Heterogeneous models for nonlinear flows on networks

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Lighthill-Whitham-Richards model:
\[
\rho_t + (\rho v(\rho))_x = 0, \quad \rho \in [0, \rho_{max}] \text{ density of cars}
\]

De Saint-Venant equation:
\[
\begin{aligned}
H_t + (H v)_x &= 0 \\
v_t + \left[\frac{1}{2} v^2 + g H\right]_x &= 0
\end{aligned}
\]

\(H\) water height, \(v\) velocity, \(g\) gravity

Isothermal Euler with friction:
\[
\begin{aligned}
\rho_t + (\rho u)_x &= 0 \\
(\rho u)_t + (\rho u^2 + a^2 \rho)_x &= -f_g \frac{\rho u|\rho u|}{2D\rho}.
\end{aligned}
\]

\(\rho\) density, \(u\) velocity
Dynamics at junctions

Rule (A) : Out. Fluxes Vector = A · Inc. Fluxes Vector

Traffic distribution matrix \( A = (\alpha_{ji}) \), \(0 < \alpha_{ji} < 1\), \( \sum_j \alpha_{ji} = 1 \)

Rule (B) : Maximize \( \| \text{Inc. Fluxes Vector} \| \)

Rule (B) is an “entropy” type rule: maximize velocity
Integration of models and scales

Car trajectories and moving bottlenecks

Mixed ODE-PDE model

Queue buffer occupancy change is given by the difference between incoming and outgoing flux

\[ \partial_t q_j(t) = f_{j-1}(\rho_{j-1}(b_{j-1}, t)) - f_j(\rho_j(a_j, t)) \]

\[ f_j(\rho_j(a_j, t)) = \begin{cases} 
\min\{f_{j-1}(\rho_{j-1}(b_{j-1}, t)), \mu_j\} & q_j(t) = 0 \\
\mu_j & q_j(t) > 0 
\end{cases} \]
Optimization of vehicular traffic

\[ J_1(t) = \sum_i \int_{I_i} v(\rho_i(t, x)) \, dx, \]

\[ J_2(t) = \sum_i \int_{I_i} \frac{1}{v(\rho_i(t, x))} \, dx. \]

\[ SGW = \int_0^T \int_{\bigcup I_i} |Dv(\rho)| \, dt \, dx. \]
Optimal control for supply chains

\[
\begin{align*}
\partial_t \rho_j (x, t) + \partial_x \min \{\mu_j, v_j \rho_j (x, t)\} &= 0 \quad j = 1, \ldots, N, \\
q_j (t) &= f_{j-1} (\rho_{j-1} (b_{j-1}, t)) - f_j^{\text{inc}} \quad j = 2, \ldots, N, \\
\rho_1 (a_1, t) &= u(t) \\
\rho_j (x, 0) &= \rho_{j,0} (x) \quad j = 1, \ldots, N, \\
q_j (x, 0) &= q_{j,0} \quad j = 2, \ldots, N,
\end{align*}
\]

Theorem.

\[
J(u) = \sum_{j=1}^n \int_0^T q_j(t) dt + \int_0^T [v_N \cdot \rho_N(b_N, t) - \psi(t)]^2 dt \equiv J_1(u) + J_2(u),
\]

Existence of solutions

Take minimizing sequence: compactness by Helly and Ascoli Arzela’ Theorem.

\[
q_n \to q \text{ in } C^0, \text{ thus } J_1(u_n) \to J_1(u)
\]

\[
\int_0^T \left( (v_N \cdot \rho_N^N(b_N, t)) - \psi(t) \right)^2 - (v_N \cdot \rho_N(b_N, t)) - \psi(t))^2 \right) dt =
\]

\[
\int_0^T \left( (v_N)^2 \left( (\rho_N^N(b_N, t))^2 - (\rho_N(b_N, t))^2 \right) + 2\psi(t)v_N (\rho_N^N(b_N, t) - \rho_N(b_N, t)) \right) dt \leq
\]

\[
||2\psi(t) + v_N(\rho_N^N(b_N, t) + \rho_N(b_N, t))||_\infty \cdot ||v_N(\rho_N^N(b_N, t) - \rho_N(b_N, t))||_{L^1}.
\]
Tangent vectors for numerical optimization

Figure 7: Supply chain with 11 arcs, case a. Left: $J_1$ versus iteration steps; right: "path" followed by the steepest descent algorithm in the plane $(t_1, t_2)$. 
Lemma. If we start from empty network, then each road presents at most one regime change for every time.

1. Make use of theoretical results to bound the number of regime changes.
2. Use simplified flux function with two characteristic speeds.
3. Track exactly the regime change (generalized characteristic) and use upwind for each zone.

Network with 5000 roads parametrized by [0,1], h space mesh size, T real time.

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CPU time:

- $T = 10$ s
- $T = 30$ s
- $T = 60$ s
- $T = 90$ s

Free phase, congested phase.
Thank you for your attention!

1. G. Bastin, A. Bayen, C. D'Apice, X. Litrico, B. Piccoli, Open problems and research perspectives for irrigation channels, Networks and Heterogeneous Media, 4 (2009), i-v.
Real data

Problems:
1. Dimensionality: big networks
2. Data: measurements and elaboration

1500 arcs network

NETWORK of SALERNO
Lighthill-Whitham-Richards model

The flux is given by the density times the average velocity

\[ f(t, x) = \rho(t, x) \cdot v(t, x) \]

If we assume that the average velocity depends only on density

\[ v(t, x) = v(\rho(t, x)) \]

Lighthill-Whitham-Richards model:

\[ \rho_t + (\rho v(\rho))_x = 0, \quad \rho \in [0, \rho_{max}] \]

Time \( t=0 \)

Non unique (weak) solutions

Finite time

Entropy (gas dynamics)

(disorder is increasing, stable shocks)
Networks and Re di Roma square

More incoming than exiting road

Priority parameters

Bifurcations, merging, complicate junctions, traffic circles

**Theory**: existence of solutions on networks for BV initial data.

Road flux total variation in $x$ \sim Junctions flux total variation in $t$
We essentially invert Rules (A) and (B), giving more importance to through flux than traffic distribution.

In the $n$th attempt, $(1 - P(R))P(R)^{n-1}$ packets are sent and $P(R)^n$ are lost.

Maximal fluxes on each line $\gamma_i^{max}$ and $\gamma_j^{max}$

\[
\sum_{t} (1 - P(R)) P(R)^{n-1}
\]

The through flux is $\Gamma = \min\{\sum_i \gamma_i^{max}, \sum_j \gamma_j^{max}\}$. 

The model for data networks