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Bounding Expectations of Functions of Random Vectors with Given Marginals and some Moments: Applications of the Multivariate Discrete Moment Problem

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Bounding Expectations of Functions of Random Vectors with Given Marginals and some Moments: Applications of the Multivariate Discrete Moment Problem

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Abstract. The paper shows how the bounding technique provided by the Multivariate Discrete Problem can be used for bounding the expectations of functions of random variables with known univariate marginals and some of the mixed moments. Four examples are presented. In the first example the function is a Monge or related array, in the second one it is a pseudo Boolean function. In the further examples bounds are presented for the values of multivariate generating functions and for the expectation of special utility functions of random variables. Numerical results are presented.
Keywords: discrete moment problem, Monge property, pseudo Boolean functions, generating functions, multiattribute utility functions

1 Introduction

Recently a number of papers have been published about the univariate and multivariate discrete moment problem (DMP, MDMP), where we create bounding formulas as well as algorithmic bounds for function of random variables under moment information (see Prékopa 1990, Prékopa 1998, Mádi-Nagy and Prékopa 2004). In the univariate case the moments of order up to m of a random variable are supposed to be known, and, based on this, lower and upper bounds for functions of the random variable have been proposed in Prékopa (1990). This includes the construction of bounds for probabilities, in terms of the moments of the random variable, since this latter problem is a special case of the former one. Similarly, the paper Prékopa (1998) deals with bounds for functions of random vectors, where the mixed moments of the components of total order up to m are known. The results have been generalized in Mádi-Nagy and Prékopa (2004), assuming that, in addition to the knowledge of moments of total order up to m, some further moments of the univariate marginals are also known.

Sometimes, when bounds for expectations of function of random vectors are constructed, all univariate marginals are completely known and the stochastic dependencies are characterized by some of the mixed moments, e.g., the covariances. This is the case in the paper by Hou and Prékopa (2006), where a bounding technique, different from the one in MDMP is used.

The purpose of the present paper is to give a number of examples for the application of the MDMP technique, to bounding expectations of functions of random vectors, where the univariate marginals and some of the mixed moments are known.

Let $\boldsymbol{X} = (X_1, \ldots, X_s)$ be a random vector where the support of X_j is a known finite set $Z_j = \{z_{j0}, \ldots, z_{jn_j}\}$ with distinct elements, $j = 1, \ldots, s$ and introduce the notation:

$$p_{i_1 \ldots i_s} = P(X_1 = z_{1i_1}, \ldots, X_s = z_{si_s}),\ 0 \le i_j \le n_j,\ j = 1, \ldots, s. \tag{1.1}$$

We assume that the probability distribution of $\boldsymbol{X}$ is not known, but known are the univariate marginals, i.e., the distributions of the components X_j's, $j = 1, \ldots, s$. We use the following notations:

$$P(X_j = z_{ji}) = q_i^{(j)},\ i = 0, \ldots, n_j,\ j = 1, \ldots, s.$$

Our aim is to give lower and upper bounds for

$$E[f(X_1, \ldots, X_s)],$$

where $f(\boldsymbol{z})$, $\boldsymbol{z} \in Z$ is discrete function about which we will introduce some assumptions. For simplicity let $f_{i_1 \ldots i_s} = f(z_{1i_1}, \ldots, z_{si_s})$.

The $(\alpha_1, \ldots, \alpha_s)$-order moment of the random vector $(X_1, \ldots, X_s)$ is defined as

$$\mu_{\alpha_1 \ldots \alpha_s} = E[X_1^{\alpha_1} \cdots X_s^{\alpha_s}] = \sum_{i_1=0}^{n_1} \cdots \sum_{i_s=0}^{n_s} z_{1i_1}^{\alpha_1} \cdots z_{si_s}^{\alpha_s} p_{i_1 \ldots i_s},$$

where $\alpha_1, \ldots, \alpha_s$ are nonnegative integers. The sum $\alpha_1 + \cdots + \alpha_s$ is called the total order of the moment.

One important objective of the multivariate discrete moment problem (MDMP) is to give lower and upper bounds for $E[f(X_1, \ldots, X_s)]$, where some collection of the moments $\mu_{\alpha_1 \ldots \alpha_s}$ is known.

The MDMP that we use in this paper is the following:

$$
\begin{aligned}
&\min(\max) \ \sum_{i_1=0}^{n_1} \cdots \sum_{i_s=0}^{n_s} f_{i_1 \ldots i_s} p_{i_1 \ldots i_s} \\
&\text{subject to} \\
&\sum_{i_1=0}^{n_1} \cdots \sum_{i_{j-1}=0}^{n_{j-1}} \sum_{i_{j+1}=0}^{n_{j+1}} \sum_{i_s=0}^{n_s} p_{i_1 \ldots i_{j-1} i i_{j+1} \ldots i_s} = q_i^{(j)} \\
&\text{for } i = 0, \cdots, n_j, \ j = 1, \ldots, s; \text{ and} \\
&\sum_{i_1=0}^{n_1} \cdots \sum_{i_s=0}^{n_s} z_{1i_1}^{\alpha_1} \cdots z_{si_s}^{\alpha_s} p_{i_1 \ldots i_s} = \mu_{\alpha_1 \ldots \alpha_s} \\
&\text{for } 0 \leq \alpha_j, \ j = 1, \ldots, s, \alpha_u \alpha_v \neq 0 \text{ for some } u \neq v, \ \alpha_1 + \cdots + \alpha_s \leq m; \\
&p_{i_1 \ldots i_s} \geq 0, \ \text{all } i_1, \ldots, i_s,
\end{aligned}
\tag{1.2}
$$

where $(q_0^{(j)}, \ldots, q_{n_j}^{(j)})$, $j = 1, \ldots, s$ are known univariate distributions, $\mu_{\alpha_1 \ldots \alpha_s}$, $\alpha_1 + \cdots \alpha_s \leq m$ are known moments and the decision variables are $p_{i_1 \ldots i_s}$, $0 \leq i_j \leq n_j$, $j = 1, \ldots, s$. The objective function, the first set of constraints and the nonnegativity restrictions define an s-dimensional transportation problem (see Hou and Prékopa 2006). Problem (1.2) will be called extended s-dimensional transportation problem.

Since the cardinality of the support of X_j is $n_j + 1$, it follows that the moments

$$
E[X_j^k] = \sum_{i=0}^{n_j} z_{ji}^k q_i^{(j)}, \ k = 0, \ldots n_j
$$

uniquely determine its probability distribution one of the marginal distributions of $\boldsymbol{X}$, it follows that problem (1.2) is equivalent to the following:

$$
\begin{aligned}
&\min(\max) \ \sum_{i_1=0}^{n_1} \cdots \sum_{i_s=0}^{n_s} f_{i_1 \ldots i_s} p_{i_1 \ldots i_s} \\
&\text{subject to} \\
&\sum_{i_1=0}^{n_1} \cdots \sum_{i_s=0}^{n_s} z_{1i_1}^{\alpha_1} \cdots z_{si_s}^{\alpha_s} p_{i_1 \ldots i_s} = \mu_{\alpha_1 \ldots \alpha_s} \\
&\text{for } \alpha_j = 0, \ j = 1, \ldots, k-1, k+1, \ldots, s, \ 0 \leq \alpha_k \leq n_k, \ k = 1, \ldots, s \text{ and} \\
&\text{for } 0 \leq \alpha_j, \ j = 1, \ldots, s, \alpha_u \alpha_v \neq 0 \text{ for some } u \neq v, \ \alpha_1 + \cdots + \alpha_s \leq m; \\
&p_{i_1 \ldots i_s} \geq 0, \ \text{all } i_1, \ldots, i_s.
\end{aligned}
\tag{1.3}
$$

The compact matrix form of problem (1.3) will be written as:

$$\begin{array}{rrcl} & \min(\max) & & \boldsymbol{f}^T\boldsymbol{p} \\ \text{subject to} & & & \\ & \widehat{A}\boldsymbol{p} & = & \widehat{\boldsymbol{b}} \\ & \boldsymbol{p} & \geq & \mathbf{0}. \end{array} \tag{1.4}$$

The paper is organized as follows. In Section 2 we specialize our general theorems proved in Mádi-Nagy and Prékopa (2004) for the case of problem (1.3), suitable for our current application. In Section 3 we derive further results, by the use of problem (1.3), for the problem studied in Hou and Prékopa (2006), where the objective function enjoys the Monge or some related property. In Section 4 we apply the specialized MDMP technique to bounding the expectations of pseudo Boolean functions under monotonicity conditions. The next example is bounding the values of generating functions, is presented in Section 5. In Section 6 we show how the MDMP technique applies to bounding expected utility functions. Finally, we summarize the conclusions of our results.

2 Bounds When the Univariate Marginal Distributions and Moments of Total Order up to m are Known

First we state a theorem valid for a Lagrange interpolation polynomial defined in $\mathbb{R}^s$.

In what follows we will use the notations

$$\begin{array}{rcl} Z_{ji} & = & \{z_{j0}, \ldots, z_{ji}\} \\ Z'_{ji} & = & \{z_{j0}, \ldots, z_{ji}, z_j\}, \\ & & i = 0, \ldots, n_j,\ j = 1, \ldots, s. \end{array}$$

Consider the set of subscripts

$$I = I_0 \cup \left(\cup_{j=1}^{s} I_j \right), \tag{2.1}$$

where

$$I_0 = \{(i_1, \ldots, i_s) |\ 0 \leq i_j \leq m-1,\ \text{integers},\ j = 1, \ldots, s,\ i_1 + \ldots + i_s \leq m\} \tag{2.2}$$

and

$$I_j = \{(i_1, \ldots, i_s) |\ m \leq i_j \leq n_j,\ i_l = 0\ l \neq j\},\ j = 1, \ldots, s. \tag{2.3}$$

Corresponding to the points

$$Z_I = \{(z_{1i_1}, \ldots, z_{si_s}) |\ (i_1, \ldots, i_s) \in I\} \tag{2.4}$$

we assign the Lagrange polynomial, given by its Newton's form:

$$
\begin{aligned}
& L_I\left(z_1, \ldots, z_s\right) \\
= & \sum_{\substack{i_1+\ldots+i_s \leq m \\ 0 \leq i_j \leq m-1,\ j=1,\ldots,s}} \left[Z_{1 i_1} ; \cdots ; Z_{s i_s} ; f\right] \prod_{j=1}^{s} \prod_{k=0}^{i_j-1}\left(z_j-z_{j k}\right) \\
& +\sum_{j=1}^{s} \sum_{i=m}^{n_j}\left[Z_{10} ; \cdots ; Z_{(j-1) 0} ; Z_{j i} ; Z_{(j+1) 0} ; \cdots ; Z_{s 0} ; f\right] \prod_{k=0}^{i-1}\left(z_j-z_{j k}\right), \\
& \text{where, by definition, } \prod_{k=0}^{i_j-1}\left(z_j-z_{j k}\right)=1, \text{ for } i_j=0.
\end{aligned}
\tag{2.5}
$$

In (2.5) the function f is not necessarily restricted to the set Z as its domain of definition; it may be defined on any subset of $\mathbb{R}^s$ that contains Z. Next, we define the residual function:

$$
R_I(z_1, \ldots, z_s) = R_{1I}(z_1, \ldots, z_s) + R_{2I}(z_1, \ldots, z_s),
\tag{2.6}
$$

where

$$
\begin{aligned}
& R_{1I}(z_1, \ldots, z_s) \\
= & \sum_{j=1}^{s}\left[z_{10} ; \cdots ; z_{(j-1) 0} ; Z_{j n_j}^{\prime} ; z_{(j+1) 0} ; \cdots ; z_{s 0} ; f\right] \prod_{k=0}^{n_j}\left(z_j-z_{j k}\right)
\end{aligned}
\tag{2.7}
$$

and

$$
\begin{aligned}
& R_{2I}(z_1, \ldots, z_s) \\
= & \sum_{h=1}^{s-1} \sum_{\substack{i_h+\cdots+i_s=m \\ 0 \leq i_j \leq m-1,\ j=h,\ldots,s}}\left[z_1 ; \cdots ; z_{h-1} ; Z_{h i_h}^{\prime} ; Z_{(h+1) i_{h+1}} ; \cdots ; Z_{s i_s} ; f\right] \prod_{l=0}^{i_h}\left(z_h-z_{h l}\right) \\
& \times \prod_{h+1}^{s} \prod_{k=0}^{i_j-1}\left(z_j-z_{j k}\right) \\
& +\sum_{j=h+1}^{s}\left[z_1 ; \cdots ; z_{h-1} ; Z_{h 0}^{\prime} ; Z_{(h+1) 0} ; \cdots ; Z_{(j-1) 0} ; Z_{j(m-1)}^{\prime} ; Z_{(j+1) 0} ; \cdots ; Z_{s 0}\right]\left(z_h-z_{h 0}\right) \\
& \times \prod_{k=0}^{m-1}\left(z_j-z_{j k}\right).
\end{aligned}
\tag{2.8}
$$

The following theorem is a consequnece of Theorem 3.1 in Mádi-Nagy and Prékopa (2004).

Theorem 2.1 *Consider the Lagrange polynomial (2.5), corresponding to the points in Z_I. For any $\boldsymbol{z} = (z_1, \ldots, z_s)$ for which the function f is defined, we have the equality*

$$
L_I(z_1, \ldots, z_s) + R_I(z_1, \ldots, z_s) = f(z_1, \ldots, z_s).
\tag{2.9}
$$

Remark 2.1 *In the following we shall use the notion of an H-type Lagrange polinomial. It means that the set of orders in the terms, i.e., $\{(\alpha_1, \ldots, \alpha_n)\}$ is the same as the set of subscripts of the moments $\{\mu_{\alpha_1,\ldots,\alpha_n}\}$ used in the constraints of the MDMP. More precise definition and details about it and its relationship to bases in MDMP can be found in Mádi-Nagy and Prékopa (2004).*

Now we prove

Theorem 2.2 *Let $z_{j0} < z_{j1} < \cdots < z_{jn_j}$, $j = 1, \ldots, s$. Suppose that the function $f(\boldsymbol{z})$, $\boldsymbol{z} \in Z$ has nonnegative mixed divided differences of total order $m+1$.*

Under this condition $L_I(z_1, \ldots, z_s)$, defined by (2.5), is a unique H-type Lagrange polynomial on Z_I and satisfies the relation

$$f(z_1, \ldots, z_s) \geq L_I(z_1, \ldots, z_s), \ (z_1, \ldots, z_s) \in Z, \tag{2.10}$$

i.e., the set of columns $\widehat{B}$ of $\widehat{A}$ in problem (1.4), with the subscript set I, is a dual feasible basis in the minimization problem (1.4), and

$$E[f(X_1, \ldots, X_s)] \geq E[L_I(X_1, \ldots, X_s)]. \tag{2.11}$$

If $\widehat{B}$ is also a primal feasible basis in problem (1.4), then the inequality (2.11) is sharp.

If all the above mentioned divided differences are nonpositive, then (2.10) and (2.11) hold with reversed inequality signs.

Proof. The proof is similar to that of Theorem 4.1 in Mádi-Nagy and Prékopa (2004). The only difference is that here we explicit the fact

$$\prod_{k=0}^{n_j} (z_j - z_{jk}) = 0 \text{ for } z_j \in Z_j, \tag{2.12}$$

which is a trivial consequence of the definition of Z_j. □

In the next theorem we prove both lower and upper bounds for the function $f(z_1, \ldots, z_s)$, $(z_1, \ldots, z_s) \in Z$ and the expectation $E[f(X_1, \ldots, X_s)]$.

Theorem 2.3 *Let $z_{j0} > z_{j1} > \cdots > z_{jn_j}$, $j = 1, \ldots, s$. Suppose that the function $f(\boldsymbol{z})$, $\boldsymbol{z} \in Z$ has nonnegative mixed divided differences of total order $m+1$. Under this condition we have the following assertions:*

(a) If $m+1$ is even, then the Lagrange polynomial $L_I(z_1, \ldots, z_s)$, defined by (2.5), satisfies

$$f(z_1, \ldots, z_s) \geq L_I(z_1, \ldots, z_s), \ (z_1, \ldots, z_s) \in Z, \tag{2.13}$$

i.e., the set of columns $\widehat{B}$ in $\widehat{A}$, corresponding to the subscripts I, is a dual feasible basis in the minimization problem (1.4). We also have the inequality

$$E[f(X_1, \ldots, X_s)] \geq E[L_I(X_1, \ldots, X_s)]. \tag{2.14}$$

If $\widehat{B}$ is also a primal feasible basis in the LP (1.4), then the lower bound (2.14) for $E[f(X_1, \ldots, X_s)]$ is sharp.

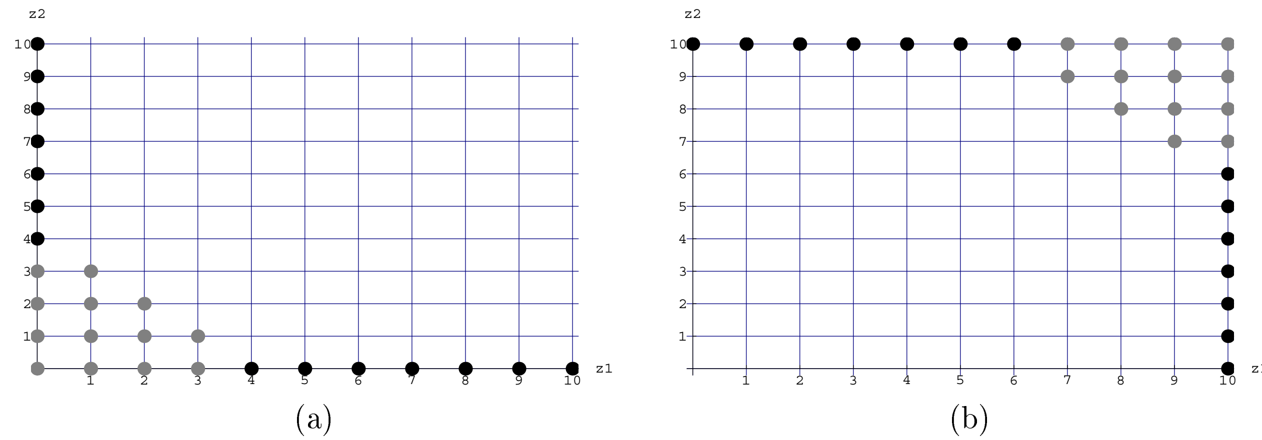


Figure 1: Dual feasible bases corresponding to Theorems 2.2 (on (a)) and 2.3 (on (b)) in case of $n_1 = n_2 = 10$, $m = 3$, $Z_1 = Z_2 = \{0, 1, \dots, 10\}$. Elements of I_0 are colored by gray while elements of I_j's are black.

(b) If $m+1$ is odd, then the Lagrange polynomial, defined by (2.5), satisfies

$$f(z_1, \dots, z_s) \le L_I(z_1, \dots, z_s), \ (z_1, \dots, z_s) \in Z, \tag{2.15}$$

i.e., the basis $\widehat{B}$ is dual feasible in the maximization problem (1.4). We also have the inequality

$$E[f(X_1, \dots, X_s)] \le E[L_I(X_1, \dots, X_s)]. \tag{2.16}$$

If $\widehat{B}$ is also a primal feasible basis in the LP (1.4), then the upper bound (2.16) for $E[f(X_1, \dots, X_s)]$ is sharp.

If all the above mentioned divided differences are nonpositive, then (2.13),(2.14),(2.15) and (2.16) hold with reversed inequality signs.

Proof. The proof is similar to that of Theorem 4.2 in Mádi-Nagy and Prékopa (2004). Here, however, we explicit the equation (2.12). □

The dual feasible structures given by the theorems above are illustrated in Figure 1.

In the two-dimensional case we can create a larger variety of dual feasible bases for problem (1.4), and produce better bounds than what we can obtain by the use of the dual feasible basis structures presented in the previous theorems.

All coefficients in the expression of $R_{2I}(z_1, z_2)$ are divided differences of order $m+1$. These divided differences are mixed. Assume all of them are nonnegative. Our aim is to arrange the elements of Z_1 and Z_2 in orders such that the products in $R_{2I}(z_1, z_2)$ are nonnegative.

These arrangements can be produced by slight modifications of the Min and Max Algorithms of Mádi-Nagy and Prékopa (2004). In this way we can get a variety of dual feasible bases that give tight bounds on $E[f(X_1, X_2)]$. Below we summarize them for the bivariate case of problem (1.3).

Consider first the case, where we construct lower bound by suitable choices of $z_{10}, \ldots, z_{1(m-1)}; z_{20}, \ldots, z_{2(m-1)}$. We present an algorithm to find these sequences. We may assume, without loss of generality, that the ordered sets Z_1 and Z_2 are the following: $Z_1 = \{0, 1, \ldots, n_1\}$, $Z_2 = \{0, 1, \ldots, n_2\}$.

Min Algorithm

Algorithm to find $z_{10}, \ldots, z_{1(m-1)}; z_{20}, \ldots, z_{2(m-1)}$.

Step 0. Initialize $t = 0$, $-1 \leq q_1 \leq m-1$, $L = (0, 1, \ldots, q_1)$, $U = (n_1, n_1 - 1, \ldots, n_1 - (m - q_1 - 2))$, $V^0 = \{$arbitrary merger of the sequences $L, U\} = (v^0, v^1, \ldots, v^{m-1})$. If $|U|$ is even, then $h^0 = 0$, $l^0 = 1$, $u^0 = n_2$, and if $|U|$ is odd, then $h^0 = n_2$, $l^0 = 0$, $u^0 = n_2 - 1$. Go to Step 1.

Step 1. If $t = m$, then go to Step 3. Otherwise go to Step 2.

Step 2. Let $V^t = (v^0, v^1, \ldots, v^{m-1-t})$, $H^t = (h^0, h^1, \ldots, h^t)$. If $v^{m-1-t} \in L$, then let $h^{t+1} = l^t$, $l^{t+1} = l^t + 1$, $u^{t+1} = u^t$, and if $v^{m-1-t} \in U$, then let $h^{t+1} = u^t$, $u^{t+1} = u^t - 1$, $l^{t+1} = l^t$. Set $t \leftarrow t + 1$ and go to Step 1.

Step 3. Stop. Let

$$(z_{10}, \ldots, z_{1(m-1)}) = V^0,$$

$$(z_{20}, \ldots, z_{2(m-1)}) = H^{m-1}.$$

Let $0, 1, \ldots, q_2, n_2, \ldots, n_2 - (m - q_2 - 2)$ be the numbers used to construct $z_{20}, z_{21}, \ldots, z_{2(m-1)}$. Then let $\{z_{jm}, z_{j(m+1)}, \ldots, z_{jn_j}\} = \{q_j + 1, q_j + 2, \ldots, n_j - (m - q_j - 1)\}$, $j = 1, 2$. They can follow the each other in any order, because they don't play role in the value of R_I, and on the other hand their order does not change the dual feasible basis structure that we finally get.

We have completed the construction of the dual feasible basis related to the subscript set I.

If we want to construct an upper bound, then only slight modification is needed in the above algorithm to find $z_{10}, \ldots, z_{1(m-1)}; z_{20}, \ldots, z_{2(m-1)}$. We only have to rewrite Step 0 and keep the other steps unchanged.

Max Algorithm

Step 0 of algorithm to find $z_{10}, \ldots, z_{1(m-1)}; z_{20}, \ldots, z_{2(m-1)}$.

Step 0. Initialize $t = 0$, $-1 \leq q_1 \leq m-1$, $L = (0, 1, \ldots, q_1)$, $U = (n_1, n_1 - 1, \ldots, n_1 - (m - q_1 - 2))$, $V^0 = \{$arbitrary merger of the sets $L, U\} = (v^0, v^1, \ldots, v^{m-1})$. If $|U|$ is odd, then $h^0 = 0$, $l^0 = 1$, $u^0 = n_2$, and if $|U|$ is even, then $h^0 = n_2$, $l^0 = 0$, $u^0 = n_2 - 1$. Go to Step 1, etc.

In the general case, where Z_1 is not necessarily $\{0, 1, \ldots, n_1\}$ and Z_2 is not necessarily $\{0, 1, \ldots, n_2\}$, we do the following. First we order the elements in both Z_1 and Z_2 in increasing order. Then, establish one-to-one correspondences between the elements of Z_1 and the elements of the set $\{0, 1, \ldots, n_1\}$ that we assume to be ordered now. We do the

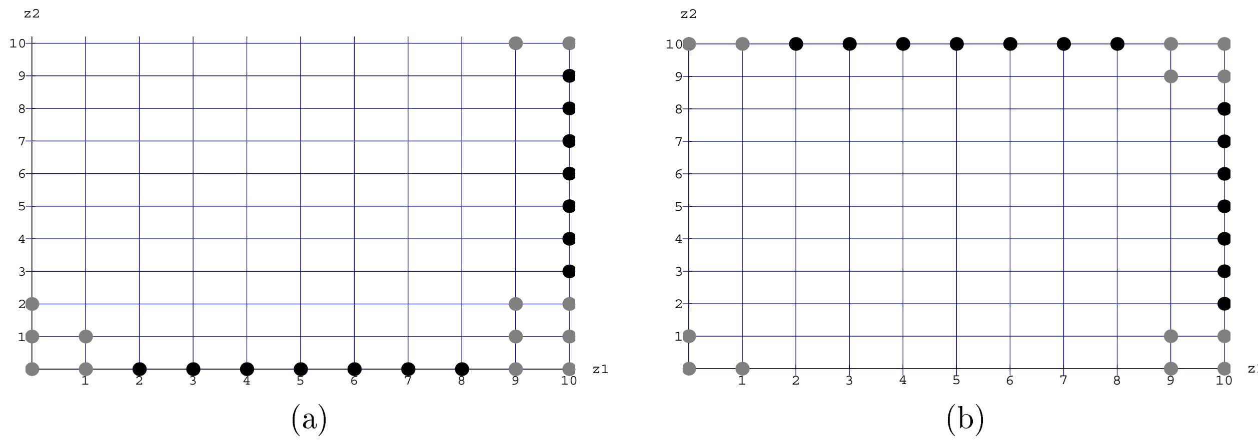


Figure 2: (a): Dual feasible basis of the min problem, where $m = 4$ and $(z_{10}, \ldots, z_{1(m-1)}) = (10, 9, 0, 1)$, $(z_{20}, \ldots, z_{2(m-1)}) = (0, 1, 2, 10)$. (b): Dual feasible basis of the max problem, where $m = 4$ and $(z_{10}, \ldots, z_{1(m-1)}) = (10, 9, 0, 1)$, $(z_{20}, \ldots, z_{2(m-1)}) = (10, 0, 1, 9)$. Elements of I_0 are colored by gray while elements of I_j's are black.

same to Z_2 and $\{0, 1, \ldots, n_2\}$. After that, we carry out the Min or Max Algorithm to find a dual feasible basis, using the sets $\{0, 1, \ldots, n_1\}$, $\{0, 1, \ldots, n_2\}$, as described in this section. Finally, we create the ordered sets Z_1 and Z_2 by the use of the above mentioned one-to-one correspondences.

Examples of dual feasible bases found by the Min and Max Algorithms are illustrated by Figure 2.

3 Monge Property and Bounding Multivariate Probability Distribution Functions with Given Marginals and Covariances

In this chapter we assume that the function f has the so-called Monge or inverse Monge or some discrete discrete higher order convexity property. First, we need the following

Definition 3.1 *An $n_1 \times \cdots \times n_s$ s-dimensional array $\boldsymbol{f} = \{f(i_1, \ldots, i_s)\}$ has the* Monge property *or is a* Monge array, *if for all entries $f(i_1, \ldots, i_s)$ and $f(j_1, \ldots, j_s)$, $1 \leq i_k, j_k \leq n_k$, $1 \leq k \leq s$, we have*

$$f(l_1, \ldots, l_s) + f(u_1, \ldots, u_s) \leq f(i_1, \ldots, i_s) + f(j_1, \ldots, j_s), \tag{3.1}$$

where $l_k = min\{i_k, j_k\}$, $u_k = max\{i_k, j_k\}$, $1 \leq k \leq s$. If the inequality (3.1) holds in reverse order, then it is called the inverse Monge property *and $\boldsymbol{f}$ is called an* inverse Monge array.

Remark 3.1 *If $f(z_1, \ldots, z_s)$, $z \in Z = Z_1 \times \cdots \times Z_s$ is a (inverse) Monge array on Z, then its second order mixed divided differences are nonpositive (nonnegative). In the two-dimensional* *case,* $f(z_1, z_2), z \in Z = Z_1 \times Z_2$ *is a (inverse) Monge array on* Z *if and only if its* $(1,1)$ *order divided differences are nonpositive (nonnegative).*

If we consider problem (1.2) in case of $m = 1$, i.e., if only the marginal distributions are known, then it can be considered as an s-dimensional transportation problem. In connection with that we have

Theorem 3.1 (Theorem 2.4 in Hou and Prékopa (2006)) *In the* s*-dimensional transportation problem any ordered sequence forms a dual feasible basis if and only if* $\boldsymbol{f}$ *is Monge.*

In case of $m = 2$, where the second order moments (covariances) are also known, all the dual feasible bases shown in the mentioned paper can be given by our theorems and the Min and Max Algorithms in a relatively simple way. In the two dimensional case our method can give additional dual feasible bases as it is shown in the following example.

Example 3.1 *Consider the minimum problem (1.2), and the equvivalent MDMP (1.3) in case of* $s = 2$ *and* $m = 2$*. Suppose that the function* $f(\boldsymbol{z})$*,* $\boldsymbol{z} \in Z$ *has nonnegative mixed divided differences of total order* 3*, i.e., the* $(1,2)$*-order and* $(2,1)$*-order divided differences are nonnegative. Apply the Min Algorithm to the problem. Below the possible dual feasible bases are listed, according to the order of the elements.*

(a) $(z_{10}, z_{11}) = V^0 := (0, 1) \Longrightarrow (z_{20}, z_{21}) = H^{m-1} = (0, 1)$

(b) $(z_{10}, z_{11}) = V^0 := (0, n_1) \Longrightarrow (z_{20}, z_{21}) = H^{m-1} = (n_2, n_2 - 1)$

(c) $(z_{10}, z_{11}) = V^0 := (n_1, 0) \Longrightarrow (z_{20}, z_{21}) = H^{m-1} = (n_2, 0)$

(d) $(z_{10}, z_{11}) = V^0 := (n_1, n_1 - 1) \Longrightarrow (z_{20}, z_{21}) = H^{m-1} = (0, n_2)$

The bases are illustrated in Figure 3. Basis (b) is the same as basis B_1 *in Figure 4.1 in Hou and Prékopa (2006) (regarding that the order of* z_j*'s there are decreasing) which was the only dual feasible basis of this problem given there.*

4 Bounding the Expectations of Pseudo Boolean Functions of Binary Random Variables

Let $A_1, \ldots, A_s$ be arbitrary events in some probability space, and inroduce the notations

$$P(A_{i_1} \cap \cdots \cap A_{i_k}) = p_{i_1 \ldots i_k},\ 1 \le i_1 < \cdots < i_k \le s. \tag{4.1}$$

We want to give bounds for $P(A_1 \cup \cdots \cup A_s)$ assuming, that some of the probabilities of (4.1) are known. The so called disaggregated problem is formulated as follows. Define

$$a_{IJ} = \begin{cases} 1 \text{ if } I \subset J, \\ 0 \text{ if } I \not\subset J, \end{cases}$$

$$
\begin{array}{llllllll}
n_2 & \bullet & \circ & \circ & \cdots & \circ & \circ & \circ \\
n_2{-}1 & \bullet & \circ & \circ & \cdots & \circ & \circ & \circ \\
n_2{-}2 & \bullet & \circ & \circ & \cdots & \circ & \circ & \circ \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
2 & \bullet & \circ & \circ & \cdots & \circ & \circ & \circ \\
1 & \ast & \ast & \circ & \cdots & \circ & \circ & \circ \\
0 & \ast & \ast & \bullet & \cdots & \bullet & \bullet & \bullet \\
 & 0 & 1 & 2 & \cdots & (n_1 -2) & (n_1 -1) & n_1
\end{array}
$$

(a)

$$
\begin{array}{llllllll}
n_2 & \ast & \bullet & \bullet & \cdots & \bullet & \bullet & \ast \\
n_2{-}1 & \ast & \circ & \circ & \cdots & \circ & \circ & \ast \\
n_2{-}2 & \bullet & \circ & \circ & \cdots & \circ & \circ & \circ \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
2 & \bullet & \circ & \circ & \cdots & \circ & \circ & \circ \\
1 & \bullet & \circ & \circ & \cdots & \circ & \circ & \circ \\
0 & \bullet & \circ & \circ & \cdots & \circ & \circ & \circ \\
 & 0 & 1 & 2 & \cdots & (n_1 -2) & (n_1 -1) & n_1
\end{array}
$$

(b)

$$
\begin{array}{llllllll}
n_2 & \ast & \bullet & \bullet & \cdots & \bullet & \bullet & \ast \\
n_2{-}1 & \circ & \circ & \circ & \cdots & \circ & \circ & \bullet \\
n_2{-}2 & \circ & \circ & \circ & \cdots & \circ & \circ & \bullet \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
2 & \circ & \circ & \circ & \cdots & \circ & \circ & \bullet \\
1 & \circ & \circ & \circ & \cdots & \circ & \circ & \bullet \\
0 & \ast & \circ & \circ & \cdots & \circ & \circ & \ast \\
 & 0 & 1 & 2 & \cdots & (n_1 -2) & (n_1 -1) & n_1
\end{array}
$$

(c)

$$
\begin{array}{llllllll}
n_2 & \circ & \circ & \circ & \cdots & \circ & \ast & \ast \\
n_2{-}1 & \circ & \circ & \circ & \cdots & \circ & \circ & \bullet \\
n_2{-}2 & \circ & \circ & \circ & \cdots & \circ & \circ & \bullet \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
2 & \circ & \circ & \circ & \cdots & \circ & \circ & \bullet \\
1 & \circ & \circ & \circ & \cdots & \circ & \circ & \bullet \\
0 & \bullet & \bullet & \bullet & \cdots & \bullet & \ast & \ast \\
 & 0 & 1 & 2 & \cdots & (n_1 -2) & (n_1 -1) & n_1
\end{array}
$$

(d)

Figure 3: Bases of Example 3.1. Elements of I_0 are denoted by $\ast$'s while elements of I_j's are denoted by $\bullet$'s.

$$v_J = P\left(\left(\bigcap_{j\in J} A_j\right) \cap \left(\bigcap_{j\notin J} \overline{A_J}\right)\right),$$

$$p_I = P\left(\bigcap_{j\in I} A_j\right)$$

for any $I, J \subset \{1,\dots,s\}$. Then we have the equation

$$\sum_{J\subset\{1,\dots s\}} a_{IJ} v_J = p_I,\ I \subset \{1,\dots,s\}.$$

We formulate the following LP:

$$\begin{aligned}
& \min(\max) \sum_{\emptyset\neq J\subset\{1,\dots,s\}} x_J \\
\text{subject to} & \\
& \sum_{J\subset\{1,\dots s\}} a_{IJ} x_J = p_I,\ I \subset \{1,\dots,s\} \\
& \text{for } |I| \le m, \\
& x_j \ge 0, J \subset \{1,\dots,s\}.
\end{aligned} \tag{4.2}$$

Problem (4.2) can be reformulated as an MDMP. Consider the event sequence $A_1,\dots,A_s$ and define the random vector $\boldsymbol{X} = (X_1,\dots,X_s)$ such that X_j is the characteristic random variable of event A_j, $j = 1,\dots,s$, i.e.

$$X_j = \begin{cases} 1 \text{ if } A_j \text{ occurs,} \\ 0 \text{ otherwise.} \end{cases}$$

Let us define $f(z_1,\dots,z_s)$, $(z_1,\dots,z_s) \in Z = \{0,1\} \times \cdots \times \{0,1\}$ in the following way:

$$f(z_1,\dots,z_s) = \begin{cases} 0 \text{ if } (z_1,\dots,z_s) = (0,\dots,0), \\ 1 \text{ otherwise.} \end{cases} \tag{4.3}$$

It is easy to check that all divided differences of any order of the function (4.3) are nonnegative.

The equvivalent MDMP is the following:

$$\begin{aligned}
& \min(\max) \sum_{i_1=0}^{1} \cdots \sum_{i_s=0}^{1} f_{i_1\dots i_s} p_{i_1\dots i_s} \\
\text{subject to} & \\
& \sum_{i_1=0}^{1} \cdots \sum_{i_s=0}^{1} z_{1i_1}^{\alpha_1} \cdots z_{si_s}^{\alpha_s} p_{i_1\dots i_s} = \mu_{\alpha_1\dots\alpha_s} \\
& \text{for } \alpha_j = 0,1;\ j = 1,\dots,s;\ \alpha_1 + \cdots + \alpha_s \le m \\
& p_{i_1\dots i_s} \ge 0,\ \text{all } i_1,\dots,i_s.
\end{aligned} \tag{4.4}$$

We can see that the objective function is indeed the probability of the union of the events while the constraints are the same as in (4.2)

Now, let us consider problem (4.4) with an arbitrary function $f(z_1,\dots,z_s)$, defined on $(z_1,\dots,z_s)\in\{0,1\}\times\{0,1\}$. Problem (4.2) can be rewritten in a more compact form:

$$\begin{array}{rrcl} & \min(\max) & \boldsymbol{f}^T\boldsymbol{p} & \\ \text{subject to} & & & \\ & \breve{A}\boldsymbol{p} & = & \breve{\boldsymbol{b}} \\ & \boldsymbol{p} & \geq & \mathbf{0}. \end{array} \tag{4.5}$$

Let us define the subscript set

$$I=\{(i_1,\dots,i_s)|\ 0\le i_j\le 1,\ \text{integers},\ j=1,\dots,s,\ i_1+\ldots+i_s\le m\}. \tag{4.6}$$

Corresponding to the points Z_I we assign the Lagrange polynomial, given by its Newton's form

$$\begin{aligned} & L_I(z_1,\dots,z_s) \\ &= \sum_{\substack{i_1+\ldots+i_s\le m\\ 0\le i_j\le 1,\ j=1,\dots,s}} [Z_{1i_1};\cdots;Z_{si_s};f]\prod_{j=1}^{s}\prod_{k=0}^{i_j-1}(z_j-z_{jk}), \\ & \text{where, by definition, } \prod_{k=0}^{i_j-1}(z_j-z_{jk})=1,\ \text{for } i_j=0. \end{aligned} \tag{4.7}$$

The residual function is defined as:

$$\begin{aligned} & R_I(z_1,\dots,z_s) \\ &= \sum_{h=1}^{s}\Bigg(\sum_{\substack{0+i_{h+1}\cdots+i_s=m\\ 0\le i_j\le 1,\ j=(h+1),\dots,s}} \Big[z_1;\cdots;z_{h-1};Z'_{h0};Z_{(h+1)i_{h+1}};\cdots;Z_{si_s};f\Big](z_h-z_{h0}) \\ &\qquad \times\prod_{h+1}^{s}\prod_{k=0}^{i_j-1}(z_j-z_{jk}) \\ &\quad + \sum_{\substack{1+i_{h+1}+\cdots+i_s\le m\\ 0\le i_j\le 1,\ j=(h+1),\dots,s}} \Big[z_1;\cdots;z_{h-1};Z'_{h1};Z_{(h+1)i_{h+1}};\cdots;Z_{si_s};f\Big]\prod_{l=0}^{1}(z_h-z_{hl}) \\ &\qquad \times\prod_{h+1}^{s}\prod_{k=0}^{i_j-1}(z_j-z_{jk})\Bigg). \end{aligned} \tag{4.8}$$

Theorem 4.1 *Consider the Lagrange polynomial (4.7), corresponding to the points Z_I. For any $\boldsymbol{z}=(z_1,\dots,z_s)$ for which the function f is defined, we have the equality*

$$L_I(z_1,\dots,z_s)+R_I(z_1,\dots,z_s)=f(z_1,\dots,z_s). \tag{4.9}$$

Proof. The assertion can be proved similarly as the proof of Theorem 4.1 in Prékopa (1998).

Theorem 4.2 *Let* $0 = z_{j0} < z_{j1} = 1,\ j = 1, \ldots, s$. *Suppose that the function* $f(\boldsymbol{z}),\ \boldsymbol{z} \in Z$ *has nonnegative mixed divided differences of total order* $m+1$.

Under these conditions $L_I(z_1, \ldots, z_s)$, *defined by (4.7), is a unique suitable* H*-type Lagrange polynomial on* Z_I *and satisfies the relations*

$$f(z_1, \ldots, z_s) \geq L_I(z_1, \ldots, z_s),\ (z_1, \ldots, z_s) \in Z, \tag{4.10}$$

i.e., the set of columns $\breve{B}$ *of* $\breve{A}$ *in problem (4.5), with the subscript set* I, *is a dual feasible basis in the minimization problem (4.5), and*

$$E[f(X_1, \ldots, X_s)] \geq E[L_I(X_1, \ldots, X_s)]. \tag{4.11}$$

If $\breve{B}$ *is also a primal feasible basis in problem (4.5), then the inequality (4.11) is sharp.*

If all the above mentioned divided differences are nonpositive, then (4.10) and (4.11) hold with reversed inequality signs.

Proof. Similar to the proof of Theorem 2.2. □

Theorem 4.3 *Let* $1 = z_{j0} > z_{j1} = 0,\ j = 1, \ldots, s$. *Suppose that the function* $f(\boldsymbol{z}),\ \boldsymbol{z} \in Z$ *has nonnegative mixed divided differences of total order* $m+1$. *Under these conditions we have the following assertions:*

(a) If $m+1$ *is even, then the Lagrange polynomial* $L_I(z_1, \ldots, z_s)$, *defined by (4.7), satisfies*

$$f(z_1, \ldots, z_s) \geq L_I(z_1, \ldots, z_s),\ (z_1, \ldots, z_s) \in Z, \tag{4.12}$$

i.e., the set of columns $\breve{B}$ *in* $\breve{A}$, *corresponding to the subscripts* I, *is a dual feasible basis in the minimization problem (4.5). We also have the inequality*

$$E[f(X_1, \ldots, X_s)] \geq E[L_I(X_1, \ldots, X_s)]. \tag{4.13}$$

If $\breve{B}$ *is also a primal feasible basis in the LP (4.5), then the lower bound (4.13) for* $E[f(X_1, \ldots, X_s)]$ *is sharp.*

(b) If $m+1$ *is odd, then the Lagrange polynomial, defined by (4.7), satisfies*

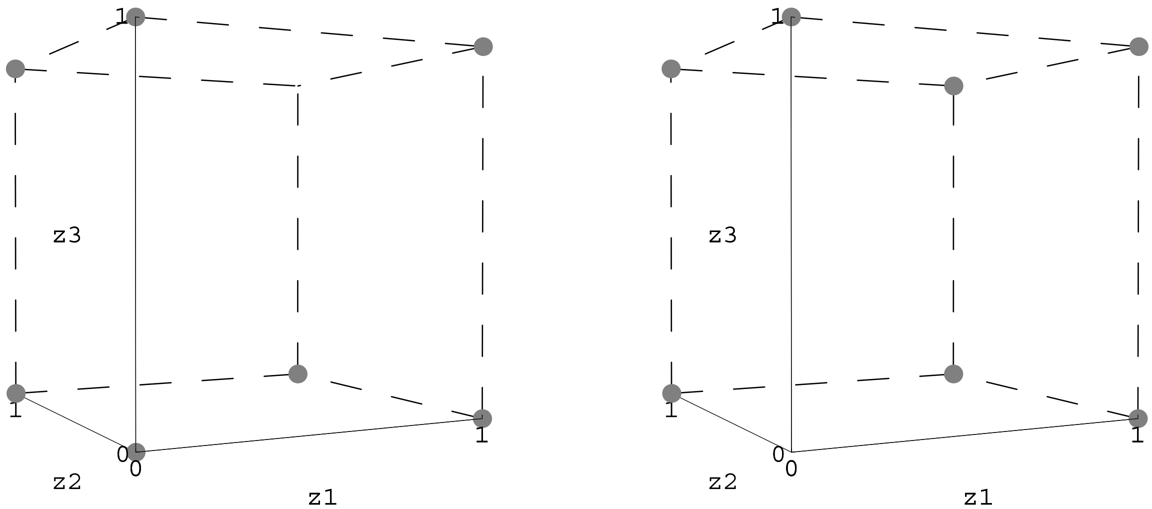
$$f(z_1, \ldots, z_s) \leq L_I(z_1, \ldots, z_s),\ (z_1, \ldots, z_s) \in Z, \tag{4.14}$$

i.e., the basis $\breve{B}$ *is dual feasible in the maximization problem (4.5). We also have the inequality*

$$E[f(X_1, \ldots, X_s)] \leq E[L_I(X_1, \ldots, X_s)]. \tag{4.15}$$

If $\breve{B}$ *is also a primal feasible basis in the LP (4.5), then the upper bound (4.15) for* $E[f(X_1, \ldots, X_s)]$ *is sharp.*

If all the above mentioned divided differences are nonpositive, then (4.12),(4.13),(4.14) and (4.15) hold with reversed inequality signs.



(a) (b)

Figure 4: Dual feasible bases of Theorems 4.2 (figure (a)) and 4.3 (figure (b)), where $m = 2$, $s = 3$

Proof. Similar to the proof of Theorem 2.3. □

The dual feasible bases that appear in Theorems 4.2 and 4.3 are illustrated in Figure 4, for the three-dimensional case.

Function (4.3) has all nonnegative divided differences. It follows from this, that the bases in Theorems 4.2 and 4.3 are dual feasible in problem (4.4), with that objective function. This also means that we found dual feasible bases to the disaggregated problem (4.2), which can serve for bounding the probability of the union of events.

If we want to create bounds for the probability of the intersection, i.e., for $P(A_1 \cap \cdots \cap A_n)$, then we work with the same constraints in the MDMP (4.4) but in this case the objective function is defined as:

$$f(z_1, \ldots, z_s) = \begin{cases} 1 \text{ if } (z_1, \ldots, z_s) = (1, \ldots, 1), \\ 0 \text{ otherwise.} \end{cases} \tag{4.16}$$

If $m + 1$ is even (odd) then all divided differences of the function (4.16) of total order $m + 1$ are nonpositive (nonnegative). Using this we can construct dual feasible bases for the new MDMP (4.4), where the objective function is given by (4.16).

Example 4.1 *The following example is taken from Kuai, Alajaji and Takahara (2000) and Prékopa and Gao (2005). There are* 6 *events:* $A_1, \ldots, A_6$*,* 15 *possible outcomes with proba-*

bilities given in the table below.

Outcomes x	$p(x)$	A_1	A_2	A_3	A_4	A_5	A_6
x_0	0.012	$\times$		$\times$		$\times$	
x_1	0.022		$\times$		$\times$		
x_2	0.023	$\times$		$\times$		$\times$	
x_3	0.033		$\times$				
x_4	0.034	$\times$				$\times$	$\times$
x_5	0.044		$\times$	$\times$		$\times$	
x_6	0.045		$\times$			$\times$	$\times$
x_7	0.055			$\times$	$\times$		$\times$
x_8	0.056	$\times$		$\times$			
x_9	0.066				$\times$	$\times$	
x_{10}	0.067		$\times$		$\times$	$\times$	
x_{11}	0.077		$\times$				
x_{12}	0.078	$\times$			$\times$		$\times$
x_{13}	0.088		$\times$				
x_{14}	0.089	$\times$		$\times$		$\times$	$\times$

We give sharp lower and upper bounds for the probablitity of the union of the events, using the information of probabilities of certain intersections of A_i*'s. I.e., we solve problem (4.4) with the function (4.3). We use the bases of Theorems 4.2 and 4.3 as an initial bases of the dual method of CPLEX. The results depending on the parameter* m *of (4.4) are:*

m	*Minimum*	*Iteration*	*Maximum*	*Iteration*
2	0.789	13	0.955	27
3	0.789	11	0.789	20
4	0.789	0	0.789	12
5	0.789	0	0.789	1

Kuai, Alajaji and Takahara (2000) gives 0.7222 *as a lower bound, using probabilities of the single events and intersections of pairs events. Prékopa and Gao (2005) gives* 0.73145 *as a lower bound and* 0.8038333 *as an upper bound. They use intersections up to three events. Bounds given by formulas of these probabilities also can be found in Bukszár and Prékopa (2001).*

The aim of the example is to show that the known dual feasible bases of the connected MDMP can give an initial basis of the dual method of linear programming. That usually means less numerical difficulty and running time.

5 Bounding Multivariate Moment Generating Functions

Let X be a random variable taking values in a subset of $\mathbb{R}$. The moment generating function of X is the function M defined by

$$M(t) = E[e^{tX}],\ t \in \mathbb{R}.$$

The moment generating function shares many of the important properties. E.g., if the $M(t)$ is finite for t in an open interval J about 0, then M completely determines the distribution of X. On the other hand, M has derivatives of all orders in J and $M^{(n)}(t) = E[X^n e^{tX}],\ t \in J$. That means that $M^{(n)}(0) = E[X^n], n = 1, 2, \ldots$.

The joint moment generating function is defined as

$$M(t_1, \ldots, t_s) = E[e^{t_1 X_1 + \cdots + t_s X_s}]$$

If the joint moment generating function is finite in an open neighborhood of the origin then this function completely determines the distribution of $\boldsymbol{X} = (X_1, \ldots, X_s)$. Other interesting properties are, e.g., $M(0, \ldots, 0, t_i, 0, \ldots, 0) = M_i(t_i)$,

$$\frac{\partial^{\alpha_1 + \cdots \alpha_s} M}{\partial {t_1}^{\alpha_1} \cdots \partial {t_s}^{\alpha_s}}(0, \ldots, 0) = \mu_{\alpha_1 \ldots \alpha_s}.$$

More details about (joint) generating function can be found e.g., in S. M. Ross (2002).

If we assume that the random vector $\boldsymbol{X}$ has a finite support, then we can use MDMP to bounding the value of the joint moment generating function for certain values of $(t_1, \ldots, t_s)$ in terms of the (mixed) power moments of $\boldsymbol{X}$. Recently, e.g., Ibrahim and Mugdadi (2005) gave bounds of (univariate) moment generating functions by the aid of the moments.

For any fixed $(t_1, \ldots, t_s) \geq \mathbf{0}$ all divided differences of the function $e^{t_1 z_1 + \cdots + t_s z_s}$ are nonnegative . It is also true that the $m + 1$st divided differences are nonnegative (nonpositive) for $(t_1, \ldots, t_s) \leq \mathbf{0}$ if $m + 1$ is even (odd). In these cases the methods of Section 2 can be applied as it shown in the following

Example 5.1 *We shall give lower and upper bounds for the joint generating function $M(t_1, t_2)$, where $t_1 = 0.04$ and $t_2 = 0.05$, by the use of the Min and Max Algorithms of Section 2. We use programs written in Wolfram's Mathematica. Let the random variables X_1, X_2 has uniform distribution on the supports $Z_1 = Z_2 = \{0, \ldots, 14\}$. Regarding the mixed moments will be taken into account, we are generating them by the multivariate unique discrete distribution on Z.*

We are computing different problems according to the maximum order of the mixed moments taken into account.

Considering the given marginal distributions and mixed moments up to the order m, we obtained the following results:

m	*Lower*	*CPU*	*Upper*	*CPU*
2	1.91194	0.28	1.98564	0.28
3	1.94560	0.58	1.95640	0.56
4	1.95009	1.18	1.95108	1.19
5	1.95051	2.59	1.95060	2.59
6	1.95053	6.11	1.95056	6.13

In the example above we were enable to carry out the dual method by the use of CPLEX. In most cases it reported infeasibility of the primal problem, even though the moments that have used in the problem allow for feasibility, by construction. This stresses the importance of the Min and Max Algorithms of Section 2.

6 Bounding Expected Utilities

The most general definition of a von Neumann-Morgenstern type utility function $u(z),\ z \geq 0$ only requires that it should be an increasing function, i.e., $u'(z) > 0$. It is called risk averse, if we also have $u''(z) < 0$ which means that the function is also concave.

More generally, we may require:

$$(-1)^{n-1} u^{(n)}(z) > 0,\ n = 1, 2, \ldots. \tag{6.1}$$

Utility functions satisfying (6.1) are called *mixed* by Caballe and Pomansky (1996). For economic justification see Ingersoll (1987). Relation (6.1) means that $u(-z)$ is a *completely monotone function.* Examples of mixed utility functions are:

$$u(z) = a \log\left(1 + \frac{z}{b}\right),\ u(z) = -ae^{-bz},$$

where $a > 0$, $b > 0$.

In multiattribute utility theory (MAU) the well-known multiplicative form of Keeney and Raiffa (1976) is the following:

$$Ku(z_1, \ldots, z_s) + 1 = \prod_{i=1}^{s} (Kk_i u_i(z_i) + 1) \tag{6.2}$$

with $K \neq 0$. (The case $K = 0$ leads to a weighted addititive from.)

The risk averse multiattribute utility function may be defined in such a way that $u(z_1, \ldots, z_s)$ is increasing in each variable and concave as an s-variate function.

In addition, we may require

$$(-1)^{i_1+\cdots+i_s-1} \frac{\partial^{i_1+\cdots+i_s} u(z_1, \ldots, z_s)}{\partial z_1^{i_1} \cdots \partial z_s^{i_s}} > 0,\ 1 \leq i_1 + \cdots + i_s. \tag{6.3}$$

This is a multivariate counterpart of relations (6.1).

These properties are usually not true for functions (6.2). However, it is easy to see that the following is valid for (6.2) in case of $s = 2$:

$$(-1)^{i_1+i_2} \frac{\partial^{i_1+i_2} u(z_1, z_2)}{\partial z_1^{i_1} \partial z_2^{i_2}} > 0, \; 1 \leq i_1 \text{ and } 1 \leq i_2, \tag{6.4}$$

assuming that u_1 and u_2 are mixed utility functions with property (6.1).

A class of multiattribute utility functions which fulfills the concavity as well as property (6.3) is given in Prékopa and Mádi-Nagy (2006) by

Definition 6.1 *Let $k \geq 1$ and D an open convex set. We define the utility function u as:*

$$u(z_1, \ldots, z_s) := log \left[k(e^{g_1(z_1)} - 1) \cdots (e^{g_s(z_s)} - 1) - 1 \right], \tag{6.5}$$

where for every $(z_1, \ldots, z_n) \in D$ the following conditions hold:

$$e^{g_j(z_j)} > 2, \; j = 1, \ldots, s, \tag{6.6}$$

$$\begin{array}{rcl} g_j'(z_j) & > & 0 \\ g_j^{(i)}(z_j) & \geq & 0, \; \textit{if } i > 1 \textit{ and is odd} \\ g_j^{(i)}(z_j) & \leq & 0, \; \textit{if } i \textit{ is even} \\ & & j = 1, \ldots, s. \end{array} \tag{6.7}$$

Let $\boldsymbol{X} = (X_1, \ldots, X_s)$ be a random vector where the support of X_j is a known finite set $Z_j = \{z_{j0}, \ldots, z_{jn_j}\}$. Assume that the marginal distributions and the collection of mixed moments $\mu_{\alpha_1 \ldots \alpha_s}$, $\alpha_1 + \cdots \alpha_s \leq m$ are known, and we would like to bounding the expected utility

$$E[u(X_1, \ldots, X_s)].$$

This means exactly the problem (1.2) (and the equivalent MDMP form). If additionaly property (6.3) or (6.4) holds, then we are able to apply the methods of Section 2 as we show in the following examples.

Example 6.1 *Let the random variables X_1, X_2 has uniform distribution on the supports $Z_1 = Z_2 = \{0, 0.1, 0.2, \ldots, 1\}$. Regarding the mixed moments will be taken into account, we are generating them by the multivariate unique discrete distribution on Z.*

Consider the following univariate mixed utility functions:

$$u_1(z_1) = \frac{\log(1 + z_1)}{\log 2},$$

$$u_2(z_2) = \frac{1 - e^{-z_2}}{1 - \frac{1}{e}}.$$

The multiattribute utility function composed by the multiplicative form is the following:

$$u(z_1, z_2) = k_1 u_1(z_1) + k_2 u_2(z_2) + K k_1 k_2 u_1(z_1) u_2(z_2). \tag{6.8}$$

We are bounding the expected utilitity

$$E[u(X_1, X_2)]$$

in case of $k_1 = 0.3$ *and* $k_2 = 0.2$ *(*$K = 1 - k_1 - k_2$*), where the marginal distributions and the mixed moments up to mth order are known. The function* $u(z_1, z_2)$ *fulfills (6.4), hence lower and upper bounds can be given by the use of the Min and Max Algorithms of Section 2. We use programs written in Wolfram's Mathematica. The results are:*

m	*Lower*	*CPU*	*Upper*	*CPU*
2	0.405218	0.08	0.467877	0.08
3	0.432214	0.16	0.445082	0.16
4	0.437435	0.36	0.439693	0.36
5	0.438281	0.86	0.438624	0.86
6	0.438418	2.30	0.438473	2.28

Example 6.2 *Let* $Z_1 = Z_2 = \{0, \ldots, 19\}$*. Consider the* X, Y_1, Y_2 *random variables having Poisson distributions with* λ *parameters* $3, 4, 5$*, respectively. We generate the moments of the random vector*

$$(X_1, X_2) := (min(X + Y_1, 19), min(X + Y_2, 19))$$

Note that X_1, X_2, X_3 *are stochastically dependent.*

Considering the bivariate utility function

$$u(z_1, z_2) = log[(e^{\alpha z_1 + a} - 1)(e^{\beta z_2 + b} - 1) - 1], \tag{6.9}$$

defined for

$$e^{\alpha z_1 + a} > 2, \ e^{\beta z_2 + b} > 2.$$

The function (6.9) is a special case of (6.5), hence satisfies (6.3). This means that we can apply the Min and Max Algorithms of Section 2 in order to bound the expected utility. Let $\alpha = 0.75$, $\beta = 1.25$, $a = 2$, $b = 3$*. We get the following results:*

m	*Lower*	*CPU*	*Upper*	*CPU*
2	20.2456980	0.88	20.2458790	0.81
3	20.2456982	1.75	20.2458790	1.70
4	20.2456989	3.45	20.2458790	3.45
5	20.2457012	2.63	20.2458790	7.20
6	20.2457066	15.44	20.2458790	15.47

Example 6.3 *We calculate the expected value of the following utility function*

$$u(z_1, z_2, z_3) = log\left[(e^{\alpha_1 z_1 + a_1} - 1)(e^{\alpha_2 z_2 + a_2} - 1)(e^{\alpha_3 z_3 + a_3} - 1) - 1\right] \quad (z_1, z_2, z_3) \in Z, \tag{6.10}$$

where

$$Z = (0, 1, 2, 3, 4, 5, 6, 7, 8, 9) \times (0, 1, 2, 3, 4, 5, 6, 7, 8, 9) \times (0, 1, 2, 3, 4, 5, 6, 7, 8, 9)$$

with parameters $\alpha_1 = \alpha_2 = \alpha_3 = a_1 = a_2 = a_3 = 1$. It is easy to see that the function is a special case of (6.5).

Assume that X_1, X_2, X_3 are independent and each one has uniform distribution on $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. The moments and marginal distributions presented in the following are those of the random variables X_1, X_2, X_3.

We use the dual method of CPLEX with the initial dual feasible bases of Section 2. Carrying out the dual method for the problems below without these initial bases CPLEX reports infeasility. This means that the use of these bases as initial bases in the dual method can give not only shorter running time, but it can reduce the numerical difficulties, as well.

The results are summarized below.

m	*Minimum*	*Iteration*	*Maximum*	*Iteration*
2	16.272644221	375	16.294708615	65
3	16.279932313	428	16.294702240	515
4	16.288384112	779	16.294688894	391
5	16.290690088	1121	16.294643748	1326
6	16.292421158	1605	16.294587574	1198

7 Conclusions

We have shown how the MDMP technique for bounding functions of random variables can efficiently be used to some special bounding problems. In these problems the univariate marginals and moments of total order up to m are assumed to be known. We have obtained results not mentioned in Hou and Prékopa (2006), for bounding expectations of Monge arrays (as functions) of random variables. We have presented an efficient method for bounding pseudo Boolean functions of binary random variables, where function has a special monotonicity property and known are the joint distributions of up to m random variables. In two further examples bounds for multivariate generating functions and for the expectation of a special utility function. Sometimes the bounds are given in terms of formulas, sometimes in terms of algorithms. In the latter case the dual algorithm of linear programming is adapted for the problems at hand.

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