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Bounding the Probability of the Union of Events by the Use of Aggregation and Disaggregation in Linear Programs

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Abstract. Given a sequence of n arbitrary events in a probability space, we assume that the individual probabilities as well as some or all joint probabilities of up to m events are known, where $m < n$. Using this information we give lower and upper bounds for the probability of the union. The bounds are obtained as optimum values of linear programming problems or objective function values corresponding to feasible solutions of the dual problems. If all joint probabilities of the k-tuples of events are known, for k not exceeding m, then the LP is the large scale Boolean probability bounding problem. Another type of LP is the binomial moment problem, where we assume the the knowledge of some of the binomial moments of the number of events which occur. The two LP's can be obtained from each other by aggregation/disaggregation procedures. In this paper we define LP's which are obtained by partial aggregation/disaggregation from these two LP's. This way we can keep the size of the problem low but can produce very good bounds in many cases. One of our new results is a generalization of Hunter's upper bound for the probability of the union of events. Numerical examples are presented.

1 Introduction

The problem to compute or approximate the probability of the union of events, i.e., the event that at least one of them occurs, frequently comes up in applications of probability theory, statistics, reliability theory, stochastic programming, and other stochastic sciences. Typical example is the reliability problem of communication networks, where the arcs randomly work or fail and we want to compute or approximate the node-to-node or the all terminal reliability of the system. The former one means the probability that there exists at least one path, connecting two designated nodes such that all arcs in it are working. The latter one looks for the probability of the existence of at least one spanning tree consisting of all working arcs. For descriptions of a number of applications the reader is referred to Barlow and Proschan (1975), N.H. Roberts et al. (1981), F. Roberts et al. (eds., 1991).

Sometimes we want to compute or approximate the probability of the intersection of events. In this case the formula $P(A_1 \cap ... \cap A_n) = 1 - P(\bar{A}_1 \cup ... \cup \bar{A}_n)$ can be used along with a method to compute or approximate the probability of the union. In probability theory and statistics we are frequently facing with the problem to approximate values of the joint probability distribution function of some random variables $X_1, ..., X_n$. This means that the probability of the joint occurence of the events $X_i \leq z_i,\ i = 1, ..., n$ has to be approximated. Similar problem comes up in stochastic programming, where constraints of the type $P(g_i(x) \geq X_i,\ i = 1, ..., n) \geq p$ are formulated and the solution methods require the subsequent evaluation of the constraining function values (see Prékopa (1995)). As multivariate probability distributions are more and more important in many applications, more and more emphasis is on finding good approximation methods for the probability of the union.

A classical formula, that gives the probability of the union of events $A_1, ..., A_n$ in terms of the intersection probabilities of the same events, is the inclusion-exclusion formula.

$$P(A_1 \cup ... \cup A_n) = S_1 - S_2 + ... + (-1)^{n-1} S_n,$$

where

$$S_k = \sum_{1 \leq i_1 \leq ... \leq i_k \leq n} P(A_{i1} \cap ... \cap A_{ik}).$$

The use of this formula in practice is limited, however, if the number of events is large. In this case the calculation of $S_1,\ S_2, ...$ breaks down at some point, leaving us with the task to create lower and upper bounds for $P(A_1 \cup ... \cup A_n)$, based on $S_1, ..., S_m$, for some $m < n$.

The well-known Bonferroni (1937) bounds, stating that

$$P(A_1 \cup ... \cup A_n) \geq S_1 - S_2 + ... + (-1)^{m-1} S_m,$$

if m is even, and

$$P(A_1 \cup ... \cup A_n) \leq S_1 - S_2 + ... + (-1)^{m-1} S_m,$$

if m is odd, are, on the other hand, weak, in general.

To obtain the best possible, or sharp lower and upper bounds in terms of $S_1, ..., S_m$, Prékopa (1988) has formulated linear programming problems with objective function to be minimized or maximized, respectively. These yield lower and upper bounds of the form

$$\sum_{i=1}^{m} x_i S_i \ \le \ P(A_1 \cup ... \cup A_n) \ \le \ \sum_{i=1}^{m} y_i S_i,$$

where $x = (x_1, ..., x_m)^T$ and $y = (y_1, ..., y_m)^T$ are the optimal dual solutions to the minimization and maximization problems, respectively. We also know (see Prékopa (1988) and Boros and Prékopa (1989)) that the components of x and y have alternating signs, starting with $+$, and $|x_1| \ge ... \ge |x_m|$, $|y_1| \ge ... \ge |y_m|$.

The bounds for the probability of the union become better, if we use the available information in more detailed form, i.e., we use individually the probabilities in the sums S_k, $k = 1, ..., m$. A classical result in this respect is Hunter's upper bound (Hunter(1976)) which states that if we create the n-node complete graph and assign the weight $P(A_i \cap A_j)$ to the arc $\{i, j\}$, then

$$P(A_1 \cup ... \cup A_n) \le S_1 - \sum_{(i,j) \in T} P(A_i \cap A_j),$$

where T is the heaviest spanning tree in the graph.

Bounding problems, where we use $S_1, ... S_m$, will be called aggregated problems and those, where we use the individual probabilites in the sums, will be called disaggregated problems. The purpose of the paper is to improve on existing bounds for the probability of the union of the events by the use of partial aggregation/disaggregation.

Let $A_1, ..., A_n$ be arbitrary events in some probability space, and introduce the notations

$$P(A_{i_1} \cap ... \cap A_{i_k}) \ = \ p_{i_1 ... i_k}, \quad 1 \le i_1 < ... < i_k \le n,$$

then we have

$$S_k \ = \sum_{1 \le i_1 < ... < i_k \le n} p_{i_1 ... i_k}, \quad k = 1, ..., n.$$

Let $S_0 = 1$, by definition. If ν designates the number of those events (among $A_1, ..., A_n$) which occur, then we have the relations (see, e.g., Prékopa (1995)):

$$E\left[\binom{\nu}{k}\right] \ = \ S_k, \quad k = 0, ..., n. \tag{1}$$

The equations (1) can be written in the more detailed form

$$\sum_{i=0}^{n} \binom{i}{k} v_i \ = \ S_k, \quad k = 0, ..., n,$$

where $v_i = P(\nu = i)$, $i = 0, ..., n$.

The values S_k are called the binomial moments of ν. If we know all binomial moments of ν, then the probabilities $v_0, ..., v_n$, and also $P(A_1 \cup ... \cup A_n) = v_0 + ... + v_n$ can be determined. If, however, we only know $S_1, ..., S_m$, where $m < n$, then we look for bounds on the probability of the union. To obtain them we formulate two closely related types of linear programming problems (see Prékopa (1988, 1990, 1995)) :

$$\begin{aligned} & \min(\max) \sum_{i=1}^{n} x_i \\ & \text{subject to} \\ \sum_{i=1}^{n} \binom{i}{k} x_i &= S_k, \quad k = 1, ..., m \\ x_i &\geq 0, \quad i = 1, ..., n, \end{aligned} \tag{2}$$

and

$$\begin{aligned} & \min(\max) \sum_{i=0}^{n} x_i \\ & \text{subject to} \\ \sum_{i=0}^{n} \binom{i}{k} x_i &= S_k, \quad k = 0, ..., m \\ x_i &\geq 0, \quad i = 0, ..., n. \end{aligned} \tag{3}$$

Problems (2) can be obtained from problems (3) by removing the constraint corresponding to $k = 0$ and the variable x_0. The optimal values of problems (3) will be called the sharp lower and upper bounds for the probability of the union. The optimal values of problems (2) and (3) are in the following simple relationship. The optimum values of the two minimization problems coincide. If V_{max} is the optimum value of the maximization problem (2), then the optimum value of the maximization problem (3) is $min(V_{max}, 1)$.

The reason why we look at problems (2) too, and not only at problems (3), is that it is easier to generate the bounds by the use of problems (2).

In addition to the optimum values of the above problems, any dual feasible basis of any of the minimization (maximization) problems provides us with a lower (upper) bound for the optimum value, hence also for the probability of the union. A basis in a linear program is called dual feasible if the optimality condition, written up with that basis, is satisfied. For a brief presentation of the main concepts and algorithms of linear programming the reader is referred to Prékopa (1996).

Lower and upper bounds for the probability that at least one out of n events occurs, based on the knowledge of $S_1, ..., S_m$, were found by Bonferroni (1937). These bounds are not sharp. For the case of $m = 2$, sharp lower bound for this probability was proposed by Dawson and Sankoff (1967). For the case of $m = 2$ other results are due to Galambos (1977), and Sathe, Pradhan and Shah (1980). By the use of linear programming, Kwerel (1975a,b) derived sharp bounds for the case of $m \leq 3$. For a general m, the linear programs (2) and (3) have been formulated and analyzed by Prékopa (1988, 1990). He also presented simple dual type algorithms to solve the problems. Boros and Prékopa (1989) utilized these results and presented closed form sharp bounds, for the case of $m \le 4$, and other closed form bounds. The closed form bounds for $m \le 4$ are the following:

Lower bound, using S_1, S_2 (Dawson, Sankoff (1967)):

$$P(A_1 \cup ... \cup A_n) \ \ge \ \frac{2}{h+1} S_1 \ - \ \frac{2}{h(h+1)} S_2, \tag{4}$$

where

$$h \ = \ 1 + \left\lfloor \frac{2S_2}{S_1} \right\rfloor.$$

Upper bound, using S_1, S_2 (Kwerel (1975a), Sathe, Pradhan and Shah (1980)):

$$P(A_1 \cup ... \cup A_n) \ \le \ \min\{S_1 \ - \ \frac{2}{n} S_2, \ 1\}. \tag{5}$$

Lower bound, using S_1, S_2, S_3 (Kwerel (1975b), Boros and Prékopa (1989)):

$$P(A_1 \cup ... \cup A_n) \ \ge \ \frac{h+2n-1}{(h+1)n} S_1 \ - \ \frac{2(2h+n-2)}{h(h+1)n} S_2 \ + \ \frac{6}{h(h+1)n} S_3,$$

where

$$h \ = \ 1 \ + \ \left\lfloor \frac{-6S_3 + 2(n-2)S_2}{-2S_2 + (n-1)S_1} \right\rfloor. \tag{6}$$

Upper bound, using S_1, S_2, S_3 (Kwerel (1975b), Boros and Prékopa (1989)):

$$P(A_1 \cup ... \cup A_n) \ \le \ \min \left(S_1 \ - \ \frac{2(2h-1)}{h(h+1)} S_2 \ + \ \frac{6}{h(h+1)} S_3, \ 1 \right), \tag{7}$$

where

$$h \ = \ 2 \ + \ \left\lfloor \frac{3S_3}{S_2} \right\rfloor.$$

Upper bound, using S_1, S_2, S_3, S_4 (Boros and Prékopa (1989)):

$$\begin{aligned} P(A_1 \cup ... \cup A_n) \ \le \ \min \Bigg(S_1 \ &- \ \frac{2((h-1)(h-2) + (2h-1)n)}{h(h+1)n} S_2 \\ &+ \ \frac{6(2h+n-4)}{h(h+1)n} S_3 \ - \ \frac{24}{h(h+1)n} S_4, \ 1 \Bigg), \end{aligned} \tag{8}$$

where

$$h \ = \ 1 \ + \ \left\lfloor \frac{(n-2)S_2 \ + \ 3(n-4)S_3 \ - \ 12S_4}{(n-2)S_2 \ - \ 3S_3} \right\rfloor.$$

A closed form lower bound for the last case also exists, but it is too complicated, therefore we disregard its presentation.

Problems (2) and (3) use the probabilities $p_{i_1...i_k}$ in aggregated forms, i.e., $S_1, ..., S_m$ are used rather than the probabilities in these sums. This way we trade information for simplicity and size reduction of the problems. We call (2) and (3) aggregated problems.

The linear programs which make us possible to use the probabilities $p_{i_1...i_k}$, $1 \le i_1 < ... < i_k \le n$ individually, will be called disaggregated, and can be formulated as follows. Let D_1 be the $n \times 2^n - 1$ matrix, the columns of which are formed by all 0,1-component vectors which are different from the zero vector.

Let us call the collection of those columns of D_1, which have exactly k components equal to 1, the k^{th} block, $1 \le k \le n$. Assume that the columns in D_1 are arranged in such a way that first come all vectors in the first block, then all those in the second block, etc. Within each block the vectors are assumed to be arranged in a lexicographic order, where the 1's precede the 0's. Let $d_1, ..., d_n$ designate the rows of D_1, and define the matrix D_k, $2 \le k \le m$, as the collection of all rows of the form: $d_{i_1}...d_{i_k}$, where the product of the rows $d_{i_1}, ..., d_{i_k}$ is taken componentwise. Assume that the rows in D_k are arranged in such a way that the row subscripts $(i_1, ..., i_k)$ admit a lexicographic ordering, where smaller numbers precede larger ones. Let

$$A = \begin{pmatrix} D_1 \\ \cdot \\ \cdot \\ \cdot \\ D_m \end{pmatrix}.$$

In addition, we define the matrix $\hat{A}$ by

$$\hat{A} = \begin{pmatrix} 1 & \mathbf{1}^T \\ 0 & D_1 \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ 0 & D_m \end{pmatrix},$$

where $\mathbf{1}$ is the $2^n - 1$-component vector, all components of which are 1, and the zeros in the first column mean zero vectors of the same sizes as the numbers of rows in the corresponding D_i matrices.

Let $p^T = (p_{i_1...i_k},\ 1 \le i_1 < ... < i_k \le n,\ k = 1, ..., m)$, where the order of the components follow the order of the rows in A, and $\hat{p}^T = (1, p^T)$. The disaggregated problems are:

$$\begin{aligned} \min(\max) f^T x& \\ \text{subject to}& \\ Ax &= p \\ x &\ge 0, \end{aligned} \tag{9}$$

and

$$\begin{aligned} \min(\max) \hat{f}^T \hat{x} \\ \text{subject to} \\ \hat{A}\hat{x} \;=\; \hat{p} \\ \hat{x} \;\geq\; 0, \end{aligned} \tag{10}$$

where $f^T = (1, ..., 1)$, and $\hat{f}^T = (0, f^T)$, $\hat{x}^T = (x_0, x^T)$.

The duals of the above problems are:

$$\begin{aligned} \max(\min) p^T y \\ \text{subject to} \\ A^T y \;\leq (\geq)\; f, \end{aligned} \tag{11}$$

and

$$\begin{aligned} \max(\min) \hat{p}^T \hat{y} \\ \text{subject to} \\ \hat{A}^T \hat{y} \;\leq (\geq)\; \hat{f}, \end{aligned} \tag{12}$$

where $\hat{y}^T = (y_0, y^T)$. Since the dual vector y multiplies the vector p in problem (11), it is appropriate to designate the components of y by $y_{i_1...i_k}$, $1 \leq i_1 < .. < i_k \leq n$, $k = 1, ..., m$.

The more detailed form of problems (11) is the following:

$$\begin{gathered} \max(\min) \sum_{k=1}^{m} \sum_{1 \leq i_1 < ... < i_k \leq n} p_{i_1...i_k} y_{i_1...i_k} \\ \text{subject to} \\ \sum_{k=1}^{m} \sum_{1 \leq i_1 < ... < i_k \leq n} y_{i_1...i_k} \;\leq (\geq)\; 1. \end{gathered} \tag{13}$$

and the more detailed form of problems (12) can be written as:

$$\begin{gathered} \max(\min) \left\{ y_0 \;+\; \sum_{k=1}^{m} \sum_{1 \leq i_1 < ... < i_k \leq n} p_{i_1...i_k} y_{i_1...i_k} \right\} \\ \text{subject to} \\ y_0 \;+\; \sum_{k=1}^{m} \sum_{1 \leq i_1 < ... < i_k \leq n} y_{i_1...i_k} \;\leq (\geq)\; 1. \end{gathered} \tag{14}$$

The probability bounding schemes (13)-(14) can be attributed to George Boole (1854). A detailed account on it was presented by Hailperin (1965). Kounias and Marin (1976) made use of problem (14) to generate bounds for the case of $m = 2$.

The optimum values of the minimization problems (9) and (10) are the same. The optimum values of the maximization problems (9) and (10) are also the same provided that the optimum value corresponding to (9) is smaller than or equal to 1. Otherwise, the optimum value of problem (10) is 1.

Another way to come to problems (9) and (10) is the following. Define

$$a_{IJ} = \begin{cases} 1, & \text{if } I \subset J \\ 0, & \text{if } I \not\subset J \end{cases}$$

$$v_J = P((\cap_{j\in J} A_j) \cap (\cap_{j\in \bar{J}} \bar{A}_j))$$
$$p_I = P(\cap_{j\in J} A_j)$$

for any $I, J \subset \{1, ..., n\}$. Then we have the equation

$$\sum_{J\subset\{1,...,n\}} a_{IJ} v_J = p_I, \quad I \subset \{1, ..., n\}.$$

With these notations problems (9) can be written up in the following way

$$\min(\max) \sum_{\emptyset\neq J\subset\{1,...,n\}} x_J$$
$$\text{subject to}$$
$$\sum_{\emptyset\neq J\subset\{1,...,n\}} a_{IJ} x_J = p_I, \quad |\, I \,|\leq m$$
$$x_J \geq 0, \quad J \subset \{1, ..., n\}.$$

The new form of problems (10) can be obtained from here if we remove the restriction $\emptyset \neq J$ from the constraints..

In Section 2 we outline the connection between the aggregated and disaggregated problems. In Sections 3, 4, 5, 6 we present two different aggregation/disaggregation methods and numerical examples. In Sections 7 and 8 graph structures are exploited to obtain bounds. Finally, in Section 9 and 10 we make some examples and present conclusions.

2 Connection Between the Aggregated and Disaggregated Problems

Any feasible solution of problems (9) gives rise, in a natural way, to a feasible solution of problems (2). Similarly, any feasible solution of problems (10) gives rise to a feasible solution of problems (3).

Conversely, any feasible solution of the aggregated problem (2) or (3) gives rise to a feasible solution of the corresponding disaggregated problem. In fact, we obtain problems (9) or (10) from problems (2) or (3) in such a way that we split rows and columns. Splitting a column in the aggregated problem means its representation as a sum of columns taken from the corresponding disaggregated problem.

Another question is that which bases in the aggregated problem produce bases, by splitting columns, in the disaggregated problem. We can create simple examples in case of the minimization problem (2), when $m = 2$. The corresponding disaggregated problem we has $n + \binom{n}{2}$ rows. The i^{th} and j^{th} columns in problem (2) split into $\binom{n}{i}$ and $\binom{n}{j}$ columns, respectively. A necessary condition that these columns form a basis in problem (9) is that $\binom{n}{i} + \binom{n}{j} = n + \binom{n}{2}$, where $i < j$. This condition holds if $i = 1$ and $j = 2$, or $i = n - 2$ and $j = n - 1$. On the other hand these are in fact bases in problem (9), as it is easy to see.

The structures of the dual feasible bases of problems (2) and (3) have been discovered by Prékopa (1988, 1990). We recall the relevant theorem concerning problem (2).

Theorem 2.1 *Let $a_1, ..., a_n$ designate the columns of the matrix of problems (2) and $I \subset \{1, ..., n\}$, $\mid I \mid= m$. Then, $\{a_i,\ i \in I\}$ is a dual feasible basis if and only if I has the structure:*

	m even	*m odd*
min *problem*	$h, h+1, ..., j, j+1$	$h, h+1, ..., j, j+1, n$
max *problem*	$1, h, h+1, ..., j, j+1, n$	$1, h, h+1, ..., j, j+1.$

In view of this theorem, the first $n + \binom{n}{2}$ columns of the matrix of problem (11), i.e. the columns in the first two blocks, form a dual feasible basis in the minimization problem (2). Similarly, the $n + \binom{n}{2}$ columns in the second to the last, and third to the last blocks of problem (9) form a dual feasible basis in the same problem. The corresponding dual vectors can be computed from the equations produced by the aggregated problems:

$$(y_1, y_2)(a_1, a_2) = (1, 1),$$

and

$$(y_1, y_2)(a_{n-2}, a_{n-1}) = (1, 1),$$

respectively. The detailed forms of these equations are:

$$\begin{aligned} y_1 \qquad\quad &= 1 \\ 2y_1 + y_2 &= 1 \end{aligned}$$

and

$$\begin{aligned} (n-2)y_1 + \binom{n-2}{2} y_2 &= 1 \\ (n-1)y_1 + \binom{n-1}{2} y_2 &= 1, \end{aligned}$$

respectively. The first system of equations gives $y_1 = 1$, $y_2 = -1$, and the second one gives: $y_1 = 2/(n-1)$, $y_2 = -2/(n-1)(n-2)$. If we assign $y_1 = 1$ to all vectors in the first block and $y_2 = -1$ to all vectors in the second block of problem (9), then we obtain the dual vector corresponding to the first dual feasible disaggregated basis. Similarly, if we assign $y_1 = 2/(n-1)$ to all vectors in block $n-2$ and $y_2 = -2/(n-2)(n-1)$ to all vectors in block $n-1$ of problem (9), then we obtain the dual vector to the other dual feasible disaggregated basis. The first dual vector gives the Bonferroni lower bound:

$$P(A_1 \cup ... \cup A_n) \;\geq\; \sum_{i=1}^{n} p_i \;-\; \sum_{1\leq i<j\leq n} p_{ij} \;=\; S_1 - S_2.$$

The second dual vector gives the lower bound

$$P(A_1 \cup ... \cup A_n) \;\geq\; \frac{2}{n-1}S_1 \;-\; \frac{2}{(n-2)(n-1)}S_2.$$

The optimal lower bound, produced by the aggregated problem (2), corresponds to that dual feasible basis (a_h, a_{h+1}) which is also primal feasible. This gives $h = 1 + \lfloor 2S_2/S_1 \rfloor$, and the bound is the same as given by (4).

If we want to find the sharp lower bound for $P(A_1 \cup ... \cup A_n)$, by the use of problem (9), for $m = 2$, then we may start from any of the above mentioned two dual feasible bases and use the dual method of linear programming, to solve the problem. Since we want lower bound, we have a minimization problem. This suggests that the second dual feasible basis is a better one to serve as an initial dual feasible basis. The reason is that in blocks $n-2$, and $n-1$ the coefficients of the variables are larger, and since it is a minimization problem, we may expect that we are closer to the optimal basis than in case of the first dual feasible basis.

Numerical Example. Let $n = 6$, and assume that

$$\begin{array}{llllll}
p_1 = 0.30 & p_2 = 0.35 & p_3 = 0.55 & p_4 = 0.40 & p_5 = 0.30 & p_6 = 0.35 \\
p_{12} = 0.15 & p_{13} = 0.25 & p_{14} = 0.05 & p_{15} = 0.15 & p_{16} = 0.15 & \\
p_{23} = 0.25 & p_{24} = 0.15 & p_{25} = 0.15 & p_{26} = 0.05 & & \\
p_{34} = 0.25 & p_{35} = 0.25 & p_{36} = 0.25 & & & \\
p_{45} = 0.15 & p_{46} = 0.15 & & & & \\
p_{56} = 0.15 & & & & &
\end{array}$$

We used the dual method to solve the minimization problem (9). As initial dual feasible basis we chose the collection of vectors in blocks $n-2 = 4$ and $n-1 = 5$. These vectors have indices $42, ..., 62$. After twenty iterations an optimal basis was found, the indices of which are:

12 ,13 , 16 ,20 ,22 , 25 ,28 ,29 ,36 ,38 ,39 ,43 ,44 ,45 ,48 , 50 ,51 ,52 , 54 ,55 ,56.

The basic components of the primal optimal solution are:

$x_{12} = 0.01$, $x_{13} = 0.02$, $x_{16} = 0.10$, $x_{20} = 0.03$, $x_{22} = 0.08$, $x_{25} = 0.01$, $x_{28} = 0.03$, $x_{29} = 0.03$, $x_{36} = 0.05$, $x_{38} = 0.02$, $x_{39} = 0.05$, $x_{43} = 0.02$, $x_{44} = 0.03$, $x_{45} = 0.01$, $x_{48} = 0.01$, $x_{50} = 0.08$, $x_{51} = 0.00$, $x_{52} = 0.06$, $x_{54} = 0.05$, $x_{55} = 0.01$, $x_{56} = 0.01$.

The components of the dual optimal solution are:

$y_1 = 0.4$, $y_2 = 0.6$, $y_3 = 0.6$, $y_4 = 0.8$, $y_5 = -0.2$, $y_6 = 0.0$, $y_{12} = -0.2$, $y_{13} = -0.2$, $y_{14} = -0.4$, $y_{15} = 0.2$, $y_{16} = -0.2$, $y_{23} = -0.2$, $y_{24} = -0.4$, $y_{25} = 0.0$, $y_{26} = -0.2$, $y_{34} = -0.4$, $y_{35} = 0.0$, $y_{36} = -0.2$, $y_{45} = 0.2$, $y_{46} = -0.4$, $y_{56} = 0.0$.

The optimum value equals 0.71. The optimum value corresponding to the aggregated problem is 0.70.

In this example we generated the right-hand side vector p for problem (9) in such a way that we defined $x^0 = (x_j^0,\ j = 1,...,63)^T$, where x_j^0 is different from zero only if $j = 4k,\ k = 1,...,15$, and for these j values we made the assignments $x_j^0 = 0.05$; then we set $p = Ax^0$. In this case $\sum_{j=1}^{63} x_j^0 = 0.75$ and $x_0^0 = 0.25$. The optimum value of the maximization problem (9) is 1.

3 Bounds, using partial aggregation and disaggregation

Let $X_i = 1$, if A_i occurs and $X_i = 0$, otherwise, $i = 1,...,n$. Then we have

$$\nu = X_1 + ... + X_n$$

and (see e.g., Prékopa (1995)):

$$\binom{\nu}{k} = \sum_{1\le i_1<...<i_k\le n} X_{i_1}...X_{i_k} \tag{15}$$

which implies equation (1). A simple consequence of equation (15) is

Theorem 3.1 *For $k \ge 1$ we have*

$$X_i \binom{\nu-1}{k-1} = \sum_{\substack{1\le i_1<...<i_k\le n \\ i\in\{i_1,...,i_k\}}} X_{i_1}...X_{i_k} \tag{16}$$

and

$$E\left[X_i \binom{\nu-1}{k-1}\right] = \sum_{\substack{1\le i_1<...<i_k\le n \\ i\in\{i_1,...,i_k\}}} p_{i_1...i_k}. \tag{17}$$

Let us introduce the relation

$$x_{ij} = P(X = 1, \nu = j).$$

In view of (17) we have the equations

$$\sum_{j=1}^{n} \binom{j-1}{k-1} x_{ij} = \sum_{\substack{1 \le i_1 < \ldots < i_k \le n \\ i \in \{i_1, \ldots, i_k\}}} p_{i_1 \ldots i_k}. \tag{18}$$

If we introduce the new variables $y_{ij} = x_{ij}/j$, then (18) can be rewritten as

$$\sum_{j=1}^{n} \binom{j}{k} y_{ij} = \frac{1}{k} \sum_{\substack{1 \le i_1 < \ldots < i_k \le n \\ i \in \{i_1, \ldots, i_k\}}} p_{i_1 \ldots i_k}. \tag{19}$$

In addition, we have the following simple theorem

Theorem 3.2 *The following equation holds:*

$$\sum_{i=1}^{n} \sum_{j=1}^{n} y_{ij} = P(A_1 \cup \ldots \cup A_n). \tag{20}$$

We omit the proof.

By the use of (19) and (20) we formulate linear programming problems for bounding the probability $P(A_1 \cup \ldots \cup A_n)$:

$$\begin{aligned} \min(\max) \quad & \textstyle\sum_{i=1}^{n} \sum_{j=1}^{n} y_{ij} \\ & \text{subject to} \\ \textstyle\sum_{j=1}^{n} \binom{j}{k} y_{ij} &= S'_{ik}, \quad k = 1, \ldots, m_i \\ y_{ij} &\ge 0, \quad i, j = 1, \ldots, n, \end{aligned} \tag{21}$$

where $1 \le m_i \le n$, $i = 1, \ldots, n$ and

$$S'_{ik} = \frac{1}{k} \sum_{\substack{1 \le i_1 < \ldots < i_k \le n \\ i \in \{i_1, \ldots, i_k\}}} p_{i_1 \ldots i_k}, \quad i, k = 1, \ldots, n.$$

Let $L(U)$ designate the optimum value of the minimization (maximization) problem (21). Also, let $L_i(U_i)$ designate the optimum value of the minimization (maximization) problem:

$$\begin{aligned} \min(\max) \quad & \textstyle\sum_{j=1}^{n} y_{ij} \\ & \text{subject to} \\ \textstyle\sum_{j=1}^{n} \binom{j}{k} y_{ij} &= S'_{ik}, \quad k = 1, \ldots, m_i \\ y_{ij} &\ge 0, \quad j = 1, \ldots, n. \end{aligned} \tag{22}$$

The minimization (maximization) problem (21) splits into the n minimization (maximization) problems (22). The matrix of the equality constraints in (21) has the matrices of (22) in its main diagonal and the objective function in (21) is the sum of the objective functions in (22). This implies that

$$\begin{aligned} L &= L_1 + ... + L_n \\ U &= U_1 + ... + U_n \end{aligned} \tag{23}$$

$$L \leq P(A_1 \cup ... \cup A_n) \leq U. \tag{24}$$

In addition, if L_i (U_i) is defined as the objective function value corresponding to any dual feasible basis in the minimization (maximization) problem (22), $1 \leq i \leq n$, then with the L and U, defined by (23), the relations (24) hold true as well (but may not be as good as those, corresponding to the optimum values).

Problems (21) can be obtained from problems (9) by aggregation (of variables and constraints) and each problem (22) can be obtained from problem (3) by disaggregation. This implies that both bounds in (24) are at least as good as the corresponding binomial moment bounds.

The following examples are based on known binomial moment bounds (see Prékopa (1995), Sections 6.2.1-6.2.5).

Example 1. We mentioned in Section 2 the binomial moment lower bound for the case of $m = 2$ (Dawson-Sankoff bound). Using this and (24) we obtain

$$P(A_1 \cup ... \cup A_n) \geq \sum_{i=1}^{n} \left(\frac{2}{h_i + 1} S'_{i1} - \frac{2}{h_i(h_i + 1)} S'_{i2} \right), \tag{25}$$

where

$$h_i = 1 + \left\lfloor \frac{2S'_{i2}}{S'_{i1}} \right\rfloor, \quad i = 1, ..., n.$$

This is the lower bound obtained by Kuai, Alajaji and Takahara (2000) which generalizes De Caen's (1999) lower bound.

Example 2. If $m = 2$, then the optimum value of the maximization problem (3) equals

$$\min\{S_1 - \frac{2}{n} S_2, 1\}.$$

This implies that

$$P(A_1 \cup ... \cup A_n) \leq \min \left(\sum_{i=1}^{n} (S'_{i1} - \frac{2}{n} S'_{i2}), 1 \right). \tag{26}$$

Example 3. If $m = 3$, then the optimum value of the minimization problem (3) equals

$$\frac{h + 2n - 1}{(h + 1)n} S_1 - \frac{2(2h + n - 2)}{h(h + 1)n} S_2 + \frac{6}{h(h + 1)n} S_3,$$

where

$$h = 1 + \left\lfloor \frac{-6S_3 + 2(n-2)S_2}{-2S_2 + (n-1)S_1} \right\rfloor . \tag{27}$$

Thus we have the lower bound

$$P(A_1 \cup ... \cup A_n) \geq \sum_{i=1}^{n} \left(\frac{h_i + 2n - 1}{(h_i+1)n} S'_{i1} - \frac{2(2h_i + n - 2)}{h_i(h_i+1)n} S'_{i2} + \frac{6}{h_i(h_i+1)n} S'_{i3} \right), \tag{28}$$

where

$$h_i = 1 + \left\lfloor \frac{-6S'_{i3} + 2(n-2)S'_{i2}}{-2S'_{i2} + (n-1)S'_{i1}} \right\rfloor , \quad i = 1, ..., n.$$

Example 4. By the use of the optimum value of the maximization problem (3), for the case of $m = 3$, we obtain the new bound:

$$P(A_1 \cup ... \cup A_n) \leq \min \left(\sum_{i=1}^{n} \left(S'_{i1} - \frac{2(2h_i - 1)}{h_i(h_i+1)} S'_{i2} + \frac{6}{h_i(h_i+1)} S'_{i3} \right), 1 \right), \tag{29}$$

where

$$h_i = 2 + \left\lfloor \frac{3S'_{i3}}{S'_{i2}} \right\rfloor , \quad i = 1, ..., n.$$

Example 5. If we use the optimum value of the maximization problem (3) for the case of $m = 4$, then the new bound is obtained as

$$P(A_1 \cup ... \cup A_n) \leq \min \left(\sum_{i=1}^{n} \left(S'_{i1} - 2\frac{(h_i-1)(h_i-2) + (2h_i-1)n}{h_i(h_i+1)n} S'_{i2} + 6\frac{2h_i + n - 4}{h_i(h_i+1)n} S'_{i3} - \frac{24}{h_i(h_i+1)n} S'_{i4} \right), 1 \right), \tag{30}$$

where

$$h_i = 1 + \left\lfloor \frac{-12S'_{i4} + 3(n-4)S'_{i3} + (n-2)S'_{i2}}{(n-2)S'_{i2} - 3S'_{i3}} \right\rfloor , \quad i = 1, ..., n.$$

4 Numerical Examples for Section 3

In this section we present two examples to show how the new method improves on the bounds. In both examples we subdivide the collection of events $A_1, ..., A_n$ into two groups. For those events A_i which belong to the first group we create lower and upper bounds based

$$R = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 1 \end{pmatrix} \tag{31}$$

on $S'_{i1}, S'_{i2}, S'_{i3}$ and the linear programs (22). For those events A_j which are in the second group we use S'_{j1}, S'_{j2} and the linear programs (22) to create the bounds. The values (23) and (24) provide us with the overall bounds. If we apply suitable subdivision of the collection of events, we may be able to save a lot of computing time. Below we discuss and present numerical results in connection with three subdividing strategies that we call order, greedy and passive. The order method means that we enlist each event that belongs to the first half of the sequence, written up in the original order, into the first group and all other events go to the second group. To describe the other two methods first we arrange the events in such a way that their probabilities form a decreasing sequence. The greedy method means that the first (second) half of the events belongs to the first (second) group. The passive method does just the opposite, The first (second) half of the events belongs to the second (first) group.

Example1. First we look at one of the examples presented in Kuai, Alajaji and Takahara (2000, p157). There are 6 events in the example: $A_1, ..., A_6$. The sample space has 15 elements $1, 2, ..., 15$ with probability $x_1, x_2, ..., x_{15}$ respectively. The events are defined by the matrix $R = (r_{ij})$ in (31), where $r_{ij} = 1$, if $i \in A_j$, otherwise $r_{ij} = 0$. We have $x_1 = 0.012, x_2 = 0.022, x_3 = 0.023, x_4 = 0.033, x_5 = 0.034, x_6 = 0.044, x_7 = 0.045, x_8 = 0.055, x_9 = 0.056, x_{10} = 0.066, x_{11} = 0.067, x_{12} = 0.077, x_{13} = 0.078, x_{14} = 0.088, x_{15} = 0.089$.

The lower and upper bounds for the system are presented in Table 1.

Column 1 in Table 1 contains the bounds obtained by (4) and (5). Column 2 contains the bounds presented in (25) and (26). Column $3, 4, 5$ contain the bounds obtained by the passive, order and greedy subdivisions of the events into two groups. The word "Mixture" refers to the fact that in each of these bounds two and three binomial moments are used

Table 1: Results for Example 1

Bound	S_1, S_2	$S_{i1}^{'}, S_{i2}^{'}$	Mixture (passive)	Mixture (order)	Mixture (greedy)	$S_{i1}^{'}, S_{i2}^{'}, S_{i3}^{'}$
Lower	0.6933333	0.7221667	0.7221667	0.7314500	0.7314500	0.7314500
Upper	1.0000000	1.0000000	1.0000000	0.8038333	0.8038333	0.8038333

$$R_1 = \begin{pmatrix}
1&0&0&0&1&0&0&0&0&0&0&1&0&0&1&0&0&1&0&1\\
0&1&0&1&0&0&1&0&0&1&0&0&0&1&0&0&0&1&0&0\\
1&0&1&0&1&0&0&1&0&0&0&0&0&0&0&1&0&0&1&0\\
0&1&0&0&0&1&0&0&1&0&0&1&0&0&1&0&0&1&0&0\\
1&0&1&0&1&0&0&1&0&0&1&0&0&1&0&0&1&0&1&0\\
0&1&0&0&1&0&1&0&1&0&1&1&0&0&1&0&1&0&0&1\\
1&0&0&1&0&1&0&1&0&0&1&0&1&0&0&1&0&0&0&0\\
0&1&0&1&0&0&0&0&1&0&1&0&0&0&0&1&0&1&0&1\\
1&0&1&0&0&1&0&1&0&1&0&1&0&1&0&0&0&1&0&1\\
1&0&0&1&0&0&1&0&0&1&0&0&1&0&0&0&1&0&0&0\\
0&1&0&0&1&0&1&0&1&0&0&1&0&1&0&1&0&1&0&1\\
0&0&0&1&0&0&0&1&0&0&1&0&0&1&0&0&1&0&0&0\\
1&0&0&1&0&1&0&0&0&1&0&0&1&0&1&0&0&1&0&1\\
0&1&0&0&0&1&0&0&1&0&0&0&1&0&0&1&0&0&1&0\\
0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1
\end{pmatrix} \tag{32}$$

in a mixed manner, computed by (25), (26), (28) and (29). Column 6 contains the bounds where in case of each A_i three binomial moments are used. We use the same notations in other tables in this section.

Table 1 shows that the passive method does not prove to be a good strategy. More information is used to compute the bounds in Column 3, but there is no increase in them as compared to those in column 2. On the other hand both the order and the greedy method produce the same bounds as what are contained in the last column.

Example2. This is an extension of example 1. We define three event sequences A_{kj} , $j = 1, ..., 20$, $k = 1,\ 2,\ 3$. The elementary events are again $1, ..., 15$ and $x_1, ..., x_{15}$ are the corresponding probabilities respectively. Define the matrices $R_k = (r_{k,ij})$ where $r_{k,ij} = 1$ if $i \in A_{kj}$ and $r_{k,ij} = 0$ if $i \notin A_{kj}$, $k = 1, 2, 3$.

System 1: $x_1 = 0.01221377$, $x_2 = 0.02223128$, $x_3 = 0.02328652$, $x_4 = 0.03397571$, $x_5 = 0.03476138$, $x_6 = 0.04458161$, $x_7 = 0.04594259$, $x_8 = 0.05518453$, $x_9 = 0.0564044$, $x_{10} = 0.06631682$, $x_{11} = 0.06768523$, $x_{12} = 0.07737555$, $x_{13} = 0.07864836$, $x_{14} = 0.08887805$, $x_{15} = 0.2925142$.

System 2: $x_1 = 0.008964634$, $x_2 = 0.02492217$, $x_3 = 0.02109813$, $x_4 = 0.03779353$, $x_5 = 0.0463261$, $x_6 = 0.04284324$, $x_7 = 0.07804262$, $x_8 = 0.02536991$, $x_9 = 0.01916672$, $x_{10} = 0.06340085$, $x_{11} = 0.07315289$, $x_{12} = 0.07732742$, $x_{13} = 0.0224802$, $x_{14} = 0.09164494$, $x_{15} = 0.3674566$.

$$R_2 = \begin{pmatrix}
0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\
1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\
1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0
\end{pmatrix} \tag{33}$$

System 3: $x_1 = 0.1017688$, $x_2 = 0.112992$, $x_3 = 0.01514044$, $x_4 = 0.05684733$, $x_5 = 0.03270125$, $x_6 = 0.1005075$, $x_7 = 0.07306695$, $x_8 = 0.01743922$, $x_9 = 0.06284498$, $x_{10} = 0.05830101$, $x_{11} = 0.06833096$, $x_{12} = 0.07153743$, $x_{13} = 0.04503293$, $x_{14} = 0.03487869$, $x_{15} = 0.1486024$.

The upper bounds in all cases are equal to 1, while the lower bounds for the different systems are presented in Table 2.

In Table 2, we can see that the bounds obtained by the use of S'_{i1}, S'_{i2} are much better than those obtained by the use of S_1, S_2. We also observe that the bounds obtained by the greedy method are much better than those obtained by the use of S'_{i1}, S'_{i2}. The greedy method outperforms the other two mixture methods. Furthermore, if we compare the bounds in column 5 and 6, we see that the bounds obtained by the use of $S'_{i1}, S'_{i2}, S'_{i3}$ are only a little better than those obtained by the greedy method.

Finally, let's compare the results obtained by S_1, S_2, S_3 and the mixture methods. The results are presented in Table 3.

We notice in Table 3 that we have obtained better bounds (at least in most cases) by any of the order and greedy mixture methods than what we have obtained by the use of the binomial moments S_1, S_2, S_3.

5 Another Method of Partial Disaggregation to Generate Bounds

In this section we split the sequence of events $A_1, ..., A_n$ into subsequences and apply to the latter the bounding technique based on the multivariate binomial moment problem. The

$$R_3 = \begin{pmatrix}
0&1&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&1\\
1&0&0&1&0&0&0&0&0&1&0&0&0&0&0&0&0&0&1&0\\
0&0&1&0&0&0&1&0&0&0&1&0&1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&0&0&1&0&0&0&0&0&1&0\\
0&1&0&0&1&0&0&1&0&0&1&0&0&1&0&0&1&0&0&0\\
1&0&0&1&0&0&1&0&1&0&0&1&0&0&1&0&1&0&0&1\\
0&1&0&1&0&0&0&0&0&1&0&0&0&0&0&1&0&0&0&0\\
0&1&0&0&0&1&0&0&0&0&1&0&0&0&1&0&0&0&0&1\\
0&0&0&0&0&1&0&1&0&0&1&0&0&0&0&1&0&0&0&0\\
1&0&0&0&1&0&0&0&0&0&0&0&0&1&0&0&1&0&0&1\\
0&1&0&0&0&0&0&0&0&1&0&1&0&0&1&0&0&1&0&0\\
1&0&0&0&0&0&0&1&0&0&1&0&0&0&1&0&1&0&0&1\\
0&1&0&0&0&1&0&0&0&0&0&0&1&0&0&0&0&1&0&0\\
0&0&1&0&0&0&0&0&1&0&0&0&0&0&0&0&1&0&1&0\\
0&0&0&1&0&1&0&0&0&0&0&0&0&1&0&0&0&0&0&0
\end{pmatrix} \tag{34}$$

Table 2: Results for Example 2

System	S_1, S_2	S'_{i1}, S'_{i2}	Mixture(passive)	Mixture(order)	Mixture(greedy)	$S'_{i1}, S'_{i2}, S'_{i3}$
1	0.8275266	0.8580833	0.86123	0.8698107	0.8832994	0.886446
2	0.8658182	0.9100646	0.9111695	0.9264307	0.9343052	0.93541
3	0.8985498	0.9435812	0.9446198	0.9537189	0.9577441	0.9587778

Table 3: Comparison of the results obtained by S_1, S_2, S_3 and mixture methods

System	Bound	S_1, S_2, S_3	Mixture (passive)	Mixture (order)	Mixture(greedy)
Example1	Lower	0.7025667	0.7221667	0.73145	0.73145
Example1	Upper	0.8130000	1.0000000	0.8038333	0.8038333
Example2(1)	Lower	0.8553803	0.8612300	0.8698107	0.8832994
Example2(2)	Lower	0.8944319	0.9111695	0.9264307	0.9343052
Example2(3)	Lower	0.930407	0.9446149	0.9537189	0.9577441

efficient bounds are algorithmic rather than given by closed form formulas.

Let $E_1, ..., E_s$ be pairwise disjoint nonempty subsets of the set $\{1, ..., n\}$ exhausting the set $\{1, ..., n\}$, and introduce the notation $n_j = \mid E_j \mid,\ j = 1, ..., s$.

Out of the events $A_1, ..., A_n$ we create s event sequences, where the i^{th} one is $\{A_i,\ i \in E_j,\ 1 \le j \le s\}$. Any of the events $A_1, ..., A_n$ is contained in one and only one event sequence. For the events in these sequences we will use the alternative notations:

$$\begin{array}{c} A_{11}, ..., A_{1n_1} \\ ... \\ A_{s1}, ..., A_{sn_s}. \end{array} \tag{35}$$

Let X_{i_j} designate the number of those events which occur in the j^{th} sequence, and

$$\begin{aligned} S_{\alpha_1 ... \alpha_s} &= E\left[\binom{Xi_1}{\alpha_1} ... \binom{Xi_s}{\alpha_s}\right] \\ & 0 \le \alpha_j \le n_j, \quad j = 1, ..., s. \end{aligned} \tag{36}$$

We formulate the multivariate binomial moment problem (see Prékopa (1992, 1998)):

$$\begin{aligned} & \min(\max) \sum_{i_1=0}^{n_1} ... \sum_{i_s=0}^{n_s} f_{i_1 ... i_s} x_{i_1 ... i_s} \\ & \text{subject to} \\ \sum_{i_1=0}^{n_1} ... \sum_{i_s=0}^{n_s} \binom{i_1}{\alpha_1} ... \binom{i_s}{\alpha_s} x_{i_1 ... i_s} &= S_{\alpha_1 ... \alpha_s} \\ & \alpha_j \ge 0,\ j = 1, ..., s,\ \alpha_1 + ... + \alpha_s \le m \\ & \forall i_1, ..., i_s : x_{i_1 ... i_s} \ge 0. \end{aligned} \tag{37}$$

The $S_{\alpha_1 ... \alpha_s}$ $(\alpha_1 + ... + \alpha_s \le m)$ multivariate binomial moments can be computed from the probabilities $p_{i_1 ... i_k} (1 \le i_1 < ... < i_k \le m)$. In order to simplify the rule how to do this, assume that $E_1 = \{1, ..., n_1\}, ..., E_s = \{n_1 + ... + n_{s-1} + 1, ..., n_1 + ... + n_s\}$. Then, we have the equality

$$S_{\alpha_1 ... \alpha_s} = \sum p_{i_{11} ... i_{1\alpha_1} ... i_{s1} ... i_{s\alpha_s}},$$

where the summation is extended over those indices which satisfy the relations

$$\begin{array}{c} 1 \le i_{11} < ... < i_{1\alpha_1} \le n_1 \\ \\ ... \\ \\ n_1 + ... + n_{s-1} + 1 \le i_{s1} < ... < i_{s\alpha_s} \le n_1 + ... + n_s. \end{array}$$

For example, if $n = 6$ and $E_1 = \{1, 2, 3\}$, $E_2 = \{4, 5, 6\}$, then

$$\begin{gathered}
S_{10} = p_1 + p_2 + p_3,\ S_{01} = p_4 + p_5 + p_6, \\
S_{20} = p_{12} + p_{13} + p_{23},\ S_{02} = p_{45} + p_{46} + p_{56}, \\
S_{11} = p_{14} + p_{15} + p_{16} + p_{24} + p_{25} + p_{26} + p_{34} + p_{35} + p_{36}, \\
S_{21} = p_{124} + p_{125} + p_{126} + p_{134} + p_{135} + p_{136} + p_{234} + p_{235} + p_{236}, \\
S_{22} = p_{1245} + p_{1246} + p_{1256} + p_{1345} + p_{1346} + p_{1356} + p_{2345} + p_{2346} + p_{2356}
\end{gathered}$$

etc.

We have yet to formulate suitable objective functions for problems (37). Since we want to create bounds for the union of all events in (37), our choice is:

$$f_{i_1,\dots,i_s} = \begin{cases} 0 & \text{if } (i_1, \dots, i_s) = (0, \dots, 0) \\ 1 & \textit{otherwise.} \end{cases} \tag{38}$$

Problems (37) reduce to problems (2), if $s = 1$, and to problems (10), if $s = n$. Problems (37) are disaggregated counterparts of problems (2), and aggregated counterparts of problems (10). Further notations are presented below.

Let P designate the probability that at least one out of $A_1, \dots, A_n$ occurs. Let L (U) designate the optimum value of the minimization (maximization) problem (10) and l (u) designate the optimum value of the minimization (maximization) problem (37).

By construction, we have the following inequality:

$$l \le L \le P \le U \le u. \tag{39}$$

In fact, the problems with optimum values l and u are aggregations of problems with optimum values L and U, respectively.

The duals of problems (37) are the following:

$$\begin{gathered}
\max(\min) \sum_{\substack{\alpha_j \ge 0,\ j = 1, \dots, s \\ 1 \le \alpha_1 + \dots + \alpha_s \le m}} y_{\alpha_1 \dots \alpha_s} S_{\alpha_1 \dots \alpha_s} \\
\text{subject to} \\
\sum_{\substack{\alpha_j \ge 0,\ j = 1, \dots, s \\ 1 \le \alpha_1 + \dots + \alpha_s \le m}} y_{\alpha_1 \dots \alpha_s} \binom{i_1}{\alpha_1} \cdots \binom{i_s}{\alpha_s} \le (\ge)\ f_{i_1 \dots i_s} \\
0 \le i_j \le n_j,\ j = 1, \dots, s \\
i_1 + \dots + i_s \ge 1.
\end{gathered} \tag{40}$$

In the left-hand sides of the constraints of problems (40) there are values of a polynomial of the variables $i_1, \dots, i_s$, defined on the lattice points of the set $\times_{j=1}^{s} [0, n_j]$. Replacing

z_j for i_j, the m-degree polynomial takes the form

$$P(z_1,...,z_s) = \sum_{\substack{\alpha_j \geq 0,\ j=1,...,s \\ \alpha_1+...+\alpha_s \leq m}} y_{\alpha_1...\alpha_s} \binom{z_1}{\alpha_1} ... \binom{z_s}{\alpha_s}. \tag{41}$$

Problems (37) may serve to construct polynomials $P(z_1,...,z_s)$ of the type (41) for one sided approximation of the function $f_{z_1...z_s}$ which we will also designate by $f(z_1,...,z_s)$. Our method consists of construction of dual feasible bases to problem (37). Each dual feasible basis of problem (37) determines a dual vector satisfying the inequalities (40), hence it also determines a polynomial (41), which approximates the function f in a one-sided manner. If the basis is dual feasible in the minimization (maximization) problem, then the polynomial is entirely below (above) the function. given the polynomial, we replace X_i for z_i, $i=1,...,n$, take expectation and obtain the lower (upper) bound.

Incidentally we make two remarks. Suppose that the matrix A of the linear programming problem: $\min c^T x$, subject to $Ax = b$, $x \geq 0$, has rank equal to its number of rows m. Let T be an $m \times m$ non-singular matrix and formulate the problem: $\min c^T x$, subject to $(TA)x = Tb$, $x \geq 0$. Then a basis is primal (dual) feasible in one of these two problems if and only if it is primal (dual) feasible in the other one. In fact, if $A = (a_1,...,a_n)$, then we have the relations

$$(TB)^{-1}Tb = B^{-1}b$$

$$c_k - c_B^T (TB)^{-1} T a_k = c_k - c_B^T B^{-1} a_k,$$

which imply the assertion.

Let us associate with problem (37) a multivariate power moment problem in such a way that we replace $i_1^{\alpha_1},...,i_s^{\alpha_s}$ for $\binom{i_1}{\alpha_1} ... \binom{i_s}{\alpha_s}$ and the power moment $\mu_{\alpha_1...\alpha_s}$ for the binomial moment $S_{\alpha_1...\alpha_s}$ on the right-hand side. A single linear transformation takes the column vector in (37):

$$\left(\binom{i_1}{\alpha_1} ... \binom{i_s}{\alpha_s} :\ \alpha_j \geq 0,\ j=1,...,s;\ \alpha_1+...+\alpha_s \leq m \right)$$

into the vector

$$\left(i_1^{\alpha_1} ... i_s^{\alpha_s} :\ \alpha_j \geq 0,\ j=1,...,s;\ \alpha_1+...+\alpha_s \leq m \right).$$

The same transformation applies to the right-hand sides. The matrix of this transformation is non-singular (it is also triangular). Thus, the above remark applies, and therefore a basis in the multivariate binomial moment problem is primal (dual) feasible if and only if the corresponding basis in the multivariate power moment problem is primal (dual) feasible. Thus, we can apply, without any change, the dual feasibility theorems proved in Prékopa (1998, Theorems 4.1 and 4.2) for our multivariate binomial moment problems.

Let us associate the lattice point $(i_1, ..., i_s) \in R^s$ with the vector

$$\left(\binom{i_1}{\alpha_1} ... \binom{i_s}{\alpha_s} : \ \alpha_j \geq 0, \ j = 1, ..., s; \ \alpha_1 + ... + \alpha_s \leq m \right)$$

of the matrix of the equality constraints of problems (37). Let B_Δ and B^Δ designate the sets of vectors corresponding to the sets of lattice points

$$\{(i_1, ..., i_s) \mid i_j \geq 0, \ j = 1, ..., s; \ i_1 + ... + i_s \leq m\}, \tag{42}$$

and

$$\{(n_1 - i_1, ..., n_s - i_s) \mid i_j \geq 0, \ j = 1, ..., s; \ i_1 + ... + i_s \leq m\}, \tag{43}$$

respectively. Then both B_Δ and B^Δ are bases in problem (37). It is easy to check that all divided differences of the function (38) are nonnegative (nonpositive) if $m+1$ is odd (even). Combining this with the above mentioned results, we can state

Theorem 5.1 *The bases B_Δ and B^Δ are dual feasible bases in the following types of problems (37), where the objective function is given by (38):*

	$m+1$ even	*$m+1$ odd*
B_Δ	*max*	*min*
B^Δ	*max*	*max.*

If a bound of this type is not satisfactory (e.g, a lower bound is negative, an upper bound is greater than 1, or it is not enough close to the other bound), then we regard the basis as an initial dual feasible basis, and carry out the solution of problem (37) by the dual method of linear programming. This way we obtain the best possible bound, at least for a given subdivision $E_1, ..., E_s$ of the set $\{1, ..., n\}$.

Note that problem (10) has $1 + n + \binom{n}{2} + ... + \binom{n}{m}$ equality constraints and 2^n variables, whereas problem (37) has $\binom{s+m}{m}$ constraints and $(n_1+1)...(n_s+1)$ variables. Thus, problem (37) has a much smaller size than problem (10), in general. For example, if $n = 20$, $s = 2$, $n_1 = n_2 = 10$, $m = 3$, then problem (10) has sizes 1351 and 1,048,576, whereas problem (37) has sizes 10 and 121.

To obtain the best possible bound which can be given by our method, one has to maximize (minimize) the lower (upper) bound with respect to all subdivisions $E_1, ..., E_s$ of the set $\{1, ..., n\}$. In practice we use only a few trial subdivisions, and choose that one which provides us with the best bound.

Another possibility to create lower (upper) bound for $P(A_1 \cup ... \cup A_n)$ is that we create upper (lower) bound for $P(\bar{A}_1 \cap ... \cap \bar{A}_n)$ and then subtract from 1 the obtained values. In this case we have to write up problems (37) with new right-hand side values and new objective function. The new right-hand side values are $\bar{S}_{\alpha_1 ... \alpha_s}$, defined in the same way as we have defined $S_{\alpha_1...\alpha_s}$ but in this case we use the complementary events. The new objective function is

$$f_{i_1,...,i_s} = \begin{cases} 1 & \text{if } (i_1,...,i_s) = (n_1,...,n_s) \\ 0 & otherwise. \end{cases} \tag{44}$$

It is easy to check that all divided differences of any order of the function (44) are nonnegative. Combining this with Theorem 4.2 in Prékopa (1998) we obtain

Theorem 5.2 *The bases B_Δ and B^Δ are dual feasible bases in the following types of problems (37), where the objective function is given by (44):*

	$m+1$ *even*	$m+1$ *odd*
B_Δ	*min*	*min*
B^Δ	*min*	*max.*

The polynomials determined by the bases B_Δ and B^Δ can be taken from Prékopa (1998). They are multivariate Lagrange interpolation polynomials with base points (42) and (43), respectively. We designate them by $L_\Delta(z_1,...,z_s)$, and $L^\Delta(z_1,...,z_s)$, respectively, and present them first in Newton's form:

$$L_\Delta(z_1,...,z_s) = \sum_{\substack{i_1+...+i_s \le m \\ 0 \le i_j \le n_j, j=1,...,s}} [0,...,i_1;...;0,...,i_s;f] \prod_{j=1}^{s} \prod_{h=0}^{i_j-1} (z_j - h) \tag{45}$$

and

$$L^\Delta(z_1,...,z_s) = \sum_{\substack{i_1+...+i_s \le m \\ 0 \le i_j \le n_j, j=1,...,s}} [n_1-i_1,...,n_1;...;n_s-i_s,...,n_s;f] \prod_{j=1}^{s} \prod_{h=n_j-i_j+1}^{n_j-1} (z_j - h). \tag{46}$$

Since we are given the multivariate binomial moments rather than the power moments, we rewrite these polynomials in other forms. In case of the function (38) we have $L^\Delta(z_1,...,z_s) \equiv 1$, and

$$L_\Delta(z_1,...,z_s) = \sum_{1 \le i_1+...+i_s \le m} (-1)^{i_1+...+i_s-1} \binom{z_1}{i_1} \cdots \binom{z_s}{i_s}. \tag{47}$$

In case of the function (44) we have $L_{\Delta}(z_1, ..., z_s) \equiv 0$, and

$$L^{\Delta}(z_1, ..., z_s) = 1 + \sum_{\substack{1 \le i_1 + ... + i_s \le m \\ 0 \le i_j \le n_j,\ j = 1, ..., s}} (-1)^{i_1 + ... + i_s} \binom{n_1 - z_1}{i_1} ... \binom{n_s - z_s}{i_s}. \quad (48)$$

Theorem 5.1 and 5.2 tell us the following. If f is the function (38) and $L_{\Delta}(z_1, ..., z_s)$ is the polynomial (47), then

$$L_{\Delta}(z_1, ..., z_s) \ \ge (\le)\ f(z_1, ..., z_s), \quad (49)$$

if $m + 1$ is even (odd); if $L^{\Delta}(z_1, ..., z_s)$ is the polynomial (48), then

$$L^{\Delta}(z_1, ..., z_s) \ \ge \ f(z_1, ..., z_s), \quad (50)$$

no matter if $m + 1$ is even, or odd. If f is the function (44), then we have the inequalities

$$L_{\Delta}(z_1, ..., z_s) \ \le \ f(z_1, ..., z_s), \quad (51)$$

no matter if $m + 1$ is even, or odd, and

$$L^{\Delta}(z_1, ..., z_s) \ \le (\ge)\ f(z_1, ..., z_s), \quad (52)$$

if $m + 1$ is even (odd).

6 Numerical Examples for Section 5

We present two examples. Both are based on the knowledge of some of binomial moments S_{ij}, the numerical values of which are presented. However, we disregard the presentation of the events themselves.

Example 1. Let $n = 20,\ n_1 = n_2 = 10,\ m = 3$, and assume that we have obtained the following numbers:

$$S_{01} = S_{10} = 4.5,\ \ S_{02} = S_{20} = 12,\ \ S_{11} = 20.25,\ \ S_{03} = S_{30} = 21,\ \ S_{12} = S_{21} = 54.$$

The polynomial (47) takes the form

$$\begin{aligned} L_{\Delta}(z_1, z_2) \ \equiv \ & z_1 - \binom{z_1}{2} + \binom{z_1}{3} + z_2 - z_1 z_2 \\ & + \binom{z_1}{2} z_2 - \binom{z_2}{2} + z_1 \binom{z_2}{2} + \binom{z_2}{3}, \end{aligned} \quad (53)$$

The dual vector corresponding to the basis B_Δ equals:

$$y \;=\; (0\;1\;-1\;1\;1\;-1\;1\;-1\;1\;1)^T. \tag{54}$$

The polynomial (48) takes the form

$$L^\Delta(z_1, z_2) \;\equiv\; 1, \tag{55}$$

The dual vector corresponding to the basis B^Δ equals:

$$y \;=\; (1\;0\;0\;0\;0\;0\;0\;0\;0\;0)^T. \tag{56}$$

Note that B_Δ, and B^Δ correspond to the lattice points $\{(0,0), (0,1), (0,2), (0,3), (1,0), (1,1), (1,2), (2,0), (2,1), (3,0)\}$, and $\{(10,7), (10,8), (10,9), (10,10), (9,8), (9,9), (9,10), (8,9), (8,10), (7,10)\}$, respectively.

By (50) we have that $L^\Delta(z_1, z_2) \;\geq\; f(z_1, z_2)$, which is a trivial inequality in view of (55). Since $m + 1 = 4$ is even, by (49) we have that $L_\Delta(z_1, z_2) \;\geq\; f(z_1, z_2)$ for all (z_1, z_2). Thus, both B^Δ, and B_Δ are dual feasible bases in the maximization problem (37).

The dual vector (54) produces the trivial upper bound $y^T S = 114.75$, where

$$S \;=\; (S_{00}, S_{10}, S_{20}, S_{30}, S_{01}, S_{11}, S_{21}, S_{02}, S_{12}, S_{03})^T.$$

The dual vector (56) produces the upper bound $y^T S = 1$, which is at the same time the optimum value of the maximization problem (37), and the sharp upper bound for $P(\cup_{i=1}^{20} A_i)$.

The sharp lower bound is obtained by the solution of the minimization problem (37). We have used the dual method with initial dual feasible basis B_Δ and obtained the following optimal solution: $x_{00} = 0.11$, $x_{90} = 0.055556$, $x_{72} = 0.160714$, $x_{73} = 0$, $x_{55} = 0.33$,$x_{36} = 0.208333$, $x_{28} = 0$, $x_{09} = 0.075397$, $x_{10,9} = 0$, $x_{10,10} = 0.06$. This provides us with the lower bound:

$$P(\cup_{i=1}^{20} A_i) \;\geq\; 1 - x_{00} \;=\; 0.89.$$

The dual vector corresponding to the optimal basis is:

$$y \;=\; (0, 0.28, -0.0577, 0.0066, 0.2, -0.04, 0.0044, -0.0222, 0.0022, 0).$$

This determines the polynomial

$$\begin{aligned} L_\Delta(z_1, z_2) \;=\; & 0.28 z_1 - 0.0577 \binom{z_1}{2} + 0.0066 \binom{z_1}{3} + 0.2 z_2 - 0.04 z_1 z_2 + \\ & 0.0044 \binom{z_1}{2} z_2 - 0.0222 \binom{z_2}{2} + 0.0022 z_1 \binom{z_2}{2}, \end{aligned}$$

which satisfies $L(z_1, z_2) \leq f(z_1, z_2)$ for all (z_1, z_2).

Example 2. In this example we consider 40 events for which all binomial moments of order up to 11 have been computed. The 40 events have been subdivided into two

Table 4: **Univariate binomial moments, 40 events**

S_0	1.000
S_1	8.164
S_2	54.025
S_3	290.574
S_4	1435.025
S_5	7115.369
S_6	34884.230
S_7	158338.877
S_8	637735.541
S_9	2249527.156
S_{10}	6955762.090
S_{11}	18955303.836

20-element groups and all bivariate binomial moments of total order at most 6 have been computed.

Lower and upper bounds for the probability that at least one out of the 40 events occurs have been computed based on the two sets of data. The bounds are displayed for all lower order binomial moments, too. Thus, we have two sequences of bounds. The bounds in the first sequence are optimum values of problems (2). The bounds in the second sequence are optimum values of problems (37) with objective function (38). The latter problems are partially disaggregated problems, as compared to problems (2).

The results show that much better bounds can be obtained in the latter case. The bounds obtained from the partially disaggregated problem for $m = 6$ are better than those obtained from the aggregated problem for $m = 11$. The data and the bounds are presented in the tables 4-7.

Table 5: **Bivariate binomial moments when the 40 events are subdivided into two 20-element groups**

first group	second group 0	1	2	3	4	5	6
0	1.00	1.93	4.70	12.19	41.05	127.37	317.72
1	6.23	3.28	31.15	186.89	794.26	2541.64	
2	46.04	31.15	295.90	1775.41	7545.49		
3	216.09	186.89	1775.41	10652.46			
4	724.30	794.26	7545.49				
5	1848.66	2541.64					
6	3739.79						

The results show that by the use of the bivariate binomial moments of order up to 5 we can obtain better bounds than by the use of univariate binomial moments of order up to 10. The bivariate moments of order up to 6 produce better bounds than the univariate binomial moments of order up to 11.

Table 6: **Bounds based on univariate binomial moments**

m	lower bound	upper bound
1	0.20410	1.00000
2	0.57400	1.00000
3	0.63452	1.00000
4	0.67613	1.00000
5	0.77875	1.00000
6	0.78559	0.97028
7	0.79960	0.92088
8	0.80000	0.81438
9	0.80156	0.81185
10	0.80191	0.80671
11	0.80299	0.80638

Table 7: **Bounds based on bivariate binomial moments**

m	lower bound	upper bound
1	0.31137	1.00000
2	0.66045	1.00000
3	0.79552	0.91272
4	0.80255	0.83071
5	0.80275	0.80583
6	0.80325	0.80410

7 Upper Bounds Based on Graph Structures

The bounds given in Section 5 can be interpreted as bounds based on special hypergraphs. Let

$$\sum_{\substack{i_1+\ldots+i_s\le m\\ 0\le i_j\le n_j,\, j=1,\ldots,s}} a_{i_1\ldots i_s}\binom{z_1}{i_1}\cdots\binom{z_s}{i_s}$$

be any polynomial. Let $N=\{1,\ldots,n\}$, and $N_j\subset N$, $j=1,\ldots,m$ with $N=\cup_{j=1}^m N_j$, $\mid N_j\mid\le n_j$, and $\forall j_1\ne j_2:\ N_{j_1}\cap N_{j_2}=\emptyset$. Let $E_{i_1\ldots i_s}$ be the set of all subsets of N containing exactly i_j elements from N_j, $j=1,\ldots,s$. Then we define the hypergraph as follows:

$$H\ =\ (N,\ E),$$

where

$$E\ =\ \bigcup_{\substack{i_1+\ldots+i_s\le m\\ 0\le i_j\le n_j,\, j=1,\ldots,s}} E_{i_1\ldots i_s}.$$

All hyperedges lying in $E_{i_1\ldots i_s}$ are weighted by $a_{i_1\ldots i_s}$. These weights form a dual feasible vector of problem (9). The scalar product of that and the right-hand side vector of problem (11) provides us with the lower or upper bounds. If $m=2$, then only nodes and pairs of nodes have weights. To each node we assign the weight 1.

In Section 7, 8 and 9, the components of an $\binom{n}{2}$-vector are indexed by $11,12,\ldots,(n-1)n$ or $1,\ldots,\binom{n}{2}$ depending on which notation is more convenient. The following lemma is very simple, the proof is omitted.

Lemma 7.1 *The* $\binom{n}{2}$*-component vector* $(1,1,\ldots,1,-w_{12},\ldots,-w_{n-1,n})$ *is feasible in the minimization problem (11) if and only if for all* $S\subset N$ *containing at least two elements the inequality* $\sum_{i,j\in S,i<j}w_{ij}\le\mid S\mid-1$ *holds.*

Remark. It is easy to see, that any feasible solution to the problem (11) has $w_1,\ldots,w_n\le 1$. Lemma 7.1 implies that if $w_1=\ldots=w_n\le 1$, then $\forall 1\le i,j\le n, i\ne j$, we have $w_{ij}\le 1$.

The above lemma can be applied in the following way. Let $G^1(N,E^1)$ and $G^2(N,E^2)$ be two graphs on the vertex set N. Assume that to each $\{i,j\}$, $i,j\in N,\ i\ne j$ a real number w_{ij} is assigned and the following conditions are satisfied:
(i) $E^1\cap E^2=\emptyset$,
(ii) if $\{i,j\}\in E^1$ then $w_{ij}=1$,
(iii) if $\{i,j\}\in E^2$ then $w_{ij}\le 0$,

(iv) if $\{i, j\} \notin E^1 \cup E^2$ then $w_{ij} = 0$,
(v) if $S \subset N, \mid S \mid \geq 2$, then $\sum_{i,j \in S, i<j} w_{ij} \leq \mid S \mid -1$.

The first bound which can be discussed in the framework of the above lemma is Hunter's bound (see Hunter (1976)). Let $G^1 = T(N, E)$ be any tree, and $G^2 = (N, \emptyset)$. Let

$$w_{ij} = \begin{cases} 1 & \text{if } \{i, j\} \in E \\ 0 & \text{otherwise.} \end{cases}$$

Let $S \subset N, \mid S \mid \geq 2$ As any induced subgraph of a tree is a forest, it follows that $\sum_{i,j \in S, i<j} w_{ij} \leq \mid S \mid -1$. Thus, the conditions of the lemma are satisfied. This means that any tree determines an upper bound, and Hunter's bound is the best among them. Thus, Lemma 7.1 generalizes that bound.

Lemma 7.2 *If* $n \geq 3$ *then*

$$P(A_1 \cup ... \cup A_n) \leq S_1 - \max_{1 \leq k < l \leq n} \sum_{i \neq k,l} (p_{ik} + p_{li}) + (n-3)p_{kl}. \tag{57}$$

Proof For a fixed k and l let G^1 be the complete bipartite graph connecting k and l with all other vertices, G^2 the edge $\{k, l\}$, and $w_{kl} = 3 - n$. Thus, G^1 has $2n - 4$ edges. Let $S \subset N$ be any subset containing at least two elements. If $k, l \notin S$, then the subgraph of G^1 induced by S has no edge; thus,

$$\sum_{i,j \in S, i<j} w_{ij} = 0.$$

If S contains only one of k and l, then

$$\sum_{i,j \in S, i<j} w_{ij} = \mid S \mid -1.$$

Finally, if S contains both k and l, then

$$\sum_{i,j \in S, i<j} w_{ij} = 2 \mid S \mid - 4 + 3\, n \leq \mid S \mid - 1.$$

Thus, the conditions of Lemma 7.1 are satisfied in all cases. □

As the structure of the graph obeys the above mentioned hypergraph scheme, the polynomial

$$g(z_1, z_2) = z_1 + z_2 - z_1 z_2 + (n-3) \binom{z_2}{2}$$

satisfies the condition:

$$\forall z_1, z_2 \in Z_+ : (z_1, z_2) \neq (0, 0), \; z_1 \leq n - 2, \; z_2 \leq 2 \text{ implies } g(z_1, z_2) \geq 1.$$

Prékopa (1999) has shown that Hunter's bound can be represented as the objective function value corresponding to a dual feasible basis of problem (9).

Lemma 7.3 *Assume $n \geq 4$. Let $G^1(N, E)$ be any 1-tree and*

$$C = \{\{u_1, u_2\}, \{u_2, u_3\}, ..., \{u_{k-1}, u_k\}, \{u_k, u_1\}\}$$

be the unique simple circuit contained in G^1. Assume that $k \geq 4$. Let s, t be positive integers with $1 \leq s < t \leq k$, and $t - s \not\equiv \pm 1 (\text{mod } k)$. Let G^2 be a graph containing a single edge such that $E = \{\{u_s, u_t\}\}$. Finally, let

$$w_{ij} = \begin{cases} 1 & \text{if } i < j \text{ and } \{i,j\} \in E^1 \\ -1 & \text{if } i < j \text{ and } \{i,j\} \in E^2 \\ 0 & \text{otherwise.} \end{cases}$$

Then we have the inequality:

$$P(A_1 \cup ... \cup A_n) \leq S_1 - \sum_{\{i,j\} \in E^1, i<j} p_{ij} + p_{u_s u_t}. \tag{58}$$

Proof Any subgraph of G^1 induced by a set $S \subset N$ contains at most $| S | -1$ edges, except C, which contains as many edges as vertices. But even in this case the necessary inequality, given in Lemma 7.1, holds because of the presence of the (-1)-valued edge $\{u_s, u_t\}$, and thus, the conditions of Lemma 7.1 are satisfied. $\square$

The following method is an approximation algorithm for determining the best bound of this type. The algorithm works on the complete graph $K_n(N, E)$. The edge $\{i, j\}$ of K_n is weighted by p_{ij}.

STEP 1: Find a maximum weight spanning tree of K_n, designate it by $T(N, E_T)$.

STEP 2: For any edge $\{i, j\} \in E \setminus E_T$ let $C_{ij} = \{\{u_1^{ij}, u_2^{ij}\}, \ldots, \{u_{l_{ij}-1}^{ij}, u_{l_{ij}}^{ij}\}, \{u_{l_{ij}}^{ij}, u_1^{ij}\}\}$ be the unique simple circuit of the graph $T_{ij}(N, E_T \cup \{i, j\})$, where l_{ij} is the length of C_{ij}. Then, let

$$(i^*, j^*, s^*, t^*) = \text{argmax } \{p_{ij} - p_{u_s^{ij} u_t^{ij}} : l_{ij} \geq 4, 1 \leq s < t \leq l_{ij}, t - s \not\equiv \pm 1 \text{ mod } l_{ij}\}. \tag{59}$$

If $p_{i^*j^*} - p_{u_{s^*}^{ij} u_{t^*}^{ij}} > 0$, then the resulting bound based on the graphs $G = T_{i^*j^*}$ and $G^2(N, \{s^*, t^*\})$ is an improvement on Hunters's bound. The order of the algorithm is $O(n^4)$. In (59) the number of pairs $\{i, j\}$ to be considered is $O(n^2)$. The determination of C_{ij} is equivalent with finding the unique simple path going from i to j in T which can be done in $O(n)$ steps as the sum of the degrees of the vertices in T is $2n - 2$. Then, the selection of the best possible pair $\{s, t\}$ takes $O(n^2)$ operations.

A special case of this type of upper bound is obtained by restricting G^1 to be a Hamiltonian circuit. Let $\mathcal{H}$ be the set of all Hamiltonian circuits. In this way the following upper bound can be obtained:

$$P(A_1 \cup ... \cup A_n) \leq S_1 - \max_{H \in \mathcal{H}} \sum_{\{i,j\} \in H, i<j} p_{ij} + \min_{\{s,t\} \notin H} p_{st}. \tag{60}$$

The second term of the right-hand side is equivalent to a travelling salesman problem which is known to be NP-hard. But plenty of good and fast heuristics are available to generate approximate solutions.

8 Comparision with the Aggregated Upper Bound

The optimal value of the maximization problem (2) with $m = 2$ is

$$S_1 - \frac{2}{n} S_2$$

as it is shown by Kwerel (1975a), Sathe, Pradhan and Shah (1980), and Boros-Prékopa (1989). In this section a general lemma is proved, which makes it easy to prove for a wide class of upper bounds, that they are at least as good as the corresponding aggregated ones.

Lemma 8.1 *Let $N^1 = \{\{i,j\} \mid 1 \le i,j \le n; i \ne j\}$, $N^2 = \{1,2,...,r\}$, where $r = \binom{n}{2}$, and assume that the function $\rho : N^1 \to N^2$ defines a one-to-one correspondence between the two sets. Let $w_1, ..., w_r$ be any real numbers satisfying the equation*

$$\sum_{j=1}^{r} w_j \ = \ n-1.$$

Finally, let π be any permutation of the set $\{ 1, ..., n \}$. Then we have the inequality:

$$\max_{\pi:} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} w_{\rho(i,j)} p_{\rho(\pi(i),\pi(j))} \ \ge \ \frac{2}{n} S_2. \tag{61}$$

Proof The left-hand side of the inequality is the maximum of some numbers. The average of the same numbers is

$$\frac{1}{n!} \sum_{\pi:} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} w_{\rho(i,j)} p_{\rho(\pi(i),\pi(j))}.$$

The symmetricity of the expression implies that all p's must have the same coefficient in the sum, which is

$$\frac{\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} w_{\rho(i,j)}}{\binom{n}{2}} = \frac{2}{n},$$

as there are $n!$ permutations, the number of w's is $\binom{n}{2}$, and their sum is $n-1$. Thus, the above average is equal to the right-hand side of the inequality. Hence the statement follows immediately.

Remark The proof does not use any property of the p's, hence the statement holds for any vector $p \in R^{\frac{n(n-1)}{2}}$, and $S_2 = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} p_{ij}$.

In the statement the w's represent a fixed structure and the permutation of the p's ensures that the best sample is chosen which is isomorphic with the fixed structure. For example, the statement that Hunter's bound is at least as good as the aggregated bound, follows from the lemma in two steps. First, the vector w is fixed in such a way that it represents a certain tree structure. The best tree is selected which is isomorphic with this structure. Then, we look at all tree structures and the best of bests gives Hunter's bound. But it follows from the lemma that the best of any tree structure is at least as good as the aggregated bound.

Assume that if the vector $(1, 1, ..., 1, -w_1, ..., -w_r^T) \in R^{r+n}$ represents a dual feasible solution to problem (11). Then Lemma 8.1 is applicable and

$$S_1 - \max_{\pi:} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} w_{\rho(i,j)} p_{\rho(\pi(i),\pi(j))} \leq S_1 - \frac{2}{n} S_2.$$

9 Some Special Problem Classes

In this section we show that any upper bound mentioned in Section 7 corresponds to at least one problem class containing (for every n) a problem such that the upper bound coincides with the actual value of $P(A_1 \cup ... \cup A_n)$.

Lemma 9.1 *If the vector $(1, ..., 1, -w_{12}, ..., -w_{n-1,n})^T \in R^{\frac{n(n+1)}{2}}$ is a feasible solution of the dual of the maximization problem, and for every ij the inequality $w_{ij} > 0$ implies that $w_{ij} = 1$, then there is a problem instance such that the upper bound is equal to $P(A_1 \cup ... \cup A_n)$.*

Proof The upper bound is

$$S_1 - \sum_{i=1}^{n} \sum_{j=i+1}^{n} w_{ij} p_{ij}.$$

If $A_1, ..., A_n$ are events such that $p_i = 1/n$ $(1 \leq i \leq n)$ and if $i \neq j$ then

$$p_{ij} = \begin{cases} \frac{1}{n^2} & \text{if } w_{ij} = 1 \\ 0 & \text{if } w_{ij} \leq 0, \end{cases} \tag{62}$$

then the following equations hold

$$P(A_1 \cup ... \cup A_n) = 1 - \frac{\sum_{(i,j):i<j,w_{ij}=1} 1}{n^2} = S_1 - \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} w_{ij} p_{ij},$$

i.e., the statement of the lemma is true. These events $A_1, ..., A_n$ can be constructed in the following way. Let $\omega_1, ..., \omega_{n^2}$ be n^2 mutually exclusive events. Let the probability of each ω_i be $1/n^2$. Let $A_1 = \omega_1 \cup ... \cup \omega_n$. If $w_{12} \leq 0$, then we define $A_2 = \omega_{n+1} \cup ... \cup \omega_{2n}$, otherwise let $A_2 = \omega_1 \cup \omega_{n+1} \cup \ ... \cup \omega_{2n-1}$. Assume that $A_1, ..., A_{i-1}$ are determined and $A_j = \omega_{k_{j1}} \cup ... \cup \omega_{k_{jn}}$ if $1 \leq j \leq i-1$, i.e. the set of the indices of ω_l's contained in the composite event A_j is $\{ k_{j1}, ..., k_{jn} \}$. Let $\{ l_1, ..., l_t \} = \{ j : \ 1 \leq j \leq i-1, \ w_{ji} = 1 \}$. Then, let $A_i = \omega_{k_{l_1 i}} \cup ... \cup \omega_{k_{l_t i}} \cup \omega_{k_{i-1,n}+1} \cup ... \cup \omega_{k_{i-1,n}+n-t}$. Thus, A_i and A_j are mutually exclusive if $w_{ij} \leq 0$, otherwise $p_{ij} = 1/n^2$. $\square$

10 Conclusions

In order to create lower and upper bounds for the probability of the union of events, arranged in a finite sequence, a simple and frequently efficient method is the one provided by the discrete binomial moment problems. These are LP's, where the right-hand side numbers are some of the binomial moments $S_1, S_2, \ldots$. Since S_k is the sum of joint probabilities of k-tuples of events, these LP's are called aggregated problems. Better bounds can be obtained if we use the individual probabilities in the sums of all S_k binomial moments that turn up in the aggregated problem. However, the LP's based on these, called the disaggregated problems, may have huge sizes, in general, and we may not be able to solve them or they are computationally intensive. In the present paper we have shown that third types of problems, which can be placed in between the aggregated and disaggregated problems, can combine solvability and very good bounding performance, at least in many cases. Two general aggregation/disaggregation methods are presented. The first one creates subproblems corresponding to the individual events. The second one subdivides the event sequence into subsequences and then applies existing multivariate binomial moment bounding technique to improve on the univariate bounds. We have also presented an improvement on Hunter's upper bound, where we have used S_1 and the probabilities in the sum designated by S_2. Both Hunter's bound and its improvement can be associated with the Boolean probability bounding scheme, and some aggregation procedures.

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