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Ersoy Subasi[a] Mine Subasi[b] András Prékopa[c]

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RUTCOR
Rutgers Center for
Operations Research
Rutgers University
640 Bartholomew Road
Piscataway, New Jersey
08854-8003
Telephone: 732-445-3804
Telefax: 732-445-5472
Email: rrr@rutcor.rutgers.edu
http://rutcor.rutgers.edu/∼rrr

[a]RUTCOR, Rutgers Center for Operations Research, 640 Bartholomew Road Piscataway, NJ 08854-8003, USA. Email: esub@rutcor.rutgers.edu
[b]RUTCOR, Rutgers Center for Operations Research, 640 Bartholomew Road Piscataway, NJ 08854-8003, USA. Email: msub@rutcor.rutgers.edu
[c]RUTCOR, Rutgers Center for Operations Research, 640 Bartholomew Road Piscataway, NJ 08854-8003, USA. Email: prekopa@rutcor.rutgers.edu

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Discrete Moment Problems with Distributions Known to be Unimodal

Ersoy Subasi   Mine Subasi   András Prékopa

Abstract. Discrete moment problems with finite, given supports and distributions known to be unimodal, are formulated and used to obtain sharp lower and upper bounds for expectations of higher order convex functions of discrete random variables as well as probabilities of the union of events. The bounds are based on the knowledge of some of the power moments of the random variables involved, or the binomial moments of the number of events which occur. The bounding problems are formulated as LP's and dual feasible basis structure theorems as well as the application of the dual method of linear programming provide us with the results. Applications in PERT and reliability are presented.

1 Introduction

Let ξ be a random variable, the possible values of which are known to be the nonnegative numbers $z_0 < z_1 < ... < z_n$. Let $p_i = P(\xi = z_i), \quad i = 0, 1, ..., n$. Suppose that these probabilities are unknown but either the power moments $\mu_k = E(\xi^k), \ k = 1, ..., m$ or the binomial moments $S_k = E\left[\binom{\xi}{k}\right], \ k = 1, ..., m$, where $m < n$, are known.

The starting points of our investigation are the following linear programming problems

$$\min(\max) \sum_{i=0}^{n} f(z_i) p_i$$

subject to

$$\sum_{i=0}^{n} z_i^k p_i = \mu_k \ , \ k = 0, 1, ..., m \tag{1.1}$$

$$p_i \geq 0 \ , \ i = 0, 1, ..., n$$

and

$$\min(\max) \sum_{i=0}^{n} f(z_i) p_i$$

subject to

$$\sum_{i=0}^{n} \binom{z_i}{k} p_i = S_k \ , \ k = 0, 1, ..., m \tag{1.2}$$

$$p_i \geq 0 \ , \ i = 0, 1, .., n$$

where $\mu_0 = S_0 = 1$.

Problems (1.1) and (1.2) are called the power and binomial moment problems, respectively and have been studied extensively in [11, 12, 13, 14, 2]. The two problems can be transformed into each other by the use of a simple linear transformation (see [15], Section 5.6).

Note that if the binomial moment problem has feasible solution, then there exists a probability space and events $A_1, ..., A_n$ such that $S_1, ..., S_m$ are their binomial moments. In fact, we can take, as sample space, the set of all n-vectors with 0-1 components, form a $2^n \times n$ matrix with them and define A_i as the set of those rows of the matrix which have 1's in the ith column, $i = 1, ..., n$. Then, assign p_k as probability, to the set of those rows in which the number of 1's is k, further, split p_k arbitrarily among the elements within that set, $k = 1, ..., n$. The obtained events have the required property.

In this paper we specialize problems (1.1) and (1.2) in the following manner. We will alternatively use the notation f_k instead of $f(z_k)$.

(1) In case of problem (1.1) we assume that the function f has positive divided differences of order $m+1$, where m is some fixed nonnegative integer satisfying $0 \leq m \leq n$. The two optimum values of problem (1.1) provide us with sharp lower and upper bounds for $E[f(\xi)]$.

(2) In case of problem (1.2) we assume that $z_i = i, \ i = 0, ..., n$ and $f_0 = 0, \ f_i = 1, \ i = 1, ..., n$. The problem can be used in connection with arbitrary events $A_1, ..., A_n$, to obtain sharp lower and upper bounds for the probability of the union. In fact, if we define

$$S_k = \sum_{1 \leq i_1 < ... < i_k \leq n} P(A_{i_1} ... A_{i_k}), \quad k = 1, ..., n,$$

then by a well-known theorem (see, e.g., [15]) we have the equation

$$S_k = E\left[\binom{\xi}{k}\right], \quad k = 1, ..., n, \tag{1.3}$$

where ξ is the number of those events which occur. The equality constraints in (1.2) for $k = 1, ..., m$ are just the same as the equations in (1.3) for $k = 1, ..., m$ and the objective function is the probability of $\xi \geq 1$ under the distribution $p_0, ..., p_n$. The distribution, however, is allowed to vary subject to the constraints, hence the two optimum values of problem (1.2) provide us with the best possible lower and upper bounds for the probability $P(\xi \geq 1)$, given $S_1, ..., S_m$.

For small m values ($m \leq 4$) closed form bounds are presented in the literature. For power moment bounds see [14, 15]. Bounds for the probability of the union have been obtained by Fréchet [4], when $m = 1$, Dawson and Sankoff [3], when $m = 2$, Kwerel [9], when $m \leq 3$, Boros and Prékopa [2], when $m \leq 4$. In the last two papers bounds for the probability that at least r events occur, are also presented. We call a probability bound of order m if joint probabilities of m events, but not more than m events are used in it. For other closed form probability bounds see [6, 7, 15]. In [11, 12, 13, 14] Prékopa showed that the probability bounds based on the binomial and power moments of the number of events that occur, out of a given collection $A_1, ..., A_n$, can be obtained as optimum values of discrete moment problems (DMP). He also showed that for arbitrary m values simple dual algorithms solve problems (1.1) and (1.2) if f is of type (1) or (2) (and even for other objective function types).

In this paper we formulate and use moment problems with given finite supports and with unimodal distributions to obtain sharp lower and upper bounds for expectations of higher order convex functions of discrete random variables and for the probability that at least one out of n events occurs. We assume that the probability distribution $\{p_i\}$ is unimodal with a known modus (Type 1) and pay attention to the special cases when it is increasing (Type 2) or decreasing (Type 3). The reasoning goes along the lines presented in the above cited papers by Prékopa.

In Section 2 some basic notions and theorems are given. In Section 3 we use the dual feasible basis structure theorems in [12, 14] to obtain sharp bounds for $E[f(\xi)]$ in case of problems, where the first or the first two moments are known. In Section 4 we present a dual feasible basis structure theorem and give closed form bounds for $P(\xi \geq 1)$ in case of problems, where the first two moments are known. In Section 5 we give numerical examples to compare the bounds obtained by the binomial moment problem without shape information, and the bounds obtained by the problems: Type 1, Type 2 and Type 3. The results show that the use of the shape constraint significantly improves on the bounds. In Section 6 we present two examples for the application of our bounding technique.

2 Basic Notions and Theorems

Let f be a function on the discrete set $Z = \{z_0, ..., z_n\}, \quad z_0 < z_1 < ... < z_n$. The first order divided differences of f are defined by

$$[z_i, z_{i+1}]f = \frac{f(z_{i+1}) - f(z_i)}{z_{i+1} - z_i}, \qquad i = 0, 1, ..., n-1.$$

The kth order divided differences are defined recursively by

$$[z_i, ..., z_{i+k}]f = \frac{[z_{i+1}, ..., z_{i+k}]f - [z_i, ..., z_{i+k-1}]f}{z_{i+1} - z_i}, \qquad k \geq 2.$$

The function f is said to be kth order convex if all of its kth order divided differences are positive. For further information about divided differences see [8, 14].

The following two results are due to Prékopa [12, 14].

Theorem 1. *Suppose that all (m+1)st order divided differences of the function $f(z)$, $z \in \{z_0, z_1, ..., z_n\}$ are positive. Then, in problems* (1.1) *and* (1.2) *all bases are dual nondegenerate and the dual feasible bases have the following structures, presented in terms of the subscripts of the basic vectors:*

	$m+1$ *even*	$m+1$ *odd*
min problem	$\{j, j+1, ..., k, k+1\}$	$\{0, j, j+1, ..., k, k+1\}$
max problem	$\{0, j, j+1, ..., k, k+1, n\}$	$\{j, j+1, ..., k, k+1, n\}$

where in all parentheses the numbers are arranged in increasing order.

Consider the following problem

$$\begin{aligned} &min(max) \;\; c^T x \\ &\text{subject to} \\ &Ax = b \; , \quad x \geq 0 \end{aligned} \tag{2.1}$$

where A is an $m \times n$ matrix, c and x are n-component vectors and b is an m-component vector, where $m < n$.

Theorem 2. *Suppose that all minors of order* m *from* A *and all minors of order* $m+1$ *from* $\begin{pmatrix} A \\ c^T \end{pmatrix}$ *are positive. Then, the assertions of Theorem* 1 *hold true.*

Remark. The assumptions in Theorem 12.2 and 12.3 in [12] are slightly different than the ones in Theorem 2, but from the proofs it is obvious that the above theorem holds true.

3 The Case of the Power Moment Problem

In this section we consider the power moment problem (1.1) for the cases of $m = 1, 2$. We give lower and upper bound formulas for $E[f(\xi)]$ for three problem types: the sequence of probabilities $p_0, ..., p_n$ is (1) unimodal with a known modus, (2) increasing, (3) decreasing.

3.1 TYPE 1: The Case of a Unimodal Distribution

We assume that the distribution is unimodal with a known modus z_k, $1 < k < n$, i.e., $p_0 \leq ... \leq p_{k-1} \leq p_k \geq p_{k+1} \geq ... \geq p_n$. We also assume that f has positive divided differences of order $m+1$.

First, we remark that there are two possible representations of problem (1.1) with the shape constraint. In the first one, that we call *the forward representation*, we introduce the variables $v_i,\ i = 0, 1, ..., n$ and obtain:

$$p_0 = v_0\ ,\quad p_1 = v_0 + v_1\ ,\ ...\ ,\quad p_k = v_0 + ... + v_k$$

$$p_{k+1} = v_{k+1} + ... + v_n\ ,\ p_{k+2} = v_{k+2} + ... + v_n\ ,\ ...\ ,\quad p_n = v_n\ . \tag{3.1}$$

In the second one, that we call *the backward representation*, only the representation of p_k is different and it is: $p_k = v_k + ... + v_n$.

The forward representation of problem (1.1), with the additional information regarding $p_0, ..., p_n$, is the following:

$$min(max) \sum_{i=0}^{k} (f_i + ... + f_k) v_i + \sum_{i=k+1}^{n} (f_{k+1} + ... + f_i) v_i$$

subject to

$$\sum_{i=0}^{k} (a_i + ... + a_k) v_i + \sum_{i=k+1}^{n} (a_{k+1} + ... + a_i) v_i = b \tag{3.2}$$

$$v_0 + ... + v_k - v_{k+1} - ... - v_n \geq 0 \tag{3.2a}$$

$$v_i \geq 0\ ,\quad i = 0, 1, ..., n\ .$$

In the backward representation the problem is given as follows:

$$min(max) \sum_{i=0}^{k-1} (f_i + ... + f_{k-1})v_i + \sum_{i=k}^{n} (f_k + ... + f_i)v_i$$

subject to

$$\sum_{i=0}^{k-1} (a_i + ... + a_{k-1})v_i + \sum_{i=k}^{n} (a_k + ... + a_i)v_i = b \tag{3.3}$$

$$v_k + ... + v_n - v_0 - ... - v_{k-1} \geq 0 \tag{3.3a}$$

$$v_i \geq 0 \ , \quad i = 0, 1, ..., n \ .$$

The general method that we can apply to solve problem (3.2) (or (3.3)) is the following. Relax the problem by removing the constraint (3.2a) (or (3.3a)) and solve the relaxed problem. If m is small, then the optimum values can be obtained in closed forms. Otherwise, the dual method of linear programming, presented in [14, Section 4], solve the problem. In both cases primal-dual feasible bases are obtained. Now, looking at problem (3.2) (or (3.3)) prescribe (3.2a) (or (3.3a)) as additional constraint and use the dual method to reoptimize the problem. We also remark that if we obtain $p_k \geq p_{k+1}$ ($p_{k-1} \leq p_k$) in the optimal solutions of the first (second) relaxed problem, then reoptimization is not needed.

We do not have dual feasible basis structure theorems for problems (3.2) and (3.3) but we can derive one for the relaxed problems, i.e., for the problems without the constraints (3.2a) and (3.3a), respectively.

Note that if we designate the optimum values of problem (3.2) (or (3.3)) as min_{opt} and max_{opt} and the optimum values of the relaxed problem as min'_{opt} and max'_{opt}, then we have $min'_{opt} \leq min_{opt} \leq max_{opt} \leq max'_{opt}$.

Theorem 3. *If the constraints* (3.2a) *and* (3.3a) *are removed from problems* (3.2) *and* (3.3), *respectively, then the matrix $\tilde{A}$ of the equality constraints and the coefficient vector $\tilde{f}$ of the objective function satisfy the conditions of Theorem* 2.

Proof. We prove the assertion in case of problem (3.2). Take an $(m+2) \times (m+2)$ submatrix from $\begin{pmatrix} \tilde{A} \\ \tilde{f}^T \end{pmatrix}$. It may be entirely from the first k columns or from the last $n-k$ columns or in a mixed manner. In all cases we can apply a column subtraction procedure such that the resulting matrix has the following property: if $I_i = \{j \mid a_j$ is a term in the sum in the ith column$\}$, $i = 1, ..., m+2$, then for any pair $I_t,\ I_u,\ t < u$ we have that all elements in I_t are smaller than any of the elements in I_u. This implies that the determinant of resulting matrix splits into the sum of positive determinants since f has all positive divided differences of order $m+1$. The same column subtraction procedure can be applied to show that any $(m+1) \times (m+1)$ minor of $\tilde{A}$ is positive since $(a_0, a_1, ..., a_n)$ is a Vandermonde matrix.

The proof of the assertion in case of problem (3.3) can be done similarly. □

The bounds for $E[f(\xi)]$ ***in case of problem* (3.2)**

Below we present closed form bounds for the second relaxed problem, i.e., problem (3.2) without the additional constraint (3.2a), when $m = 1, 2$.

***Case* 1.** Let $m = 1$. Since $m + 1$ is even, by the use of Theorem 1, any dual feasible basis of the minimization problem, that we designate by B_{min}, is of the form

$$B_{min} = \{j, j+1\}, \quad 0 \le j \le n-1 \ .$$

Similarly, by Theorem 1, the only dual feasible basis of the maximization problem, designated by B_{max}, is obtained as

$$B_{max} = \{0, n\}.$$

Since B_{max} is the only dual feasible basis it must also be primal feasible (see, e.g.: [16]).

B_{min} is primal feasible if j satisfies the following condition:

$$\frac{\sum_{t=j}^{k} z_t}{k-j+1} \le \mu_1 \le \frac{\sum_{t=j+1}^{k} z_t}{k-j} \qquad \text{if} \quad j \le k-1 \ ; \tag{3.4}$$

$$\frac{\sum_{t=k+1}^{j} z_t}{j-k} \le \mu_1 \le \frac{\sum_{t=k+1}^{j+1} z_t}{j-k+1} \qquad \text{if} \quad j \ge k+1 \ ; \tag{3.5}$$

$$z_k \le \mu_1 \le z_{k+1} \qquad \text{if} \quad j = k \ . \tag{3.6}$$

Let us introduce the notations:

$$\alpha_{i,j}^2 = (n-j)\sum_{t=i}^{j} z_t^2 - (j-i+1)\sum_{t=j+1}^{n} z_t^2 \ , \quad \alpha_{i,j} = (n-j)\sum_{t=i}^{j} z_t - (j-i+1)\sum_{t=j+1}^{n} z_t \ ,$$

$$\Sigma_{i,j}^2 = i\sum_{t=i}^{j} z_t^2 - (j-i+1)\sum_{t=0}^{i-1} z_t^2 \ , \quad \Sigma_{i,j} = i\sum_{t=i}^{j} z_t - (j-i+1)\sum_{t=0}^{i-1} z_t \ ,$$

$$\sigma_{i,j}^2 = \sum_{t=i}^{j} z_t^2 - (j-i+1)z_{i-1}^2 \ , \quad \sigma_{i,j} = \sum_{t=i}^{j} z_t - (j-i+1)z_{i-1} \ , \tag{3.7}$$

$$\gamma_{i,j}^2 = \sum_{t=i}^{j} z_t - (j-i+1)z_{j+1}^2 \ , \quad \gamma_{i,j} = \sum_{t=i}^{j} z_t - (j-i+1)z_{j+1} \ .$$

The lower bound for $E[f(\xi)]$ is given as follows:

- If $j \le k-1$ and (3.4) is satisfied, then we have

$$\frac{\sum_{t=j+1}^{k}(f_j z_t - z_j f_t) - \mu_1\left[(k-j)f_j - \sum_{t=j+1}^{k} f_t\right]}{\sigma_{j+1,k}} \le E[f(\xi)] \ ; \tag{3.8}$$

- if $j \geq k+1$ and (3.5) is satisfied, then we have

$$\frac{\sum_{t=k+1}^{j}(f_{j+1}z_t - z_{j+1}f_t) - \mu_1\left[\sum_{t=k+1}^{j} f_t - (k-j)f_{j+1}\right]}{\gamma_{k+1,j}} \leq E[f(\xi)] \; ; \tag{3.9}$$

- if $j = k$ and (3.6) is satisfied, then we have

$$\frac{z_{k+1}-\mu_1}{z_{k+1}-z_k} f_k + \frac{\mu_1 - z_k}{z_{k+1}-z_k} f_{k+1} \leq E[f(\xi)] \; . \tag{3.10}$$

The upper bound for $E[f(\xi)]$ is the following:

$$E[f(\xi)] \leq \frac{\sum_{t=k+1}^{n} z_t - (n-k)\mu_1}{\Sigma_{k+1,n}} \sum_{t=0}^{k} f_t + \frac{(k+1)\mu_1 - \sum_{t=0}^{k} z_t}{\Sigma_{k+1,n}} \sum_{t=k+1}^{n} f_t \; . \tag{3.11}$$

Here $\sigma_{i,j},\ \gamma_{i,j},\ \Sigma_{i,j}$ are the values in (3.7).

Below we present the closed form bounds for the case of $m = 2$.

***Case* 2.** Let $m = 2$. In this case we assume that the third order divided differences of f are positive. The bounds for $E[f(\xi)]$ are based on the knowledge of μ_1 and μ_2. Since $m+1$ is odd, by the use of Theorem 1, any dual feasible basis for the minimization or maximization problem, respectively, in (3.2), without the additional constraint (3.2a), is of the form

$$B_{min} = \{0, i, i+1\} \quad \text{and} \quad B_{max} = \{j, j+1, n\},$$

where $1 \leq i \leq n-1, \quad 0 \leq j \leq n-2$.

The basis B_{min} is primal feasible if i satisfies the following condition:

- If $i \leq k-1$, then

$$\frac{\Sigma_{i,k}^2}{\Sigma_{i,k}} \leq \frac{(k+1)\mu_2 - \sum_{t=0}^{k} z_t^2}{(k+1)\mu_1 - \sum_{t=0}^{k} z_t} \leq \frac{\Sigma_{i+1,k}^2}{\Sigma_{i+1,k}} \; ,$$

$$\begin{aligned} &\left[(k-i+1)\Sigma_{i+1,k}^2 - (k-i)\Sigma_{i,k}^2\right]\left[(k+1)\mu_1 - \sum_{t=0}^{k} z_t\right] \\ &- \left[(k-i+1)\Sigma_{i+1,k} - (k-i)\Sigma_{i,k}\right]\left[(k+1)\mu_2 - \sum_{t=0}^{k} z_t^2\right] \leq \Sigma_{i,k}\Sigma_{i+1,k}^2 - \Sigma_{i,k}^2\Sigma_{i+1,k} \; ; \end{aligned} \tag{3.12}$$

- if $i \geq k+1$, then

$$\frac{\Sigma_{k+1,i}^2}{\Sigma_{k+1,i}} \leq \frac{(k+1)\mu_2 - \sum_{t=0}^{k} z_t^2}{(k+1)\mu_1 - \sum_{t=0}^{k} z_t} \leq \frac{\Sigma_{k+1,i+1}^2}{\Sigma_{k+1,i+1}} \; ,$$

$$\left[(i-k)\Sigma^2_{k+1,i+1} - (i-k+1)\Sigma^2_{k+1,i}\right]\left[(k+1)\mu_1 - \sum_{t=0}^{k} z_t\right] \tag{3.13}$$

$$-\left[(i-k)\Sigma_{k+1,i+1} - (i-k+1)\Sigma_{k+1,i}\right]\left[(k+1)\mu_2 - \sum_{t=0}^{k} z_t^2\right] \le \Sigma_{k+1,i}\Sigma^2_{k+1,i+1} - \Sigma^2_{k+1,i}\Sigma_{k+1,i+1} ;$$

- if $i = k$, then

$$\frac{\gamma^2_{0,k-1}}{\gamma_{0,k-1}} \le \frac{(k+1)\mu_2 - \sum_{t=0}^{k} z_t^2}{(k+1)\mu_1 - \sum_{t=0}^{k} z_t} \le \frac{\gamma^2_{0,k}}{\gamma_{0,k}} ,$$

$$\left(\gamma^2_{0,k} - \gamma^2_{0,k-1}\right)\left[(k+1)\mu_1 - \sum_{t=0}^{k} z_t\right] - \left(\gamma_{0,k} - \gamma_{0,k-1}\right)\left[(k+1)\mu_2 - \sum_{t=0}^{k} z_t^2\right] \tag{3.14}$$

$$\le \gamma_{0,k-1}\gamma^2_{0,k} - \gamma^2_{0,k-1}\gamma_{0,k} ,$$

where $\Sigma_{i,j}$, $\Sigma^2_{i,j}$, $\gamma_{i,j}$, $\gamma^2_{i,j}$ are defined as in (3.7).
The basis B_{max} is primal feasible if j satisfies the following condition:

- If $j \le k-1$, then

$$\frac{\alpha^2_{j,k}}{\alpha_{j,k}} \le \frac{(n-k)\mu_2 - \sum_{t=k+1}^{n} z_t^2}{(n-k)\mu_1 - \sum_{t=k+1}^{n} z_t} \le \frac{\alpha^2_{j+1,k}}{\alpha_{j+1,k}} ,$$

$$\left[(k-j+1)\alpha^2_{j+1,k} - (k-j)\alpha^2_{j,k}\right]\left[(n-k)\mu_1 - \sum_{t=k+1}^{n} z_t\right] \tag{3.15}$$

$$-\left[(k-j+1)\alpha_{j+1,k} - (k-j)\alpha_{j,k}\right]\left[(n-k)\mu_2 - \sum_{t=k+1}^{n} z_t^2\right] \le \alpha_{j,k}\alpha^2_{j+1,k} - \alpha^2_{j,k}\alpha_{j+1,k} ;$$

- if $j \ge k+1$, then

$$\frac{\alpha^2_{k+1,j}}{\alpha_{k+1,j}} \le \frac{(n-k)\mu_2 - \sum_{t=k+1}^{n} z_t^2}{(n-k)\mu_1 - \sum_{t=k+1}^{n} z_t} \le \frac{\alpha^2_{k+1,j+1}}{\alpha_{k+1,j+1}} ,$$

$$\left[(j-k)\alpha^2_{k+1,j+1} - (j-k+1)\alpha^2_{k+1,j}\right]\left[(n-k)\mu_1 - \sum_{t=k+1}^{n} z_t\right] \tag{3.16}$$

$$-\left[(j-k)\alpha_{k+1,j+1} - (j-k+1)\alpha_{k+1,j}\right]\left[(n-k)\mu_2 - \sum_{t=k+1}^{n} z_t^2\right]$$

$$\le \alpha_{k+1,j}\alpha^2_{k+1,j+1} - \alpha^2_{k+1,j}\alpha_{k+1,j+1} ;$$

- if $j = k$, then

$$\frac{\sigma^2_{k+1,n}}{\sigma_{k+1,n}} \;\le\; \frac{(n-k)\mu_2 - \sum_{t=k+1}^n z_t^2}{(n-k)\mu_1 - \sum_{t=k+1}^n z_t} \;\le\; \frac{\sigma^2_{k+2,n}}{\sigma_{k+2,n}} \;,$$

$$\left(\sigma^2_{k+2,n} - \sigma^2_{k+1,n}\right)\left[(n-k)\mu_1 - \sum_{t=k+1}^n z_t\right] - \left(\sigma_{k+2,n} - \sigma_{k+1,n}\right)\left[(n-k)\mu_2 - \sum_{t=k+1}^n z_t^2\right]$$
$$\le\; \sigma_{k+1,n}\sigma^2_{k+2,n} - \sigma^2_{k+1,n}\sigma_{k+2,n} \;, \tag{3.17}$$

where $\sigma_{i,j},\ \sigma^2_{i,j},\ \alpha_{i,j},\ \alpha^2_{i,j}$ are defined as in (3.7).

We have the following lower bound for $E[f(\xi)]$:

- If $i \le k-1$ and conditions (3.12) are satisfied, then

$$\frac{1}{k+1}\sum_{t=0}^k f_t \;+\; \frac{\Sigma^2_{i+1,k}[(k+1)\mu_1 - \sum_{t=0}^k z_t] - \Sigma_{i+1,k}[(k+1)\mu_2 - \sum_{t=0}^k z_t^2]}{\Sigma_{i,k}\Sigma^2_{i+1,k} - \Sigma_{i+1,k}\Sigma^2_{i,k}}\left[\sum_{t=i}^k f_t - \frac{\sum_{t=0}^k f_t}{k+1}\right]$$
$$+\frac{\Sigma_{i,k}[(k+1)\mu_2 - \sum_{t=0}^k z_t^2] - \Sigma^2_{i,k}[(k+1)\mu_1 - \sum_{t=0}^k z_t]}{\Sigma_{i,k}\Sigma^2_{i+1,k} - \Sigma_{i+1,k}\Sigma^2_{i,k}}\left[\sum_{t=i+1}^k f_t - \frac{\sum_{t=0}^k f_t}{k+1}\right] \;; \tag{3.18}$$

- if $i \ge k+1$ and conditions (3.13) are satisfied, then

$$\frac{1}{k+1}\sum_{t=0}^k f_t \;+\; \frac{\Sigma^2_{k+1,i+1}[(k+1)\mu_1 - \sum_{t=0}^k z_t] - \Sigma_{k+1,i+1}[(k+1)\mu_2 - \sum_{t=0}^k z_t^2]}{\Sigma_{k+1,i}\Sigma^2_{k+1,i+1} - \Sigma^2_{k+1,i}\Sigma_{k+1,i+1}}\left[\sum_{t=k+1}^i f_t - \frac{(i-k)\sum_{t=0}^k f_t}{k+1}\right]$$
$$+\frac{\Sigma_{k+1,i}[(k+1)\mu_2 - \sum_{t=0}^k z_t^2] - \Sigma^2_{k+1,i+1}[(k+1)\mu_1 - \sum_{t=0}^k z_t]}{\Sigma_{k+1,i}\Sigma^2_{k+1,i+1} - \Sigma^2_{k+1,i}\Sigma_{k+1,i+1}}\left[\sum_{t=k+1}^{i+1} f_t - \frac{(i-k+1)\sum_{t=0}^k f_t}{k}\right] \;; \tag{3.19}$$

- if $i = k$ and conditions (3.14) are satisfied, then

$$\frac{1}{k+1}\sum_{t=0}^k f_t \;+\; \frac{\gamma^2_{0,k}[(k+1)\mu_1 - \sum_{t=0}^k z_t] - \gamma_{0,k}[(k+1)\mu_2 - \sum_{t=0}^k z_t^2]}{\gamma_{0,k-1}\gamma^2_{0,k} - \gamma_{0,k}\gamma^2_{0,k-1}}\left[f_k - \frac{\sum_{t=0}^k f_t}{k+1}\right]$$
$$+\frac{\gamma_{0,k-1}[(k+1)\mu_2 - \sum_{t=0}^k z_t^2] - \gamma^2_{0,k-1}[(k+1)\mu_1 - \sum_{t=0}^k z_t]}{\gamma_{0,k-1}\gamma^2_{0,k} - \gamma_{0,k}\gamma^2_{0,k-1}}\left[f_{k+1} - \frac{\sum_{t=0}^k f_t}{k+1}\right] \;. \tag{3.20}$$

The upper bound for $E[f(\xi)]$ is given as follows:

- If $j \le k-1$ and conditions (3.15) are satisfied, then

$$\frac{1}{n-k}\sum_{t=k+1}^n f_t$$
$$+\frac{\alpha^2_{j+1,k}[(n-k)\mu_1 - \sum_{t=k+1}^n z_t] - \alpha_{j+1,k}[(n-k)\mu_2 - \sum_{t=k+1}^n z_t^2]}{\alpha_{j,k}\alpha^2_{j+1,k} - \alpha_{j+1,k}\alpha^2_{j,k}}\left[\sum_{t=j}^k f_t - \frac{(k-j+1)\sum_{t=k+1}^n}{n-k}\right]$$
$$+\frac{\alpha_{j,k}[(n-k)\mu_2 - \sum_{t=k+1}^n z_t^2] - \alpha^2_{j,k}[(n-k)\mu_1 - \sum_{t=k+1}^n z_t]}{\alpha_{j,k}\alpha^2_{j+1,k} - \alpha_{j+1,k}\alpha^2_{j,k}}\left[\sum_{t=j+1}^k f_t - \frac{(k-j)\sum_{t=k+1}^n}{n-k}\right] \;; \tag{3.21}$$

- if $j \geq k+1$ and conditions (3.16) are satisfied, then

$$\frac{1}{n-k}\sum_{t=k+1}^{n} f_t$$
$$+\frac{\alpha^2_{k+1,j+1}[(n-k)\mu_1-\sum_{t=k+1}^{n} z_t]-\alpha_{k+1,j+1}[(n-k)\mu_2-\sum_{t=k+1}^{n} z_t^2]}{\alpha_{k+1,j}\alpha^2_{k+1,j+1}-\alpha_{k+1,j+1}\alpha^2_{k+1,j}}\left[\sum_{t=k+1}^{j} f_t-\frac{(j-k)\sum_{t=k+1}^{n}}{n-k}\right]$$
$$+\frac{\alpha_{k+1,j}[(n-k)\mu_2-\sum_{t=k+1}^{n} z_t^2]-\alpha^2_{k+1,j}[(n-k)\mu_1-\sum_{t=k+1}^{n} z_t]}{\alpha_{k+1,j}\alpha^2_{k+1,j+1}-\alpha_{k+1,j+1}\alpha^2_{k+1,j}}\left[\sum_{t=k+1}^{j+1} f_t-\frac{(j-k+1)\sum_{t=k+1}^{n}}{n-k}\right] \; ; \tag{3.22}$$

- if $j = k$ and conditions (3.17) are satisfied, then

$$\frac{1}{n-k}\sum_{t=k+1}^{n} f_t \;+\; \frac{\sigma^2_{k+2,n}[(n-k)\mu_1-\sum_{t=k+1}^{n} z_t]-\sigma_{k+2,n}[(n-k)\mu_2-\sum_{t=k+1}^{n} z_t^2]}{\sigma_{k+1,n}\sigma^2_{k+2,n}-\sigma^2_{k+1,n}\sigma_{k+2,n}}\left[f_k-\frac{\sum_{t=k+1}^{n} f_t}{n-k}\right]$$
$$+\frac{\sigma_{k+1,n}[(n-k)\mu_2-\sum_{t=k+1}^{n} z_t^2]-\sigma^2_{k+1,n}[(n-k)\mu_1-\sum_{t=k+1}^{n} z_t]}{\sigma_{k+1,n}\sigma^2_{k+2,n}-\sigma^2_{k+1,n}\sigma_{k+2,n}}\left[f_{k+1}-\frac{\sum_{t=k+1}^{n} f_t}{n-k}\right] \; , \tag{3.23}$$

  where $\Sigma_{i,j}$, $\Sigma^2_{i,j}$, $\sigma_{i,j}$, $\sigma^2_{i,j}$, $\gamma_{i,j}$, $\gamma^2_{i,j}$, $\alpha_{i,j}$ and $\alpha^2_{i,j}$ are defined as in (3.7).

If we replace k by $k-1$ in all formulas given above, we obtain the primal feasibility conditions and the bounds in case of the second relaxed problem.

We remark that the monotonic cases are also unimodal cases. However, they can be handled without additional constraint (3.2a) (or (3.3a)). Since the reasoning and the formulas are considerably simpler than the ones in the general case, below we present the sharp bound formulas separately for these two cases.

3.2 TYPE 2: The Case of an Increasing Distribution

In this section we assume that the probability distribution is increasing, i.e., $p_0 \leq ... \leq p_n$ and f has positive divided differences of order $m+1$. If we introduce variables $v_i, \; i = 0, 1..., n$ as follows:

$$v_0 = p_0, \;\; v_1 = p_1 - p_0, \;\; ..., \;\; v_n = p_n - p_{n-1} \; ,$$

then problem (1.1), with the additional information regarding $p_0, ..., p_n$, can be written as

$$min(max)\{(f_0 + ... + f_n)v_0 + (f_1 + ... + f_n)v_1 + ... + f_n v_n\}$$

subject to

$$(a_0 + ... + a_n)v_0 + (a_1 + ... + a_n)v_1 + ... + a_n v_n = b \tag{3.24}$$

$$v_i \geq 0 \; , \quad i = 0, 1, ..., n$$

where $a_i = (1, z_i, ..., z_i^m)^T, \;\; i = 0, ..., n$ and $b = (1, \mu_1, ..., \mu_m)^T$.

It is easy to show that all minors of order $m+1$ from the matrix of the equality constraints and all minors of order $m+2$ from the matrix with the objective function coefficients in the last row, are positive. So, we can use Theorem 1 to obtain dual feasible bases for problem (3.24).

If m is small, then the optimum values of (3.24) can be given in closed forms, otherwise the dual method of linear programming, presented in [14, Section 4], can be used. Below we present the sharp bounds for $E[f(\xi)]$ for the case of $m = 1, 2$.

***Case* 1.** Let $m = 1$. If we take $k = n$ in (3.4) and (3.8), then we can obtain the primal feasibility condition for the dual feasible basis B_{min} and lower bound for $E[f(\xi)]$, respectively.

The basis B_{max} is the only dual feasible basis, hence it must also be primal feasible. In this case we get the following upper bound for $E[f(\xi)]$:

$$E[f(\xi)] \ \leq \ \frac{\mu_1 - z_n}{\gamma_{0,n-1}} \sum_{t=0}^{n} f_t + \frac{(n+1)\mu_1 - \sum_{t=0}^{n} z_t}{\gamma_{0,n-1}} f_n \ . \tag{3.25}$$

***Case* 2.** Let $m = 2$. If we take $k = n$ in formulas (3.12) and (3.18), then we can obtain the primal feasibility conditions for B_{min} and the sharp lower bound for $E[f(\xi)]$, respectively. The basis B_{max} is primal feasible if the following relations hold:

$$\frac{\gamma_{j,n-1}^2}{\gamma_{j,n-1}} \leq \frac{\mu_2 - z_n^2}{\mu_1 - z_n} \leq \frac{\gamma_{j+1,n-1}^2}{\gamma_{j+1,n-1}} \ ,$$

$$\begin{aligned} \left[(n-j+1)\gamma_{j+1,n-1}^2 - (n-j)\gamma_{j,n-1}^2\right](\mu_1 - z_n) - \left[(n-j+1)\gamma_{j+1,n-1} - (n-j)\gamma_{j,n-1}\right](\mu_2 - z_n^2) \\ \leq \gamma_{j,n-1}\gamma_{j+1,n-1}^2 - \gamma_{j,n-1}^2\gamma_{j+1,n-1} \ . \end{aligned} \tag{3.26}$$

In this case we have the following sharp upper bound for $E[f(\xi)]$:

$$\begin{aligned} E[f(\xi)] \ \leq \ & \frac{(\mu_1 - z_n)\gamma_{j+1,n-1}^2 - (\mu_2 - z_n^2)\gamma_{j+1,n-1}}{\gamma_{j,n-1}\gamma_{j+1,n-1}^2 - \gamma_{j+1,n-1}\gamma_{j,n-1}^2} \left[\sum_{s=j}^{n} f_s - (n-j+1)f_n\right] \\ & + \frac{(\mu_2 - z_n^2)\gamma_{j,n-1} - (\mu_1 - z_n)\gamma_{j,n-1}^2}{\gamma_{j,n-1}\gamma_{j+1,n-1}^2 - \gamma_{j+1,n-1}\gamma_{j,n-1}^2} \left[\sum_{s=j+1}^{n} f_s - (n-j)f_n\right] \ , \end{aligned} \tag{3.27}$$

where $\Sigma_{i,j}, \ \Sigma_{i,j}^2, \ \gamma_{i,j}, \ \gamma_{i,j}^2$ are defined as in (3.7).

3.3 TYPE 3: The Case of a Decreasing Distribution

Now, we assume that the probabilities $p_0, ..., p_n$ are unknown, but satisfy the inequalities: $p_0 \geq ... \geq p_n$. Let us introduce the variables $v_i, \ i = 0, 1, ..., n$ as follows:

$$v_0 = p_0 - p_1, \ ..., \ v_{n-1} = p_{n-1} - p_n, \ v_n = p_n.$$

If we write up problem (1.1), with the additional shape constraint, by the use of $v_0, ..., v_n$, then we obtain

$$\min(\max)\{f_0v_0 + (f_0 + f_1)v_1 + ... + (f_0 + ... + f_n)v_n\}$$

subject to

$$a_0v_0 + (a_0 + a_1)v_1 + ... + (a_0 + ... + a_n)v_n = b \tag{3.28}$$

$$v_i \geq 0 \ , \quad i = 0, 1, ..., n$$

where $a_i = (1, z_i, ..., z_i^m)^T, \quad i = 0, ..., n$ and $b = (1, \mu_1, ..., \mu_m)^T$.

One can easily show that problem (3.28) satisfies the conditions of Theorem 2. For small m values the optimum values of problem (3.28) can be given in closed forms, otherwise the dual method of linear programming, presented in [14, Section 4], can be applied.

Below we present the sharp bounds for $E[f(\xi)]$ for the case of $m = 1, 2$.

***Case* 1.** Let $m = 1$. If we take $k + 1 = 0$ in (3.5) and (3.9), then we obtain the primal feasibility condition for B_{min} and the sharp lower bound for $E[f(\xi)]$, respectively.

Since B_{max} is the only dual feasible basis, it follows that it is optimal. In this case we obtain the following upper bound for $E[f(\xi)]$:

$$\frac{(n+1)\mu_1 - \sum_{t=0}^n z_t}{(n+1)z_0 - \sum_{t=0}^n z_t} f_0 + \frac{\mu_1 - z_0}{(n+1)z_0 - \sum_{t=0}^n z_t} \sum_{t=0}^n f_t \ . \tag{3.29}$$

***Case* 2.** Let $m = 2$. The basis B_{min} is primal feasible if i is determined by the inequalities:

$$\frac{\sigma_{1,i}^2}{\sigma_{1,i}} \leq \frac{\mu_2 - z_0^2}{\mu_1 - z_0} \leq \frac{\sigma_{1,i+1}^2}{\sigma_{1,i+1}} \ ,$$

$$\begin{aligned} &\left((i+1)\sigma_{1,i+1}^2 - (i+2)\sigma_{1,i}^2\right)(\mu_1 - z_0) - \left((i+1)\sigma_{1,i+1} - (i+2)\sigma_{1,i}\right)(\mu_2 - z_0^2) \\ &\qquad \leq \sigma_{1,i}\sigma_{1,i+1}^2 - \sigma_{1,i}\sigma_{1,i+1}. \end{aligned} \tag{3.30}$$

We have the following lower bound for $E[f(\xi)]$:

$$\frac{(\mu_1 - z_0)\sigma_{1,i+1}^2 - (\mu_2 - z_0^2)\sigma_{1,i+1}}{\sigma_{1,i+1}^2\sigma_{1,i} - \sigma_{1,i+1}\sigma_{1,i}^2}\left[\sum_{t=1}^{i} f_t - if_0\right] + \frac{(\mu_2 - z_0^2)\sigma_{1,i} - (\mu_1 - z_0)\sigma_{1,i}^2}{\sigma_{1,i+1}^2\sigma_{1,i} - \sigma_{1,i+1}\sigma_{1,i}^2}\left[\sum_{t=1}^{i+1} f_t - (i+1)f_0\right]. \tag{3.31}$$

The primal feasibility conditions for B_{max} and the sharp upper bound for $E[f(\xi)]$ can be obtained by taking $k + 1 = 0$ in (3.16) and (3.22), respectively.

4 The Case of the Binomial Moment Problem

In case of the binomial moment problem (1.2) we look at the special case, where

$$z_i = i, \quad i = 0, \ ..., \ n \ , \quad f_0 = 0, \quad f_1 = ... = f_n = 1 \ .$$

We give lower and upper bounds for the probability that at least one out of n events occurs for the case of $m = 2$. We look at problem (1.2), but the constraints are supplemented by shape constraints of the unknown probability distribution $p_0, ..., p_n$.

In the following three subsections we use the same shape constraints that we have used in Section 3.1-3.3.

4.1 TYPE 1: The Case of a Unimodal Distribution

We assume that the distribution is unimodal with a known modus, i.e., we consider the following problem:

$$\min(\max) \sum_{i=1}^{n} p_i$$

subject to

$$\sum_{i=0}^{n} \binom{i}{j} p_i = S_j \ , \ j = 0, 1, ..., m \tag{4.1}$$

$$p_0 \leq ... \leq p_{k-1} \leq p_k \geq p_{k+1} \geq ... \geq p_n$$

$$p_i \geq 0 \ , \ i = 0, 1, .., n$$

where $S_j, \ j = 0, 1..., m$ are defined as in Section 1.

As in case of the power moment problem, here too there are two representations of problem (4.1). The forward representation is the following:

$$min(max) \ \{kv_0 + \sum_{i=1}^{k}(k-i+1)v_i + \sum_{i=k+1}^{n}(i-k)v_n\}$$

subject to

$$\sum_{i=0}^{k}(k-i+1)v_i + \sum_{i=k+1}^{n}(i-k)v_i = 1 \tag{4.2}$$

$$\sum_{i=0}^{k}\left[\binom{i}{j} + ... + \binom{k}{j}\right]v_i + \sum_{i=k+1}^{n}\left[\binom{k+1}{j} + ... + \binom{i}{j}\right]v_i = S_j \ , \quad j = 1, ..., m$$

$$v_0 + ... + v_k - v_{k+1} - ... - v_n \geq 0 \tag{4.2a}$$

$$v_i \geq 0 \ , \quad i = 0, ..., n \ .$$

The backward representation of problem (4.1) is given as:

$$min(max)\ \{(k-1)v_0 + \sum_{i=1}^{k-1}(k-i)v_i + \sum_{i=k}^{n}(i-k+1)v_n\}$$

subject to

$$\sum_{i=0}^{k-1}(k-i)v_i + \sum_{i=k}^{n}(i-k+1)v_i = 1 \tag{4.3}$$

$$\sum_{i=0}^{k-1}\left[\binom{i}{j}+...+\binom{k-1}{j}\right]v_i + \sum_{i=k}^{n}\left[\binom{k}{j}+...+\binom{i}{j}\right]v_i = S_j\ ,\quad j=1,...,m$$

$$v_k + ... + v_n - v_0 - ... - v_{k-1} \geq 0 \tag{4.3a}$$

$$v_i \geq 0\ ,\quad i=0,...,n\ .$$

If m is small, then the optimum values of (4.2) and (4.3), without the additional constraints (4.2a) and (4.3a), can be given in closed forms, otherwise dual methods of linear programming, presented in [14, Section 4], can be used as we have discussed it in Section 3.1.

We take problem (4.2), without the additional constraint (4.2a), and present closed form bounds for the probability that at least one out of n events occurs for the case of $m=2$.

If we take into account the equations:

$$\binom{j+1}{2} - \binom{i}{2} = \frac{(j-i+1)(i+j)}{2}\ ,\qquad 2 \leq i \leq j \leq n \tag{4.4}$$

$$\binom{i}{2} + ... + \binom{j}{2} = \frac{(j-i+1)(j^2+ij+i^2-2i-j)}{6}\ ,\qquad 2 \leq i \leq j \leq n \tag{4.5}$$

then we can write the first relaxed problem as follows:

$$max\ \{kv_0 + \sum_{i=1}^{k}(k-i+1)v_i + \sum_{i=k+1}^{n}(i-k)v_n\}$$

subject to

$$\sum_{i=0}^{k}(k-i+1)v_i + \sum_{i=k+1}^{n}(i-k)v_i = 1$$

$$\sum_{i=0}^{k}(k-i+1)(k+i)v_i + \sum_{i=k+1}^{n}(i-k)(i+k+1)v_i = 2S_1 \tag{4.6}$$

$$\sum_{i=0}^{k}(k-i+1)(k^2+ik+i^2-2i-k)v_i + \sum_{i=k+1}^{n}(i-k)(i^2+ik+k^2-1)v_i = 6S_2$$

$$v_i \geq 0 \ , \quad i = 0, ..., n \ .$$

The optimum values of (4.6) provide us with lower and upper bounds for $P(\xi \geq 1)$, where the probability distribution is unimodal.

Replacing k by $k-1$ in problem (4.6) we obtain the second relaxed problem, i.e., problem (4.3) without the additional constraint (4.3a).

We will characterize the dual feasible bases in both relaxed problems. First we remark that any basis that does not include v_0 produces an objective function value equal to 1. The following theorem characterizes the dual feasible bases in both relaxed problems.

Theorem 4. *Any dual feasible basis in any of the relaxed problems has the following structures (in terms of the subscripts of the basic vectors):*

Minimization problem	*Maximization problem*
$B_{min} = \{0, i, i+1\} \ , \quad 1 \leq i \leq n-1$	$B_{max} = \begin{cases} \{0, 1, n\} \\ \{s, t, u\} \ , \quad 1 \leq s < t < u \leq n \end{cases}$

The basis $\{s, t, u\}$, $1 \leq s < t < u \leq n$, is dual degenerate and all other bases are dual nondegenerate.

Proof. We prove the assertion in case of problem (4.6). The justification of the structures corresponding to the second relaxed problem goes along the same line.

It is easy to show that any minor of order 3 from the matrix of equality constraints in problem (4.6) is positive.

A basis B in the minimization problem (4.6) is said to be *dual feasible* if the following inequalities hold:

$$z_p = c_B^T B^{-1} a_p \leq c_p \quad \text{for any nonbasic } p \ ,$$

where c is coefficient vector of the objective function. For the maximization problem the dual feasibility of a basis is defined by the reversed inequalities. A basis B is said to be *dual degenerate* if there is at least one nonbasic p such that $c_p - z_p = 0$. Since we have

$$\begin{pmatrix} 1 & c_B^T \\ 0 & B \end{pmatrix} \begin{pmatrix} c_p - z_p \\ B^{-1}a_p \end{pmatrix} = \begin{pmatrix} c_p \\ a_p \end{pmatrix} ,$$

the first component of the solution of this equation can be expressed as

$$c_p - z_p = \frac{1}{|B|} \begin{vmatrix} c_p & c_B^T \\ a_p & B \end{vmatrix} \ .$$

Hence, the basis B is dual feasible in the minimization (maximization) problem (4.6) if $c_p - z_p \geq 0$ ($c_p - z_p \leq 0$) for every nonbasic index p. Since $|B|$ is positive, the sign of $c_p - z_p$

is determined by the sign of the other determinant standing on the right hand side of the above equation.

In case of the minimization problem (4.6) we need to show that $c_p - z_p = \begin{vmatrix} c_p & c_B^T \\ a_p & B \end{vmatrix} > 0$ for any nonbasic p and it happens if and only if B has the structure given in Theorem 4.

In a similar way, for the case of the maximization problem we can prove that $c_p - z_p = \begin{vmatrix} c_p & c_{B_{max}}^T \\ a_p & B_{max} \end{vmatrix} < 0$ for any nonbasic p if and only if $B_{max} = \{0, 1, n\}$. Finally, $B_{max} = \{s, t, u\}$, $1 \le s < t < u \le n$ has the property that $c_0 - z_0 < 0$, $c_p - z_p = 0$ for every nonbasic $p \neq 0$, hence the basis is degenerate and dual feasible in the maximization problem. □

The bounds for $P(\xi \ge 1)$ ***in case of problem* (4.6)**

In the following we consider problem (4.6) and give conditions that ensure the primal feasibility of a dual feasible basis $B_{min} = \{0, i, i+1\}$ as well as the corresponding lower bound formulas for $P(\xi \ge 1)$.

Case 1. Let $1 \le i \le k-1$. $B_{min} = \{0, i, i+1\}$ is primal feasible if

$$\begin{aligned} 2(k+i-1)S_1 - 6S_2 &\ge ki \ , \\ 2(k+i-2)S_1 - 6S_2 &\le k(i-1) \ , \\ 2(k+2i-1)S_1 - 6S_2 &\le i(2k+i+1) \ . \end{aligned} \tag{4.7}$$

In this case the lower bound, i.e, the optimum value of (4.6) is obtained as follows:

$$\frac{k(i-1)}{(i+1)(k+1)} + \frac{2(k+2i-1)S_1 - 6S_2}{i(i+1)(k+1)} \le P(\xi \ge 1) \ . \tag{4.8}$$

Case 2. Let $k+1 \le i \le n-1$. Then the primal feasibility conditions are

$$\begin{aligned} 2(i+k)S_1 - 6S_2 &\ge k(i+1) \ , \\ 2(i+k-1)S_1 - 6S_2 &\le ik \ , \\ 2(2i+k+1)S_1 - 6S_2 &\le (i+2k+2)(i+1) \ , \end{aligned} \tag{4.9}$$

and the lower bound formula is the following:

$$\frac{i(k-1)}{k(i+2)} + \frac{2(2i+k)S_1 - 6S_2}{k(i+1)(i+2)} \le P(\xi \ge 1) \ . \tag{4.10}$$

Case 3. Let $i = k$. Then $B_{min} = \{0, k, k+1\}$ is primal feasible if the conditions

$$\begin{aligned} 4kS_1 - 6S_2 &\ge k(k+1) \ , \\ 2(2k-2)S_1 - 6S_2 &\le (k-1)k \ , \\ 6kS_1 - 6S_2 &\le 3k(k+1) \end{aligned} \tag{4.11}$$

are satisfied. In this case the lower bound formula is obtained as:

$$\frac{k-1}{k+2} + \frac{6kS_1 - 6S_2}{(k-1)k(k+1)} \le P(\xi \ge 1) \ . \tag{4.12}$$

In order to obtain upper bound formula we consider the basis $B_{max} = \{0, 1, n\}$ which is primal feasible if the following conditions are satisfied:

$$\begin{gathered} 2(n+k)S_1 - 6S_2 \le (k+1)(n+1) \ , \\ 2(n+k-1)S_1 - 6S_2 \ge nk \ , \\ (k-1)S_1 \le 3S_2 \ . \end{gathered} \tag{4.13}$$

In this case the upper bound is obtained as:

$$P(\xi \ge 1) \ \le \ \frac{2(n+k)S_1 - 6S_2}{(k+1)(n+1)} \ . \tag{4.14}$$

It is easy to see that if $B_{max} = \{s, t, u\}$, $1 \le s < t < u \le n$ is optimal, then the upper bound is equal to 1.

Remark 1. The bounds for $P(\xi \ge 1)$ in case of the second relaxed problem can be obtained by taking $k = k - 1$ in all above formulas.

Remark 2. The first relaxed problem provides us with a better upper bound than the one obtained by the use of the second relaxed problem if the following condition is satisfied:

$$3S_2 \le (n-1)S_1 \ . \tag{4.15}$$

Note that the inequality $2S_2 \le (n-1)S_1$ always holds (see, e.g., [15], p. 186).

4.2 TYPE 2: The Case of an Increasing Distribution

Now we assume that the probability distribution is increasing, i.e., $p_0 \le ... \le p_n$. Let us introduce the variables v_i , $i = 0, ..., n$: $v_0 = p_0$, $v_1 = p_1 - p_0$, ..., $v_n = p_n - p_{n-1}$. Taking into account equations (4.4) and (4.5), problem (1.2), with shape constraint, can be written as

$$min(max) \ \{nv_0 + \sum_{i=1}^{n} (n-i+1)v_i\}$$

subject to

$$\sum_{i=0}^{n} (n-i+1)v_i = 1$$

$$\sum_{i=0}^{n} (n-i+1)(n+i)v_i = 2S_1$$

$$\sum_{i=0}^{n}(n-i+1)(n^2+in+i^2-2i-n)v_i = 6S_2$$

$$v_i \geq 0\ ,\quad i=0,1,...,n\ . \tag{4.16}$$

Note that the additional constraints (4.2a) and (4.3a) are not needed in case of monotonic distributions. One can easily show that the dual feasible bases for problem (4.16) are as in Theorem 4.

We can obtain the primal feasibility conditions for the basis $B_{min} = \{0, i, i+1\}$ and the sharp lower bound for $P(\xi \geq 1)$ by taking $k = n$ in formulas (4.7) and (4.8), respectively.

The basis $B_{max} = \{0, 1, n\}$ is primal feasible if the following relations hold:

$$2(2n-1)S_1 - 6S_2 \leq n(n+1)\ ,$$

$$4(n-1)S_1 - 6S_2 \geq n(n-1) \quad \text{and} \quad (n-1)S_1 \leq 3S_2\ .$$

Finally, if the probability distribution is increasing, we have the following upper sharp bound:

$$P(\xi \geq 1)\ \leq\ min\left\{1, \frac{2(2n-1)S_1 - 6S_2}{n(n+1)}\right\}\ . \tag{4.17}$$

4.3 TYPE 3: The Case of a Decreasing Distribution

In this section we assume that the probability distribution is decreasing, i.e., $p_0 \geq ... \geq p_n$. Introducing the variables v_i , $i = 0, ..., n$: $v_0 = p_0 - p_1,\ ...\ ,\ v_{n-1} = p_{n-1} - p_n,\ v_n = p_n$, and taking into account the equation (4.4), problem (1.2), with the shape constraint, can be written as

$$min(max)\sum_{i=1}^{n} iv_i$$

subject to

$$\begin{aligned}
&\sum_{i=0}^{n}(i+1)v_i = 1\\
&\sum_{i=1}^{n}(i+1)iv_i = 2S_1\\
&\sum_{i=2}^{n}(i+1)i(i-1)v_i = 6S_2\\
&v_0, ..., v_n \geq 0\ .
\end{aligned} \tag{4.18}$$

One can easily show that problem (4.18) satisfies the conditions of Theorem 2. Therefore, we can use Theorem 1 to obtain sharp lower and upper bounds for $P(\xi \geq 1)$.

Since $\binom{\nu}{1} = \nu$ and $\binom{\nu}{2} = \frac{\nu(\nu-1)}{2} = \frac{\nu^2-\nu}{2}$, substituting $\mu_1 = S_1$ and $\mu_2 = 2S_2 + S_1$ in the closed bound formulas presented in Section 3.3 for the case of $m = 2$, we obtain the following sharp bounds for $P(\xi \geq 1)$:

$$\frac{2(2i+1)S_1 - 6S_2}{(i+1)(i+2)} \leq P(\xi \geq 1) \leq \frac{nj}{(n+1)(j+2)} + \frac{2(2j+n+1)S_1 - 6S_2}{(n+1)(j+1)(j+2)} \ , \tag{4.19}$$

where i and j are determined by the following inequalities:

$$i - 1 \leq \frac{3S_2}{S_1} \leq i \ ,$$

$$2(n+j)S_1 - 6S_2 \leq n(j+1) \ , \quad 2(n+j-1)S_1 - 6S_2 \geq nj$$

$$4jS_1 - 6S_2 \leq j(j+1) \ ,$$

where $1 \leq i \leq n-1$ and $0 \leq j \leq n-2$.

5 Numerical Examples

We present four examples to show that if the shape of the distribution is known, then by the use of our bounding methodology, we can obtain tighter bounds for $P(\xi \geq 1)$ than the second order binomial bounds.

Example 1. In order to create example for S_1 and S_2 we take the following probability distribution $p_0^* = 0.4,\ p_1^* = 0.3,\ p_2^* = 0.25,\ p_3^* = 0.03,\ p_4^* = 0.02$. With these probabilities the binomial moments are

$$S_1 = \sum_{i=1}^{4} i p_i^* = 0.97 \quad \text{and} \quad S_2 = \sum_{i=2}^{4} \binom{i}{2} p_i^* = 0.46 \ .$$

In this case the S_1, S_2 bounds for $P(\xi \geq 1)$ are given by the inequalities:

$$0.51 \leq \ P(\xi \geq 1) \leq 0.74 \ .$$

Now we assume that the probability distribution is decreasing, i.e., $p_0 \geq ... \geq p_4$. The optimal bases are $B_{min} = (a_0, a_2, a_3)$ and $B_{max} = (a_1, a_2, a_4)$.

The following are the improved lower and upper bounds obtained from (4.19):

$$0.5783 \leq P(\xi \geq 1) \leq 0.6273 \ .$$

Example 2. Let $n = 5, \quad S_1 = 3.95, \quad S_2 = 7$. Based on S_1, S_2 we obtain

$$0.88 \leq \ P(\xi \geq 1) \leq 1 \ .$$

If the distribution is increasing, the optimal bases are $B_{min} = (a_0, a_4, a_5)$ and $B_{max} = (a_0, a_1, a_5)$. By the use of the formulas given in (4.17), the improved sharp lower and upper bounds for $P(\xi \geq 1)$ are as follows:

$$0.94 \leq P(\xi \geq 1) \leq 0.97 \ .$$

Example 3. Let $n = 10$, $S_1 = 8.393$, $S_2 = 34.625$. The sharp S_1, S_2 bounds for $P(\xi \geq 1)$ are obtained as follows:

$$0.909 \leq \; P(\xi \geq 1) \leq 1 \; .$$

Now assume that the distribution is increasing. The optimal basis for the minimum problem is $B_{min} = (a_0, a_9, a_{10})$. We note that $B_{max} = (a_0, a_1, a_{10})$ is not primal feasible. Thus, the upper bound for $P(\xi \geq 1)$ is 1. By the use of the formula (4.17), the improved sharp lower and upper bounds for $P(\xi \geq 1)$ are as follows:

$$0.975 \leq P(\xi \geq 1) \leq 1 \; .$$

Example 4. In the following table we present bounds for $P(\xi \geq 1)$ with and without the unimodality condition as well as in case of relaxed problems presented in Section 4.1.

				without unimodality		with unimodality		Relaxed Problem 1		Relaxed Problem 2	
n	k	S_1	S_2	LB	UB	LB	UB	LB	UB	LB	UB
10	6	5.556	16.779	0.78975	1	0.93335	1	0.93159	1	0.93335	1
4	2	1.93	1.27	0.86333	1	0.8975	1	0.8975	1	0.8975	1
4	3	2.54	2.66	0.82667	1	0.896	0.968	0.896	0.98	0.896	0.968
10	6	5.54	16.745	0.78696	1	0.9325	0.99591	0.93075	0.99753	0.9325	0.99591
10	6	5.15	14.789	0.76719	1	0.92197	0.98787	0.9209	0.98787	0.92197	0.99645
10	6	4.715	10.905	0.84467	1	0.93545	1	0.93545	1	0.93545	1
10	6	5.3564	15.4812	0.7932	1	0.9311	1	0.92938	1	0.9311	1
10	5	5.2534	14.9632	0.78844	1	0.93081	1	0.92846	1	0.93081	1
10	5	5.213	15.099	0.77043	1	0.92583	0.99691	0.92368	0.99691	0.92583	1
10	5	4.787	13.182	0.74	1	0.91111	0.97755	0.91025	0.97755	0.91111	0.99898
10	5	4.9541	13.7327	0.76152	1	0.91928	1	0.91775	1	0.91928	1
10	5	4.918	13.748	0.75048	1	0.91589	0.98564	0.91464	0.98564	0.91589	1

Here LB and UB stand for the lower and upper bounds, respectively. The bounds are obtained as the optimum values of the LP's given in Section 4.1. In both relaxed problems the bounds are obtained by the use of the closed form formulas presented in Section 4.1.

In two cases Relaxed Problem 2 provides us with better upper bounds than the ones obtained by Relaxed Problem 1, as we can see it in lines 3 and 4. In all cases the lower bounds, corresponding to Relaxed Problem 2, are better than the ones obtained by Relaxed Problem 1.

6 Applications

We present two examples for the application of our bounding technique, where shape information about the unknown probability distribution can be used.

Example 1. ***Application in PERT***

In PERT we frequently concerned with the problem to approximate the expectation or the values of the probability distribution of the length of the critical path.

In the paper by Prékopa et al. [18] a bounding technique is presented for the c.d.f. of the critical, i.e., the longest path under moment information. In that paper first an enumeration algorithm finds those paths that are candidates to become critical. Then the joint probability distribution of the path lengths is approximated by a multivariate normal distribution that serves a basis for the bounding procedure.

In the present example we look at only one path and assume that the random length of each arc follows beta distribution, as it is usually assumed in PERT. Arc lengths are assumed to be independent, thus the probability distribution of the path length is the convolution of beta distributions with different parameters.

The p.d.f. of the beta distribution in the interval $(0, 1)$ is defined as

$$f(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}, \qquad 0 < x < 1, \tag{6.1}$$

where $\Gamma(.)$ is the gamma function, $\Gamma(p) = \int_0^\infty x^{p-1}e^{-x}dx, \quad p > 0$. The kth moment of this distribution can easily be obtained by the use of the equation

$$\int_0^1 \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}dx = 1 \ .$$

In fact,

$$\begin{aligned} \int_0^1 x^k f(x)dx &= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 x^{k+\alpha-1}(1-x)^{\beta-1}dx \\ &= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \frac{\Gamma(k+\alpha)\Gamma(\beta)}{\Gamma(k+\alpha+\beta)} \\ &= \frac{\Gamma(\alpha+k)\Gamma(\alpha+k-1)...\Gamma(\alpha+1)}{\Gamma(\alpha+\beta+k)\Gamma(\alpha+\beta+k-1)...\Gamma(\alpha+\beta+1)} \ . \end{aligned} \tag{6.2}$$

If $\alpha, \ \beta$ are integers, then, using the relation: $\Gamma(m) = (m-1)!$, the above expression takes a simple form.

The beta distribution in PERT is defined over a more general interval (a, b) and we define its p.d.f. as the p.d.f. of $a+(b-a)X$, where X has p.d.f. given by (6.1). In practical problems the values $a, \ b, \ \alpha, \ \beta$ are obtained by the expert estimations of the shortest largest and most probable times to accomplish the job represented by the arc (see, e.g., [1]).

Let n be the number of arcs in a path and assume that each arc length ξ_i has beta distribution with known parameters $a_i, \ b_i, \ \alpha_i, \ \beta_i, \quad i = 1, ..., n$. Assume that $\alpha_i \geq 1, \ \beta_i \geq 1, \ i = 1, ..., n$. We are interested to approximate the values of the c.d.f. of the path length, i.e., $\xi = \xi_1 + ... + \xi_n$.

The analytic form of the c.d.f. cannot be obtained in closed form but we know that the p.d.f. of ξ is unimodal. In fact, each ξ_i has logconcave p.d.f., hence the sum ξ also has logconcave p.d.f. (for the proof of this assertion see, e.g., [15]) and any logconcave function is also unimodal.

In order to apply our bounding methodology we discretize the distribution of ξ, by subdividing the interval $(\sum_{i=1}^n a_i, \ \sum_{i=1}^n b_i)$ and handle the corresponding discrete distribution as unknown, but unimodal such that some of its first m moments are also known. In principle any order moment of ξ is known but for practical calculation it is enough to use the first few moments, at least in many cases, to obtain good approximation to the values of the c.d.f. of ξ.

The probability functions obtained by the discretizations, using equal length subintervals, are logconcave sequences. In fact, by a theorem of Fekete [4], the convolution of logconcave sequences are also logconcave (see, also Prékopa, [15], p.108) and any logconcave sequence is unimodal in the sense of Section 3.1.

In order to apply our methodology we need to know the modus of the distribution of ξ. A heuristic method to obtain it is the following. We take the sum of the modi of the terms in $\xi = \xi_1 + ... + \xi_n$ and then compute a few probabilities around it.

Example 2. ***Application in Reliability***

Let $A_1, ...,\ A_n$ be independent events and define the random variables $X_1, ..., X_n$ as the characteristic variables corresponding to the above events, respectively, i.e.,

$$X_i = \begin{cases} 1 & \text{if } A_i \text{ occurs} \ , \\ 0 & \text{otherwise} \ . \end{cases}$$

Let $p_i = P(X_i = 1),\ i = 1, ..., n$. The random variables $X_1, ..., X_n$ have logconcave discrete distributions on the nonnegative integers, consequently the distribution of $X = X_1 + ... + X_n$ is also logconcave on the same set.

In many applications it is an important problem to compute, or at least approximate, e.g., by the use of bounds, the probability

$$X_1 + ... + X_n \geq 1 \ . \tag{6.3}$$

If $I_1, ..., I_{C(n,k)}$ designate the k-element subsets of the set $\{1, ..., n\}$ and $J_l = \{1, ..., n\} \backslash I_l$, $l = 1, ..., C(n, k)$, then we have the equation

$$P(X_1 + ... + X_n \geq 1) = \sum_{k=r}^{n} \sum_{l=1}^{C(n,k)} \prod_{i \in I_l} p_i \prod_{j \in J_l} (1 - p_j) \ , \tag{6.4}$$

where $C(n, k) = \left(\begin{smallmatrix} n \\ k \end{smallmatrix} \right)$.

If n is large, then the calculation of the probabilities on the right hand side of (6.4) may be hard, even impossible. However, we can calculate lower and upper bounds for the probability on the left hand side of (6.4) by the use of the sums:

$$S_k = \sum_{1 \leq i_1 < ... < i_k \leq n} p_{i_1} ... p_{i_k} = \sum_{l=1}^{C(n,k)} \prod_{i \in I_l} p_i \ , \qquad k = 1, ..., m \ , \tag{6.5}$$

where m may be much smaller than n. Since the random variable $X_1 + ... + X_n$ has logconcave, hence unimodal distribution, we can impose the unimodality condition on the probability distribution:

$$P(X_1 + ... + X_n = k) \ , \quad k = 0, ..., n \ . \tag{6.6}$$

Then we solve both the minimization and maximization problems presented in Section 4.1, to obtain the bounds for the probability (6.3). If m is small, then the bounds can be obtained by the formulas of Section 4.1. Note that the largest probability (6.5) corresponds to

$$k_{max} = \left\lfloor (n+1)\frac{p_1 + ... + p_n}{n} \right\rfloor .$$

A formula first obtained by C. Jordan (1867) provides us with the probability (6.3), in terms of the binomial moments $S_r, ..., S_n$:

$$P(X_1 + ... + X_n \geq r) = \sum_{k=r}^{n} (-1)^{k-r} \begin{pmatrix} k-1 \\ r-1 \end{pmatrix} S_k \ . \tag{6.7}$$

However, to compute higher order binomial moments may be extremely difficult, sometimes impossible. The advantage of our approach is that we use the first few binomial moments $S_1, ..., S_m$, where m is relatively small and in many cases we can obtain very good bounds.

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