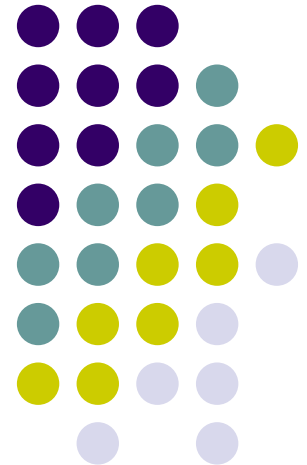


# Single Commodity Stochastic Network Design under Probabilistic Constraints with Continuously Distributed Random Variable.

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- Examples
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# Definition of a Network



## *Definition 1.*

A network  $G=(N, y)$  is a finite collection of nodes  $N$  and a capacity function  $y(i, j)$  on the set of arcs  $(i, j) \in N \times N$  which is assumed to take nonnegative values or  $\infty$ .

## *Definition 2.*

A flow is a real-valued function  $f(i, j)$   $(i, j) \in N \times N$  which satisfies the following conditions:

$$f(i, j) + f(j, i) = 0$$

$$f(i, j) \leq y(i, j), (i, j) \in N \times N$$

$$y(i, j) \geq 0 \text{ or } y(i, j) = +\infty$$

# Feasibility Conditions



## Definition 3

A demand function  $d(i), i \in N$  is a real-valued function on the set of nodes.

Demand is said to be *feasible* if there exists a flow satisfying

**Definition 2** and the relations  $f(N, i) \geq d(i), i \in N$

## Theorem 1

The demand  $d(i), i \in N$  is feasible iff the following inequalities hold:

$d(S) \leq y(\bar{S}, S), S \subset N, \text{ where } \bar{S} = N \setminus S$  (Gale-Hoffman inequalities)

# Network Optimization Problem Formulation



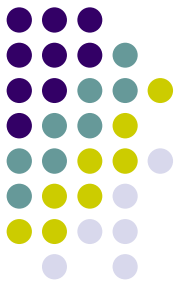
Let  $x_i$  represent capacity assigned to node  $i$

Assume it is diminished by random deficiency  $\xi_i$  (normally distributed)

We define a demand function corresponding to the network as  $d_i = \xi_i - x_i$

Then in order to satisfy the demand with the required reliability level  $p$  the following probabilistic constraints need to be satisfied:

$$P(d(S) \leq y(\bar{S}, S), \text{ all } S \subset N) \geq p$$



# Network Optimization Problem Formulation (continued)

The unknown decision variables are the node capacities  $x_i, i \in N$  and the arc capacities  $y_{ij}, (i, j) \in N \times N$ .

The static problem formulation is as follows:

$$\min \left\{ \sum_{i \in N} c_i(x_i) + \sum_{(i, j) \in N \times N} c_{ij}(y_{ij}) \right\}$$

subject to

$$P(d(S) \leq y(S, \bar{S}), S \subset N) \geq p$$

$$A_1 x + A_2 y \geq b$$

$$x, y \geq 0$$

# Stochastic Programming problem Formulation



$$\min c^T x$$

subject to

$$P(y(S, \bar{S}) \geq d(S), S \subset N_1, S \text{ non-eliminated}) \geq p$$

where  $N_1$  is a collection of the nodes with random demand,

$$y(S, \bar{S}) \geq d(S), S \subset N_2, N_2 = N \setminus N_1$$

$$d(S) = \sum_{i \in S} (\xi_i - x_i), l_i \leq x_i \leq u_i$$



# Illustrative Example

Consider the network of 3 nodes. Assume that the line capacities are symmetrical and introduce the notations:

$$y_1 = y_{12} = y_{21}, y_2 = y_{23} = y_{32}, y_3 = y_{13} = y_{31}$$

Then the system relations give the following system of inequalities:

$$\xi_1 - x_1 + \xi_2 - x_2 + \xi_3 - x_3 \leq 0,$$

$$\xi_1 - x_1 \leq y_1 + y_3,$$

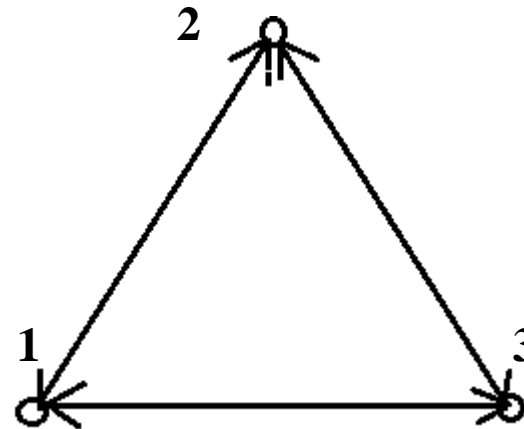
$$\xi_2 - x_2 \leq y_1 + y_2,$$

$$\xi_3 - x_3 \leq y_2 + y_3,$$

$$\xi_1 - x_1 + \xi_2 - x_2 \leq y_2 + y_3,$$

$$\xi_2 - x_2 + \xi_3 - x_3 \leq y_1 + y_3,$$

$$\xi_1 - x_1 + \xi_3 - x_3 \leq y_1 + y_2.$$





# Prékopa-Boros Elimination (by topology)



## Theorem 2

The inequality in the **Theorem 1** is redundant among the Gale-Hoffman inequalities iff at least one of the graphs  $G(S), G(\bar{S})$  is not connected.

In that case the inequality  $d(S) \leq y(\bar{S}, S)$  is the sum of other Gale-Hoffman inequalities.

Based on **Theorem 2**, we can eliminate those inequalities which are the sums of others.

We subsequently eliminate the sets  $S \subset N$  according to their cardinalities, and look at sets  $\bar{S}, S$  such that at least one of  $G(S), G(\bar{S})$  is not connected. Then eliminate the corresponding inequality.

# Mathematical Properties of the Constraint Set. Log-concavity



The set  $\{s \mid F(s) = p\}$  is an infinite set of pLEP of the continuously distributed random vector  $\xi$  and  $T_x$  is an element of a shaded region (see Figure 1, the case of  $r=2$ ) iff  $\{Tx \geq M \xi\} \geq p$

The convexity of the set is assured by the use of log-concavity theorem and the following theorem (Prékopa).

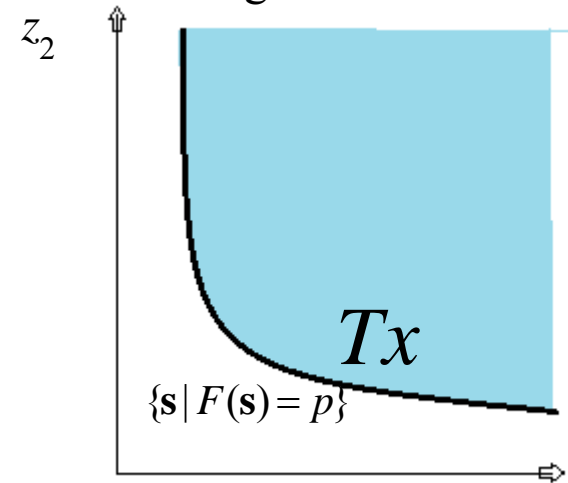


Figure 2

### Theorem 3

Let  $\xi \in \mathbf{n}$  be a random vector that has log-concave probability distribution, i.e.

$$P(\lambda A + (1 - \lambda)B) \geq P^\lambda(A)P^{1-\lambda}$$

where  $A, B \subset \mathbf{n}$  are convex sets and  $0 < \lambda < 1$ .

Let  $M$  be an  $r \times n$  matrix with real-valued entries. Then the vector  $\zeta = M\xi$  has logconcave distribution.

# Multivariate Quantiles (multivariate Value at Risk)



Let  $X = (X_1, X_2, \dots, X_r)^T$  be a continuously distributed random variable.

Its  $p$ -quantile is defined as  $F(z) = P(X \leq z), z \in \mathbb{R}^r, 0 \leq p \leq 1$   
where  $Q_p = \{z | F(z) = p\}$

If  $X$  has a non-degenerate normal distribution the Prékopa's log-concavity theory ensures that the set  $\{z | F(z) \geq p\}$  is a convex set.

In case of degenerate normal distribution (partial sums of random variables in the constraints) convexity is assured using **Theorem 3**.

# The PVB (Prékopa, Vizvári, Badics) Algorithm



PVB (Prékopa, Vizvári, Badics) algorithm

Step 0. Enumerate all  $p$ -efficient points  $s_1, \dots, s_N$ . Initialize  $k \leftarrow 0$  and go to Step 1.

Step 1. Determine the vectors  $w_1, \dots, w_h$ .

Step 2. Solve the LP:

$$\begin{aligned} \min \quad & c^T x \\ \text{subject to} \quad & Ax \geq b \\ & w_l^T (Tx - u - \bar{s}) = 0, \quad l = 1, \dots, h \\ & v_i^T (Tx - u - \bar{s}) \geq 0, \quad i = 1, \dots, k \\ & x \geq 0, \quad u \geq 0. \end{aligned}$$

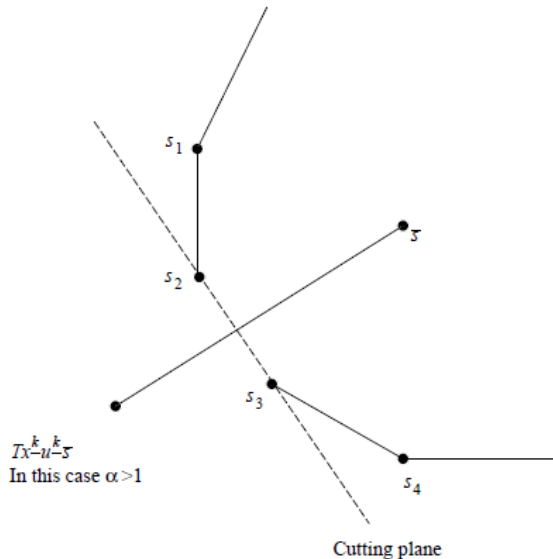
If  $k = 0$ , then ignore the constraint for  $i = 1, \dots, k$ . Let  $(x^k, u^k)$  be an optimal solution. Go to Step 3.

Step 3. For  $t = Tx^k - u^k - \bar{s}$  solve problem (28) and let  $\alpha$  be the optimum value. If  $\alpha \leq 1$ , Stop,  $(x^k, u^k)$  is an optimal solution to problem (26). Otherwise go to Step 4.

Step 4. Create the cut

$$v_{k+1}^T (Tx - u - \bar{s}) \geq 0,$$

set  $k \leftarrow k + 1$  and go to Step 2.



# The Supporting Hyperplane Algorithm

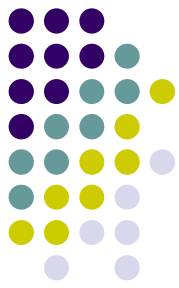


The supporting hyperplane method, developed originally by Veinott (1967) was adapted by Szantai (1988) to solve probabilistic optimization problem with continuously distributed random vector with logconcave p.d.f.

Let  $\mathbf{h}(\mathbf{x}) = P(\mathbf{T}\mathbf{x} \geq \mathbf{M}\xi) - \mathbf{p}$ . Suppose also that the convex polyhedron

$\mathbf{K}^0 = \{\mathbf{x} \mid \mathbf{A}\mathbf{x} \geq \mathbf{b}, \mathbf{x} \geq 0\}$  is bounded and that Slater's condition is satisfied:

$\exists \mathbf{x}^0 \in \mathbf{K}^0$  such that  $\mathbf{h}(\mathbf{x}^0) > 0$ .



The supporting hyperplane can be summarized as follows:

Step 0. Find  $x^0$  satisfying  $Ax^0 \geq b$ ,  $x^0 \geq 0$ ,  $h(x^0) > 0$ . Go to Step 1.

Step 1. Solve the LP:

$$\begin{aligned} & \min c^T x \\ & \text{subject to} \\ & \quad Ax \geq b \\ & \quad \nabla h(x^i)(x - x^0) \geq 0, \quad i = 1, \dots, k \\ & \quad x \geq 0. \end{aligned}$$

Let  $x^{*k}$  be an optimal solution. Go to Step 2.

Step 2. Check for the sign of  $h(x^{*k})$ . If  $h(x^{*k}) \geq 0$ , Stop, optimal solution to problem has been found. Otherwise go to Step 3.

Step 3. Find  $\lambda^k$  such that  $0 < \lambda^k < 1$  and

$$h(x^0 + \lambda^k(x^{*k} - x^0)) = 0.$$

Define  $x^{k+1} = x^0 + \lambda^k(x^{*k} - x^0)$  and go to step 4.

Step 4. Introduce the cut:

$$\nabla h(x^{k+1})(x - x^0) \geq 0,$$

set  $k \leftarrow k + 1$  and go to Step 1.



# Combination of the PVB (Prékopa, Vizvári, Badics) and the supporting hyperplane algorithms (hybrid algorithm)

The combined application of PVB and supporting hyperplane algorithms provides solution method for problem , where r.v. has continuous distribution. At each iteration both lower and upper bounds for the optimum are available, and if they are sufficiently close we may stop.

**Step 0.** Find satisfying Slater's condition. Initialize  $k \leftarrow 0$ ;

**Step 1.** Choose  $x^1 \dots x^r$  linearly independent feasible vectors such that :  $F(Tx^i) = p, i=1, \dots, r$

Set  $k \leftarrow r$

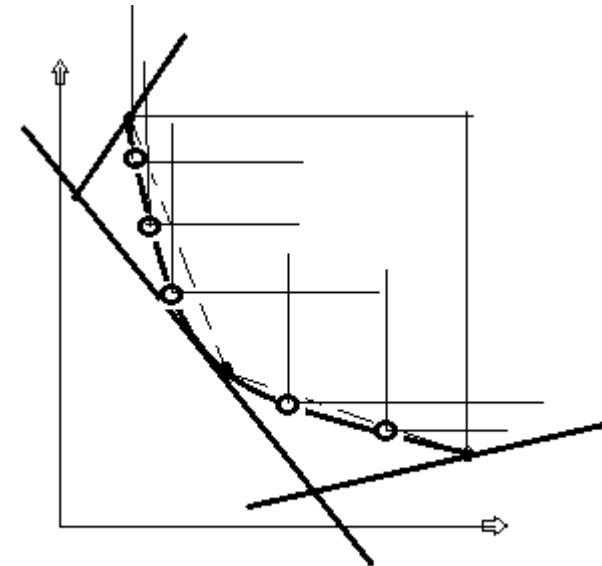
**Step 2.** Solve the LP:

$$\begin{aligned} \min c^T x \\ \text{subject to} \end{aligned} \tag{5}$$

$$Ax \geq b$$

$$\nabla h(x^i)(x - x^0) \geq 0, i = 1, \dots, k$$

$$x \geq 0$$



--- Cutting plane  
 ——— Supporting hyperplane



# Combination of the PVB (Prékopa, Vizvári, Badics) and the Supporting Hyperplane Algorithms (hybrid algorithm) (continued)

Let  $x^*$  be an optimal solution. If  $F(Tx^{*k}) \geq p$  then Stop, we found an optimal solution. Otherwise let  $x^{k+1} = x^0 + \lambda^k (x^{*k} - x^0)$  where  $\lambda^k$  is such that  $F(T(x^0 + \lambda^k (x^{*k} - x^0))) = p$ . Set  $k \leftarrow k+1$  and  $U_k = c^T x^{k+1}$

Obviously,  $U_k$  is an upper bound on the optimum value

**Step 3.** Solve the following LP by PVB algorithm:

$$\min c^T x$$

subject to

$$Ax \geq b$$

$$Tx - u - \bar{s} = \sum_{i=1}^k (s_i - \bar{s}) \quad (5)$$

$$\sum_{i=1}^k \lambda_i = 1$$

$$x \geq 0, u \geq 0, \lambda \geq 0,$$

where  $s_i = Tx^i, i=1, \dots, k; s = (1/k) \sum_{j=1}^k s_j$   
 Vectors  $s_1, \dots, s_k$  span the whole space  $r$

Let  $L_k$  be the optimum value of problem (5)

Then it gives us the lower bound for the optimum value of problem (1)





# Combination of the PVB (Prékopa, Vizvári, Badics) and the Supporting Hyperplane Algorithms (hybrid algorithm) (continued)

**Step 4.** If  $U_k - L_k \leq \varepsilon$  (tolerance level) then STOP, otherwise go to **Step 5**

**Step 5.** Having solved problem (5) the number of the vectors appended to the set  $\{s^{(1)}, \dots, s^{(k)}\}$  is the same as the number of cutting planes encountered, let it be  $l$ . They belong to the boundary of prob. constraint set. Then we identify those bases that turned optimal auxiliary LP in the course of cutting plane method, take the intersections of the normals of the corresponding cutting planes with the p=efficient surface (quantile surface). Set  $k \leftarrow k + l$  and go to **Step 2.**



# Approximation of the Boundary Surface of the Probabilistic Constraints with Normally Distributed Random Variable

One of the most important problems encountered during the implementation of the hybrid algorithm for the case of normally distributed  $\xi$  is approximation of the boundary of probabilistic constraint. The choice of approximating function was a hyperboloid-like surface.

Consider the function defining the prob. constraint:

$$G(x) = P(g_1(x, \xi) \geq 0, \dots, g_r(x, \xi) \geq 0) = P(Tx \geq M\xi)$$

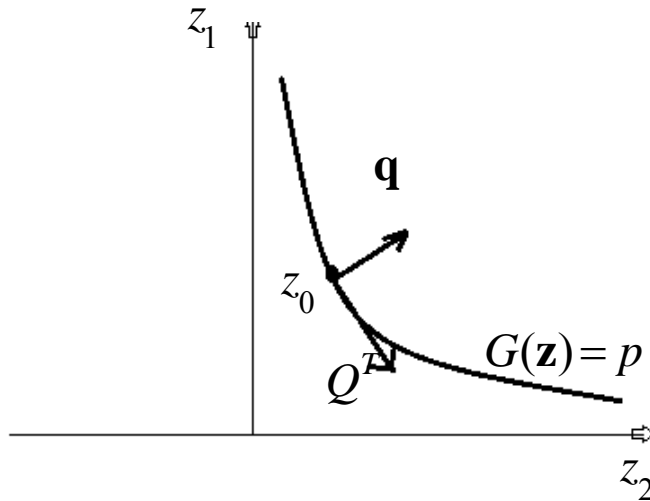
We will construct the approximation of the p-efficient set of the normal distribution in the form of a set of inequalities:

$$q^T(z - z_0) \geq \sqrt{(z - z_0)^T Q^T A Q (z - z_0) + k^2} - k$$

where  $Q$  is an orthogonal matrix of change of coordinates.



# Construction of the Approximating Function



$$\mathbf{z} \in \mathbb{R}^m, \mathbf{x} \in \mathbb{R}^{m-1}$$

$$G(\mathbf{x}) = G(Q\mathbf{z}) = \langle \mathbf{q}, \mathbf{z} \rangle$$

After finding the gradient and Hessian of the function  $G$  we can evaluate all the important coefficients in the approximation function.

They are as follows:

$$A = \frac{(\min \mu_i)^4}{4M^2} \mathbf{I}, \text{ where } \min \mu_i \text{ is the smallest eigenvalue of } \nabla^2 G(0);$$

$$k = \frac{5}{24} \frac{(\min \mu_i)^3}{M^2}; M \text{ is an approximation of the derivative of the curvature of } G$$

# Network optimization problem (simple numerical example)



We tested the hybrid algorithm on various networks, here is the example of a power network between 4 cities

The input parameters are as follows:

Demand  $\mu$  : [2 3 4 5]

Demand  $\sigma$  : [1.4142 1.7321 2 2.2361]

Cost of production: [2 2 2 2]

Cost of transmission capacities: [1 1 1 1 1 1]

**p: 0.9**

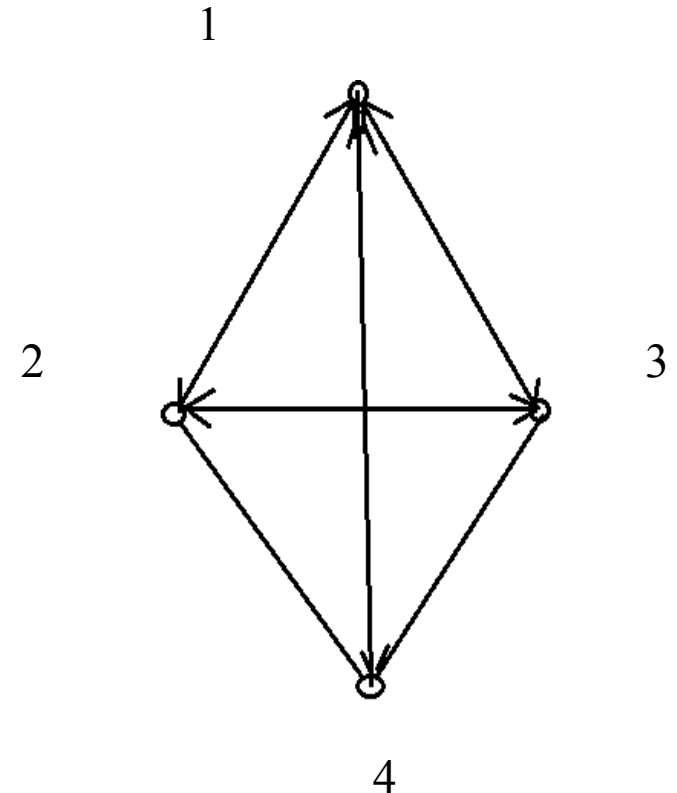
**Optimal solution found:**

**Optimal production amount:**

**3.0958 4.1878 5.5071 6.6923**

**Optimal line capacities:**

**0.7406 0.9512 0.9003 1.0110 1.0708 1.0722**



# Network optimization (numerical example 2)

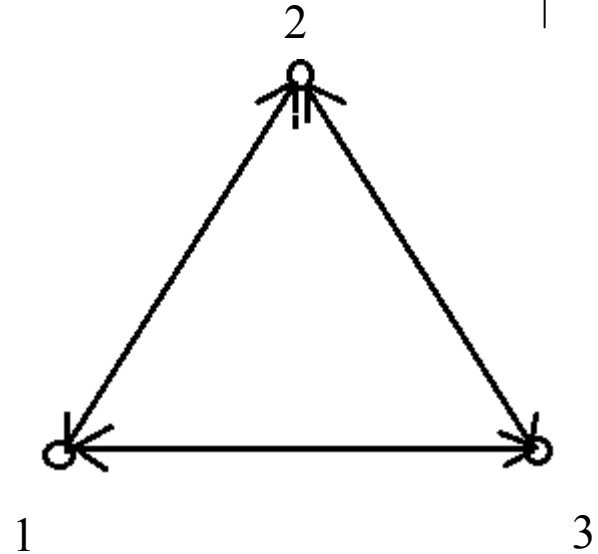


demand  $\mu$  : [2 3 4]  
demand  $\sigma$  : [1 1 1]  
cost\_production: [2 2 2]  
cost\_capacity: [1 1 1]  
**p: 0.95**

**Optimal solution found:**

**Optimal production amount:  
3.0337 4.0335 5.0335**

**Optimal line capacities:  
0.7301 0.7300 0.7300**





## Future work

- Consider other distributions
- Work on online algorithm
- Improve on timing of the p-quantile approximations in higher dimensions ( $\geq 50$ )

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