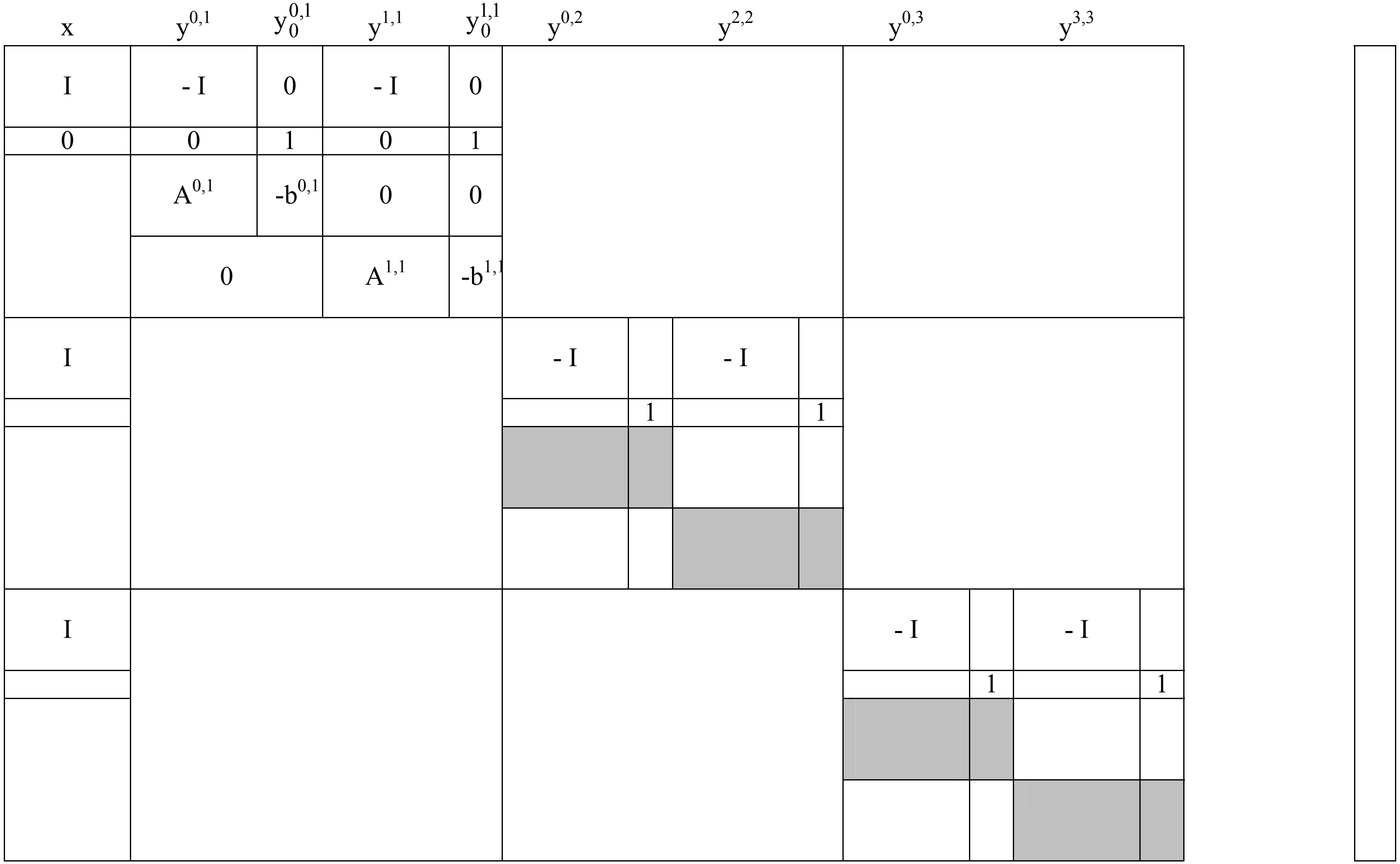
$Z_0^{\emptyset,\{i\}}$ is a substitute for F_1 (Ø, {i})

$Z_j^{\emptyset,\{i\}}$ is a substitute for x_j F_1 (Ø, {i})

The resulting linearized system defining $P^{1}_{L\&P}$ is :

$$(II')\quad \begin{cases} \sum_{j=1}^{n+m} a^j Z_J^{J,\{i\}} - b\, Z_0^{J,\{i\}} \le 0 & \forall i \in E,\ \forall J \subseteq \{i\} \\ Z_j^{J,\{i\}} - Z_0^{J,\{i\}} \le 0 & \forall j \in E, \forall i \in E, \forall J \subseteq \{i\} \\ Z_j^{J,\{i\}} \ge 0\ Z_0^{J,\{i\}} \ge 0 & \forall j \in N, \forall i \in E, \forall J \subseteq \{i\} \\ Z_0^{\{i\},\{i\}} + Z_0^{\emptyset,\{i\}} = 1 & \forall i \in E \\ Z_j^{\{i\},\{i\}} + Z_j^{\emptyset,\{i\}} = x_j & \forall j \in N \end{cases}$$

We note that in the above linear representation, for any pair $i \in E$, $j \in E$, $i \neq j$ both variables $Z_j^{\{i\},\{i\}}$ and $Z_i^{\{j\},\{j\}}$ formally correspond to the product $x_i x_j$ but they have to be considered as *distinct variables* (i.e. they are not requested to take on equal values).


x
y0,1
y0 0,1
y1,1
y0 1,1
y0,2
y2,2
y0,3
y3,3
I
- I
0
- I
0
0
0
1
0
1
A0,1
-b0,1
0
0
0
A1,1
-b1,1
I
- I
- I
1
1
I
- I
- I
1
1

2.2. AN APPLICATION TO MAX-2-SAT PROBLEMS

Let F be a 2-SAT formula in CNF form involving n Boolean variables $\alpha_1, \alpha_2, \ldots, \alpha_n$ and m clauses

$C = C_1 \cup C_2 \cup C_3$

Where C_1	is the set of clauses of the form	$(\alpha_i \vee \alpha_j)$	$(i < j)$
C_2	" " "	$(\overline{\alpha}_i \vee \alpha_j)$	$(i \neq j)$
C_3	" " "	$(\overline{\alpha}_i \vee \overline{\alpha}_j)$	$(i < j)$

A natural formulation of MAX-2-SAT as a 0-1 MIP is :

$$(\text{M2S}) \begin{cases} \text{Min } \sum_{s=1}^{m} z_s \\ \text{s.t.:} \\ -\alpha_i - \alpha_j - z_s \leq -1 & \forall s \in C_1 \\ \alpha_i - \alpha_j - z_s \leq 0 & \forall s \in C_2 \\ \alpha_i + \alpha_j - z_s \leq 1 & \forall s \in C_3 \\ \alpha \in \{0,1\}^n, z \in [0,1]^m \end{cases}$$

The computational experiments show that for (M2S) the relaxation provided by $P^1_{L\&P}$ is fairly strong and makes possible the efficient solution of fairly large problems.

						CPLEX 9.0			Iterated Lift & Project			TOOLBAR(*)	
# var	# clauses	Opt	$P^1_{L\&P}$	Sol time	# nodes	Sol time	Best Bound	# nodes	Bound	Sol Time	# nodes	Time	# nodes
75	525	61	585	39	49	115.9		10^5	47.3	50.8	1562	57	~ 10^6
75	525	65	60.9	57	275	137		>10^5	45	73.3	5289	111	2.2 10^6
75	550	70	65.7	62.1	230	624		~5.10^5	47.2	141.6	11176	323	7.1.10^6
75	600	75	70.5	63.6	245	602		~5.10^5	53	75.8	4673	249	5.7 10^6
100	700	86	79.9	155	763	> 2h	80.5	~5.10^6	55.2	423	39255	> 2h	> 10^7
150	850	86	79.2	336	990	> 2h	62.5	~2.10^6	58.4	7200	~3.10^5	> 2h	> 10^7
150	850	85	79.3	262	567	> 2h	62.2	~3.10^6	60.1	1959	92025	> 2h	> 10^7
200	1000	94	83.8	3030	13700	> 2h	55.5	~ 10^6	62.3	> 2h	223000	> 2h	> 10^7
200	1000	96	85.9	4189	18725	–	–	–	59.7	> 2h	190000	> 2h	> 10^7
200	1000	92	83.3	1150	4569	> 2h	54	~ 2.10^6	65.2	> 2h	134 000	> 2h	> 10^7

Sample results from BONAMI & M.M. (2006)

(*) de GIVRY, LARROSA, MESEGUER, SCHIEX (2003), "Solving MAX-SAT as weighted CSP"

2.3. THE RANK-1 SHERALI-ADAMS RELAXATION

The linear representation of P^{1}_{RLT} is derived in a similar way, using the *same reformulation step*, but a slightly different type of *linearization* is applied to the resulting nonlinear system (II).

More specifically (II) is linearized by introducing the $\frac{n(n+1)}{2}+nm+m$ variables w_0^J $(\forall\ J \subseteq E \ \ |J| \le 2)$, $w_j^{\emptyset}$ $(\forall\ J \in N \setminus E)$ and $w_j^{\{i\}}$ $(\forall\ i \in E, \forall\ j \in N \setminus E)$

where :

$$
\begin{cases}
w_0^J \text{ is a substitute for } F_{|J|}(J,\emptyset) & (\forall\, J \subset E, |J| \le 2) \\
w_j^{\emptyset} \text{ is a substitute for } x_j & (\forall\, j \in N \setminus E) \\
w_j^{\{i\}} \text{ is a substitute for } x_j F_1(\{i\},\emptyset) & (\forall\, j \in N \setminus E, \forall\, i \in E)
\end{cases}
$$

and it is assumed that $w_0^{\emptyset} = 1$.

By introducing the notation :

$$W_0^{J,\{i\}} = F_1(J, \{i\} \setminus J) \qquad \forall\, i \in E,\ J \subseteq \{i\}$$

$$W_j^{J,\{i\}} = x_j\, F_1(J, \{i\} \setminus J) \qquad \forall\, i \in E,\ J \subseteq \{i\},\ j \in N$$

the nonlinear system (II) can be rewritten as :

$$\text{(III)} \begin{cases} \sum_{j=1}^{n+m} a^j\, W_j^{J,\{i\}} - b\, W_0^{J,\{i\}} \leq 0 & \forall\, i \in E, \forall\, J \subseteq \{i\} \\ W_j^{J,\{i\}} - W_0^{J,\{i\}} \leq 0 & \forall\, i \in E, \forall\, J \subseteq \{i\},\ \forall\, j \in E \\ W_j^{J,\{i\}} \geq 0 & \forall\, i \in E, \forall\, J \subseteq \{i\},\ \forall\, j \in N \\ W_0^{J,\{i\}} \geq 0 & \forall\, i \in E, \forall\, J \subseteq \{i\} \end{cases}$$

The *linearized* version of (II) in terms of the w variables (denoted (III)') is then deduced from (III) by carrying out the following substitutions :

$$W_0^{J,\{i\}} = \sum_{J \subseteq H \subseteq \{i\}} (-1)^{|H \setminus J|} w_0^H \quad \forall\, i \in E, \forall\, J \subseteq \{i\} \tag{5}$$

(this yields $W_0^{\{i\},\{i\}} = w_0^{\{i\}}$ for J = {i}

and $W_0^{\emptyset,\{i\}} = w_0^{\emptyset} - w_0^{\{i\}} = 1 - w_0^{\{i\}}$ for J = Ø).

$$W_j^{J,\{i\}} = \sum_{J \subseteq H \subseteq \{i\}} (-1)^{|H \setminus J|} w_0^{H \cup \{j\}} \quad \forall\, i \in E, \forall\, J \subseteq \{i\}, \forall\, j \in E \tag{6}$$

$$W_j^{J,\{i\}} = \sum_{J \subseteq H \subseteq \{i\}} (-1)^{|H \setminus J|} w_j^H \quad \forall\, i \in E, \forall\, J \subseteq \{i\}, \forall\, j \in N \setminus E \tag{7}$$

As an immediate property of the above linearization we note the so-called ***« symmetry » condition*** satisfied by the W variables in P^{1}_{RLT} :

$$\forall\, i \in E, \forall\, j \in E : \qquad W_j^{\{i\},\{i\}} = W_i^{\{j\},\{j\}} \qquad (8)$$

(both values being equal to $w_0^{\{i, j\}}$)

On the other hand it is easily seen that $P^1_{L\&P}$ is a relaxation of P^1_{RLT} :

Proposition 1 : $P^1_{RLT} \subseteq P^1_{L\&P}$

Proof :

Let $\overline{w}$ denote a solution to the linearized system (III)' and $\overline{W} = (\overline{W}_j^{J,\{i\}})$ the values of the W variables corresponding to $\overline{W}$ through (5) (6) (7).

Then $Z = \overline{W}$ is a solution to (II)' i.e. belongs to $P^1_{L\&P}$.

The following result shows that there is a simple relationship between $P^1_{L\&P}$ and P^1_{RLT}.

Proposition 2

Let (II)'' be the linear system deduced from (II)' by adding all « symmetry » conditions of the form :

$$Z_j^{\{i\},\{i\}} = Z_i^{\{j\},\{j\}} \qquad \forall\, i \in E,\ \forall\, j \in E.$$

Then (II)'' is a linear representation of P^1_{RLT}.

Proof : see BALAS et al. (1993), BONAMI & MX (2005).

QUESTION :

DOES THIS SIMPLE RELATIONSHIP EXTEND TO RANK ≥ 2 ?

TO INVESTIGATE THIS ISSUE WE WILL INTRODUCE A NEW HIERARCHY OF RELAXATIONS (DENOTED SRL*) :

$$\overline{P} \supseteq P^{1}_{SRL*} \supseteq P^{2}_{SRL*} \cdots \supseteq P^{n}_{SRL*} \equiv P^{n}_{RLT} \equiv P.$$

3. THE RANK-d SRL* RELAXATION FOR ARBITRARY d, AND CONNECTIONS WITH $\mathbf{P^d_{RLT}}$ AND $\mathbf{P^d_{L\&P}}$

All three relaxations are obtained via linearization of the nonlinear system deduced from (1)-(3) by multiplication of each inequality by every possible d-factor :
$\forall\ J^d \subseteq E,\ \left|J^d\right| = d,\ \forall\ J \subseteq J^d$:

$$\left\{\begin{array}{lll} \sum_{j=1}^{n+m} a^j x_j F_d(J, J^d \setminus J) - b F_d(J, J^d \setminus J) \leq 0 & & (9) \\ x_j F_d(J, J^d \setminus J) - F_d(J, J^d \setminus J) \leq 0 & \forall\ j \in E & (10) \\ x_j F_d(J, J^d \setminus J) \geq 0 & \forall\ j \in N & (11) \\ F_d(J, J^d \setminus J) \geq 0 & & (12) \end{array}\right.$$

The rank-d RLT (Sherali-Adams) relaxation

The linear description of P^1_{RLT} is obtained by linearizing (9)-(12) by introducing new variables w_0^J and w_j^J with the following interpretation :

$$\begin{cases} w_0^J \text{ is a substitute for } F_{|J|}(J,\emptyset) & \forall J \subseteq E,\ J \le \mathrm{Min}\{d+1, n\} \\ w_j^J \text{ is a substitute for } x_j F_{|J|}(J,\emptyset) & \forall j \in N \setminus E, \forall J \subseteq E, |J| \le d \end{cases}$$

(we agree to set : $w_0^{\emptyset} = F_0(\emptyset, \emptyset) = 1$; $w_0^{\{i\}} = x_i, \forall i \in E$; $w_j^{\emptyset} = x_j \quad \forall j \in N \setminus E$)

The linear system describing P^1_{RLT} is then obtained by carrying out the following substitution, $\forall J^d \subseteq E$, $|J^d| = d$, $\forall J \subseteq J^d$:

$$\begin{cases} x_j F_d(J, J^d \setminus J) \text{ is replaced with the expression: } \sum_{J \subseteq H \subseteq J^d} (-1)^{|H \setminus J|} w_0^{H \cup \{j\}} & \forall j \in E \\ x_j F_d(J, J^d \setminus J) \text{ is replaced with the expression } \sum_{J \subseteq H \subseteq J^d} (-1)^{|H \setminus J|} w_j^{H} & \forall j \in N \setminus E \\ F_d(J, J^d \setminus J) \text{ is replaced with the expression } \sum_{J \subseteq H \subseteq J^d} (-1)^{|H \setminus J|} w_0^{H} & \end{cases}$$

The rank-d SRL* relaxation

The linear description of P^d_{SRL*} is obtained by linearizing the nonlinear terms in (9)-(12) by introducing new variables θ_0^J and θ_j^J with the following interpretation :

$$\begin{cases} \theta_0^J \text{ is a substitute for } F_{|J|}(J, \emptyset) & \forall J \subseteq E, |J| \leq d \\ \theta_j^J \text{ is a substitute for } x_j F_{|J|}(J, \emptyset) & \forall j \in N \setminus J, J \subseteq E, |J| \leq d \end{cases}$$

(we agree to set : $\theta_0^{\emptyset} = F_0(\emptyset, \emptyset) = 1$ and $\theta_j^{\emptyset} = x_j$, $\forall j \in N$, moreover θ_j^J is identified with θ_0^J for $j \in J$)

The linear system describing P^d_{SRL*} is then obtained by carrying out the following substitutions, $\forall J^d \subseteq E$, $|J^d| = d$, $\forall J \subseteq J^d$:

$$\begin{cases} x_j F_d(J, J^d \setminus J) \text{ is replaced with the expression: } \sum_{J \subseteq H \subseteq J^d} (-1)^{|H \setminus J|} \theta_j^H & \forall j \in N \\ F_d(J, J^d \setminus J) \text{ is replaced with the expression: } \sum_{J \subseteq H \subseteq J^d} (-1)^{|H \setminus J|} \theta_0^H \end{cases}$$

DIFFERENCES BETWEEN SRL* AND RLT

→ RLT avoids using the variables w_j^H for $j \in E$ because the identification $w_j^H = w_0^{H \cup \{j\}}$ is carried out implicitly
(note however that the variables w_j^H for all $j \in N \setminus E$ are required in RLT)

→ By contrast SRL* involves all the variables θ_j^H, for all $j \in N \setminus H$ (even if $j \in E$)
As a result for $j \in E$, θ_j^H and $\theta_0^{H \cup \{j\}}$ are allowed to take on distinct values.

Thus RLT uses fewer variables, as confirmed by the comparison in terms of
# of variables :

$$\text{RLT}: \quad \underbrace{\sum_{k=1}^{\min\{d+1,n\}} \binom{n}{k}}_{(\#\, w_0^J)} + m \underbrace{\sum_{k=0}^{d} \binom{n}{k}}_{(\#\, w_j^J)} \qquad \text{SRL*}: \quad \underbrace{\sum_{k=2}^{d} \binom{n}{k}}_{(\#\, \theta_0^J)} + \underbrace{\sum_{k=0}^{d} (n+m-k) \binom{n}{k}}_{(\#\, \theta_j^J)}$$

DIFFERENCES (continued)

Another essential difference between SRL* and RLT is that, contrary to RLT,

SRL* features decomposable structure

Indeed in the rank d RLT closure there is only one variable associated with the pair

(j, J^d) when $j \in E \setminus J^d$ and $J^d \subseteq E$ which is : $w_0^{J^d \cup \{j\}}$

whereas, in rank d SRLT* there are d + 1 distinct variables associated with this pair,

namely all the variables $\theta_k^{\left(J^d \cup \{j\}\right) \setminus \{k\}}$ for $k \in J^d \cup \{j\}$

Each block in the rank d SRL* closure corresponds to a cardinality d subset $H \subseteq E$ and

involves the subset of all variables θ_j^H having the same superscript H.

The generalized « symmetry » conditions

Consider the P^d_{SRL*} relaxation for the MIP set P expressed in terms of the θ^J_0 and θ^J_j variables above.

Definition

For any positive integer $p \leq d$ we call generalized « symmetry » conditions at rank p the set of equality constraints of the form :

$$\theta^J_j = \theta_i^{(J\setminus\{i\})\cup\{j\}} \qquad \forall\, J \subset E,\ |J| = p,\ i \in J,\ j \in E \setminus J.$$

Observe that the validity of the above set of constraints with respect to P follows from the identity $x_j\, F_p\, (J, \emptyset) = x_i\, F_p\, ((J \setminus \{i\} \cup \{j\}), \emptyset)$ (when $p = |J|$).

We denote S_p the set of all equality constraints expressing the generalized symmetry conditions at rank p.

Now, the following result shows a simple connection between the linear representation of P^d_{SRL*} and P^d_{RLT} :

Theorem : (M.M. & H.O. 2008)

The linear description of P^{d}_{SRL*} strengthened with S_d yields a linear description of P^{d}_{RLT}.

(P^{d}_{SRL*} thus appears as a relaxation of P^{d}_{RLT}).

Remark

It can be shown (M.M. & H.O. 2008) that all the generalized « symmetry » conditions up to rank d-1 are implicitly satisfied by all θ vectors solving the linear description of P^{d}_{SRL*}.

In view of this it is enough to include the constraints in S_d to ensure that all conditions in S_1, S_2, …, S_d are satisfied.

Now, an interesting question is : how does $P^d_{L\&P}$ relate to P^d_{SRL*} and/or P^d_{RLT} ?

It turns out that $P^d_{L\&P} \equiv P^d_{SRL*}$ ***is only true for d = 1***.

In other words, for $d \geq 2$, strengthening $P^d_{L\&P}$ by adding all « symmetry » conditions up to rank d is not enough to yield P^d_{RLT}.

Indeed it can be shown that $P^d_{SRL*} \subseteq P^d_{L\&P}$ for all $d \geq 2$ with strict inclusion holding in the general case.

Example :

Consider the pure 0-1 set $P \subset \{0, 1\}^5$ defined by the three linear inequalities :

$$\begin{cases} 18x_1 + 15x_2 + 17x_3 + 5x_4 + 13x_5 \le 54 \\ 17x_1 + 22x_2 + 13x_3 + 9x_4 + 25x_5 \le 63 \\ 17x_1 + 19x_2 + 3x_3 + 7x_4 + 11x_5 \le 89 \end{cases}$$

It can be shown that in this case $P^2_{RLT} \subset P^2_{SRL*} \subset P^2_{L\&P}$

for instance considering the objective function :

$z = 1\,900\ x_1 + 500\ x_2 + 200\ x_3 + 100\ x_4 + 300\ x_5$, to be maximized, the optimal values of the three relaxations are 2 652.27, 2 653.99, and 2 668.1 respectively.

(Note that, in the above example, the P^2_{SRL*} bound significantly improves over the $P^2_{L\&P}$ bound).

A FEW PRELIMINARY COMPUTATIONAL EXPERIMENTS

We consider a series of multidimensional knapsack problems with n 0-1 variables and p inequality constraints for (n, p) ranging from (25, 20) to (50, 5) :

- for each size, 5 randomly generated instances are solved.
- The Chu-Beasly (1998) random generation procedure is used.

Number of variables and rows in rank 2 L&P, SRL* and S&A relaxations :

Instance		$P^2_{L\&P}$		P^2_{SRL*}		$P^2_{S\&A}$	
n	m	nv	nc	nv	nc	nv	Nc
25	20	31225	642400	7225	81600	2625	81600
35	20	85715	157080	20265	209440	7175	209440
40	10	127960	194220	30460	274560	10700	274560
50	5	249950	341775	60075	504700	20875	504700

Comparing strengths of rank 2 Lift-and-Project, SRL* and Sherali-Adams relaxations

Instance	Nbr int vars	Nbr rows	LP optimum	MIP optimum	$P^2_{L\&P}$			P^2_{SRL*}			$P^2_{S\&A}$		
					Optimum	time (sec)	gap (%)	Optimum	time (sec)	gap (%)	Optimum	time (sec)	gap (%)
mknap25-20-1	25	20	5267.23	4550.52	5092.31	15.37	24.41	5025.21	24.35	33.77	5000.05	38.50	37.28
mknap25-20-2	25	20	5518.78	4808.26	5318.28	12.94	28.22	5261.49	24.39	36.21	5238.14	46.19	39.50
mknap25-20-3	25	20	5514.51	4882.83	5345.26	15.78	26.79	5258.88	21.09	40.47	5234.73	40.23	44.29
mknap25-20-4	25	20	5124.31	4543.78	4936.32	22.13	32.38	4876.02	25.79	42.77	4850.14	41.05	47.23
mknap25-20-5	25	20	5463.80	4669.15	5292.61	12.71	21.54	5219.96	21.86	30.69	5195.32	43.69	33.79
mknap35-20-1	35	20	7474.18	6833.80	7374.71	63.26	15.53	7316.85	120.67	24.57	7297.53	424.64	27.58
mknap35-20-2	35	20	7423.12	6844.35	7293.34	99.57	22.42	7238.12	116.62	31.96	7214.38	405.30	36.07
mknap35-20-3	35	20	7540.20	7032.55	7419.73	80.19	23.73	7363.08	111.81	34.89	7340.73	443.21	39.29
mknap35-20-4	35	20	7865.69	7110.52	7712.40	92.07	20.30	7661.08	133.29	27.10	7640.63	490.50	29.80
mknap35-20-5	35	20	7460.34	6859.09	7318.78	59.96	23.54	7258.52	128.89	33.57	7237.11	425.60	37.13
mknap40-10-1	40	10	8909.07	8487.67	8803.25	266.64	25.11	8765.02	174.64	34.18	8753.57	1480.32	36.90
mknap40-10-2	40	10	8473.43	8087.61	8414.31	155.06	15.32	8383.69	156.47	23.26	8369.33	1157.23	26.98
mknap40-10-3	40	10	9140.92	8702.34	9050.65	175.74	20.58	9002.66	134.49	31.52	8986.65	1339.67	35.17
mknap40-10-4	40	10	8693.70	8276.43	8615.76	156.70	18.68	8574.76	186.70	28.50	8562.51	1249.36	31.44
mknap40-10-5	40	10	8950.21	8495.40	8850.56	273.93	21.91	8810.29	132.65	30.76	8798.76	1393.64	33.30
mknap50-5-1	50	5	11806.96	11505.21	11772.20	300.82	11.52	11762.21	190.51	14.83	11756.57	9714.17	16.70
mknap50-5-2	50	5	12262.04	11917.85	12201.18	370.16	17.68	12180.94	257.07	23.56	12175.98	11532.94	25.00
mknap50-5-3	50	5	11930.67	11703.91	11874.63	313.44	24.72	11852.02	264.50	34.69	11846.04	11551.98	37.32
mknap50-5-4	50	5	11725.95	11553.96	11692.77	619.62	19.29	11675.91	373.95	29.10	11668.50	10345.33	33.40
mknap50-5-5	50	5	11161.87	10815.27	11086.81	571.44	21.65	11058.60	318.50	29.79	11048.98	11253.49	32.57

4. THE CASE OF LINEARLY CONSTRAINED PSEUDOBOOLEAN FUNCTION OPTIMISATION PROBLEMS

Let f be a pseudo-boolean function of degree d in n variables indexed in E = {1, 2, …, n}, of the form :

$$F(x) = c_0 + \sum_{\substack{J \subset E \\ |J| \leq d}} c_J \left(\prod_{j \in J} x_j \right)$$

where c_0 and c_J ($J \subset E$, $|J| \leq d$) are given reals.

$K \subseteq \mathbb{R}^n$ being a polyhedron (specified by a given set of linear equality / inequality system) we are interested in the following linearly constrained pseudo-boolean optimization problem :

$$(\text{PBO}) \begin{cases} \text{Minimize } f(x) \\ \text{s.t.:} \\ x \in K \cap \{0,1\}^n \end{cases}$$

The above problem is classically reformulated as the following MIP with linear objective function :

$$
(\text{MIP-PBO})\begin{cases}
\text{Min } c_0 + \sum\limits_{\substack{J \subset E \\ |J| \le d}} c_J\, u_J \\
\text{s.t.:} \\
\quad \sum\limits_{k \in J} x_k - u_J \le |J| - 1 & \forall J \subseteq E, |J| \le d \\
\quad u_J \le x_j & \forall J \subseteq E, |J| \le d, j \in J \\
\quad x \in K \cap \{0,1\}^n
\end{cases}
$$

The linear relaxation ($\overline{\text{MIP}-\text{PBO}}$) of the above is obtained by replacing $x \in K \cap \{0, 1\}^n$ with $x \in K \cap [0, 1]^n$.

Then we have :

Proposition (M.M. & H.O. 2008)

For any integer $k \leq d$, both rank k relaxations P^k_{SRL*} and P^k_{RLT} of (MIP-PBO) coincide.

[In the special case $d = 2$ (quadratic pbf optimization) the coincidence between P^1_{RLT} and $P^1_{L\&P}$ was already pointed out in Bonami & Mx (2006)]

Moreover, it can be shown (M.M. & H.O. 2008) that the presence of all constraints of the form :

$$\begin{cases} \sum_{k \in J} x_k - u_J \le |J| - 1 & \forall\, J \subseteq E,\ |J| \le d \\ u_J \le x_j & \forall\, J \subseteq E,\ |J| \le d,\ j \in J \end{cases}$$

characterizes those MIPs for which P^{d}_{SRL*} and P^{d}_{RLT} coincide.

5. PRELIMINARY COMPUTATIONAL EXPERIMENTS ON LINEARLY CONSTRAINED PBO

We consider the problem of minimizing a quadratic submodular pseudoboolean function in n variables under a double-sided constraint of the form :

$$\alpha n \leq \sum_{j=1}^{n} x_j \leq (1-\alpha)\, n$$

(with α chosen in the range $[0, \frac{1}{2}]$).

This problem is known to be NP-hard (GAREY & JOHNSON, 1979, p. 210).

Comparing strengths of rank 2 SRL*, RLT and L&P relaxations for MIN-QPBF with cardinality constraints ($0.4\,n \le \Sigma\, x_i \le 0.6\,n$)

[α-β]	Instance	# var	# quad terms	MIP opt	LP relax	L&P			SRL*			RLT		
						Opt. Val.	Time (sec.)	Gap closed (%)	Opt. Val.	Time (sec.)	Gap closed (%)	Opt. Val.	Time (sec.)	Gap closed (%)
0.4-0.6	15-1	15	78	734	0 .00	734	111	100	734	26	100	734	25	100
	15-2	15	78	686	0.00	686	75	100	686	23	100	686	23	100
	15-3	15	78	818	0.00	818	108	100	818	24	100	818	23	100
	15-4	15	78	719	0.00	719	68	100	719	24	100	719	24	100
	15-5	15	78	619	0.00	619	68	100	619	24	100	619	23	100
0.4-0.6	25-1	25	150	1238	0.00	1209.70	1413	97.7	1210	327	97.7	1210	312	97.7
	25-2	25	150	1118	0.00	1118	1061	100	1118	1060	100	1118	1054	100
	25-3	25	150	1148	0.00	1148	1206	100	1148	521	100	1148	1019	100
	25-4	25	150	1297	0.00	1256.60	1492	96.9	1256.8	335	96.9	1256.8	324	96.9
	25-5	25	150	1309	0.00	1239.34	1293	94.7	1239.6	673	94.7	1239.6	680	94.7
0.45-0.55	25.1	25	150	1279	0.00	1263.66	1695	98.8	1264.3	244	98.8	1264.3	757	98.8
	25.2	25	150	1207	0.00	1203.02	2275	99.7	1203.8	868	99.7	1203.8	1380	99.7
	25-3	25	150	1264	0.00	1225.5	1750	97	1226	686	97	1226	759	97
	25-4	25	150	1358	0.00	1310.63	1633	96.5	1311.2	234	96.5	1311.2	838	96.5
	25-5	25	150	1385	0.00	1294.32	1593	93.4	1294.7	496	93.5	1294.7	909	93.5

Rank-2 SRL* relaxation for MIN-QPBF instances with cardinality constraints

Instance	Nbr nodes	Nbr edges	α	β	SRL*			
					Optimum	Time (sec.)	Gap (%)	Time (sec)
Mincut-45-50-1.rdy	45	495	0.4	0.6	7741.84	0 : 06 : 18	87.36	0 :16 :50
Mincut-45-50-2.rdy	45	495	0.4	0.6	7788.73	0 : 00 : 47	88.70	0 :16 :49
Mincut-45-50-3.rdy	45	495	0.4	0.6	8129.05	0 : 00 : 47	87.47	0 :16 :49
Mincut-45-50-4.rdy	45	495	0.4	0.6	7634.36	0 : 03 : 07	85.96	0 :16 :50
Mincut-45-50-5.rdy	45	495	0.4	0.6	8147.92	0 : 05 : 41	87.41	0 :16 :49
Mincut-55-25-1.rdy	55	371	0.4	0.6	4926.36	0 : 02 : 24	87.30	0 :46 :21
Mincut-55-25-2.rdy	55	371	0.4	0.6	4967.47	0 : 02 : 16	87.39	0 :46 :19
Mincut-55-25-3.rdy	55	371	0.4	0.6	4767.53	0 : 02 : 15	89.95	0 :46 :19
Mincut-55-25-4.rdy	55	371	0.4	0.6				
Mincut-55-25-5.rdy	55	371	0.4	0.6	4760.89	0 : 02 : 36	92.27	0 :46 :19

RANK 2 SRL* FOR QUADRATICALLY CONSTRAINED PROBLEMS

(2 quadratic constraints + cardinality constraints)

Instance	[α, β]	CPLEX BRANCH & BOUND				SRL*		
		best bound	Best int sol	Time (sec.)	# nodes	Opt. Val.	Time (sec.)	Gap closed (%)
45.1	[0.4 , 06]	6421.5	8089	3600	180757	6987.39	250	86.4
45.2	[0.4 , 06]	6845.5	7539	3600	150668	6973.74	298	92.5
45.3	[0.4 , 06]	6468.26	7759	3600	185655	6948.02	113	89.5
45.4	[0.4 , 06]	6709.2	8103	3600	179806	7061.34	55	87.1
45.5	[0.4 , 06]	7123	8549	3600	164323	7505.94	64	87.8
55.1	[0.4 , 06]	7500.71	12516	7200	144511	10208.99	152	81.6
55.2	[0.4 , 06]	7094.05	11906	7200	193971	9761.26	152	82
55.3	[0.4 , 06]	8030.00	12698	7200	154964	10424.08	168	82.1
55.4	[0.4 , 06]	8228.55	13314	7200	194101	10792.79	488	81.1
55.5	[0.4 , 06]	8364.53	13189	7200	153789	10850.53	167	82.3