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On the Hungarian Inventory Control Model

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Abstract. In this paper we recall and further develop an inventory model formulated by the author (1965), (1973a) and M. Ziermann (1964) that has had wide application in Hungary and elsewhere. The basic assumption made in connection with this model is that the delivery of the ordered amount takes place in an interval, according to some random process, rather than at one time epoch. The problem is to determine that minimum level of safety stock, that ensures continuous production, without disruption, by a prescribed high probability. The model is further developed first by its combination with another inventory control model, the order up to S model and then, by the formulations of a static and a dynamic type stochastic programming models.

Keywords: inventory control, stochastic programming, dynamic programming.
AMS Subject Classification: 90B05, 90C15, 90C39.

1 Introduction

It was forty years ago when the president of the National Planning Board of Hungary, a mathematically educated engineer proposed the problem to the Mathematical Institute of the Hungarian Academy of Sciences, to explain and remedy the problem of superfluously high inventory levels at Hungarian industrial plants. The problem was formulated in such a way that 1% increase in production necessitates how much percent increase in the inventory levels? Supply department managers usually increased the inventory levels proportionally with production increase and this policy resulted in huge inventories nationwide.

The Operations Research Department (called Applications of Mathematics in Economics at that time), headed by the author, followed up the problem. After a long lasting (almost one year) diagnostic activity, A. Prékopa and M. Ziermann, who had been working on the problem, found that most of the costs, used in existing inventory control models in the international literature, were unknown and the high levels of the superfluous inventories were the results of the uncertainties in the deliveries. The delivery that followed an order took place in an interval, rather than at one time epoch, at random times in random quantities. We call it random interval delivery.

Inventory control theory had already existed at the time we conducted our investigations. Since the pioneering paper by Arrow, Harris and Marschak (1951) many papers appeared in that area and the publication of the comprehensive book by Hadley and Whitin (1963) was underway. Our problem, however, did not fit into the existing models and we had to look for solution elsewhere. Finally we realized that order statistics was the branch of science that could provide us at least with a starting point to describe the random delivery processes. That was the first major step. Later on some of the order statistical theorems have been generalized, and, by the use of them, very good solutions to the practical problems have been obtained.

The obtained models and methods, that allowed for the calculation of the safety stock in a simple way, became widely applied in Hungary, in a short time. Inventory levels of thousands of raw materials and semi-finished products had been set, by the use of the new formulas, and huge savings had been reported (the unofficial figure of the savings was four billion HUF in the middle of the 1960-ies which can be about the same amount of today's USD).

The applications of the model constructions worked out in connection with the random interval delivery problem went beyond the boundaries of Hungary (signals about them came from Czechoslovakia, Germany, etc.) but the models have not become as widely known as they deserve it. In fact, the problem seems to exist in many different contexts. For example Segal (1997) reported about its existence in the paper industry and Morris et al. (1988) in the utility industry.

The purpose of this paper is to call the attention to this model system and present some new variants of it. In Section 2 we describe the original Hungarian inventory control model.

In Sections 3 and 4 we show that it can be combined with more traditional inventory control models. We take the "order up to S model" as an example. In Section 5 more general stochastic programming formulation of the original model is presented. Finally, in the last Section 6 we present a dynamic type stochastic programming model for the solution of inventory control problems with interval type deliveries and safety constraints.

2 The Hungarian Inventory Control Model

Consider a finite time interval $[0, T]$. Assume that prior to time 0 Company A and B agree that Company A will deliver a given amount of some raw material or semi-finished product (briefly material) to Company B, that the latter will use (consume) for production of some product(s). There is no agreement, however, regarding the scheduling of the delivery. The amount of material delivered in $[0, T]$ is supposed to be equal to the amount Company B consumes in the same time interval. The problem is to find that minimal safety stock level M that ensures continuous production (consumption), without disruption, by probability $1 - \epsilon$, where ϵ is a given small number.

2.1 The basic model

In connection with the delivery and consumption processes we assume the following.

(a) Deliveries take place at discrete times, the number of which is fixed and is equal to n, that we can estimate from past history.

The n delivery times are random, and their joint probability distribution is the same as that of n random points, chosen independently and according to uniform distribution, from the interval $[0, T]$.

(b) The delivered amounts are equal to each other.

(c) Company B uses the material with constant intensity. Let c designate the consumption in unit time. Then the consumption in the whole time interval $[0, T]$ is cT. We also call cT the demand in the interval $[0, T]$.

(d) The total delivered amount in $[0, T]$ is equal to the total consumption: cT.

The above conditions imply that the amount of material delivered at a given time is cT/n. Let $X(t)$ designate the delivered amount in $[0, t]$, where $0 \leq t \leq T$. A safety stock M ensures continuous consumption, without disruption, if and only if $M + X(t) - ct \geq 0$ for any $0 \leq t \leq T$. If we want it to happen with probability $1 - \epsilon$, then M is the solution of the equation, called reliability equation,

$$P\left(\inf_{0 \leq t \leq T} (M + X(t) - ct) \geq 0\right) = 1 - \epsilon, \tag{2.1}$$

where $0 < \epsilon < 1$. In practice ϵ can be ,e.g., $0.2, 0.1, 0.05$ etc. The corresponding safety levels that ensure continuous production are $0.8, 0.9, 0.95$ etc., respectively.

Under assumptions (a)–(d) there is a unique solution, with respect to M, of the equation (2.1). If more general assumptions are used, than equation (2.1) can be replaced by the optimization problem:

$$\begin{gathered}\text{Min M}\\ \text{subject to}\\ P\left(\inf_{0\le t\le T}(M+X(t)-ct)\ge 0\right)\ge 1-\epsilon\\ M\ge 0.\end{gathered}$$

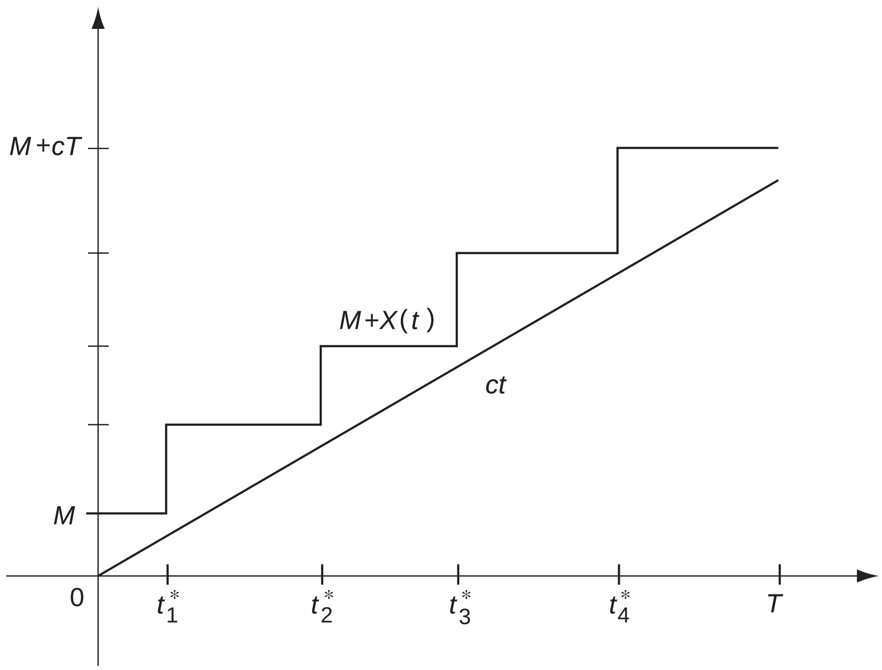

Figure 1:
The delivery times are $t_1^*, t_2^*, t_3^*, t_4^*$. Up to time t the delivered amount is $X(t)$. No disruption takes place in consumption if and only if $M + X(t) \ge ct$ for $0 \le t \le T$.

In Prékopa (1965) and Ziermann (1964) it is shown that equation (2.1) is equivalent to

$$P\left(\sup_{0\le t\le T}\left(\frac{t}{T}-\frac{1}{cT}X(t)\right)\le\frac{M}{cT}\right)=1-\epsilon \tag{2.2}$$

and that $\frac{1}{cT}X(t)$, $0 \le t \le T$ can be regarded as the empirical cumulative distribution function (c.d.f.) of a sample of size n taken from a population uniformly distributed in $[0, T]$. Since $\frac{t}{T}$,

$0 \leq t \leq T$ is the theoretical c.d.f. of the same population, a well-known formula of Bernstein (1946) and Birnbaum and Tingey (1951), for the supremum of the difference between the theoretical and empirical c.d.f.'s could be used to write up the probability in (2.2) in closed form.

The result is that formula (2.2) is the same as

$$\begin{aligned} &P\left(\sup_{0\leq t\leq T}\left(\frac{t}{T}-\frac{1}{cT}X(t)\right)\leq\frac{M}{cT}\right) \\ &= 1-\frac{M}{cT}\sum_{i=0}^{\lfloor n\left(1-\frac{M}{cT}\right)\rfloor}\binom{n}{i}\left(1-\frac{M}{cT}\right)^{n-i}\left(\frac{M}{cT}+\frac{i}{n}\right)^{i-1}=1-\epsilon. \end{aligned} \tag{2.3}$$

¿From here we can determine numerically the value of M. An approximation of the value of M can be obtained by the use of Smirnov's (1939) theorem:

$$\lim_{n\to\infty} P\left(\sqrt{n}\sup_{0\leq t\leq T}\left(\frac{t}{T}-\frac{1}{cT}X(t)\right)\leq y\right)=1-e^{-2y^2}, \tag{2.4}$$

where the convergence holds uniformly in y. If we replace $y=\sqrt{n}\frac{M}{cT}$, and use the limiting distribution (2.4) in the reliability equation (2.3), as an approximation, we obtain the relation

$$1-e^{-2\left(\sqrt{n}\frac{M}{cT}\right)^2}\approx 1-\epsilon,$$

from where we conclude

$$M\approx cT\sqrt{\frac{1}{2n}\log\frac{1}{\epsilon}}. \tag{2.5}$$

If we assume that n is a linear function of c, then (2.5) implies (writing equality instead of approximate equality)

$$M=K\sqrt{c}, \tag{2.6}$$

where K is a constant. It follows that

$$\frac{dM}{M}:\frac{dc}{c}=\frac{dM}{dc}\frac{c}{M}=\frac{1}{2}.$$

We interpret the result in such a way that 1% increase in the production implies only 0.5% increase in the safety stock. Formula (2.6) is the counterpart, in our inventory model, of the celebrated square root formula in classical deterministic inventory control theory:

$$Q=\sqrt{\frac{c_1}{c_2}c}=\text{const}\,\sqrt{c},$$

where c_1 = cost of placing and order, c_2 = time and quantity proportional inventory holding cost, c = consumption intensity, Q = optimal amount to be ordered periodically (see Figure 2).

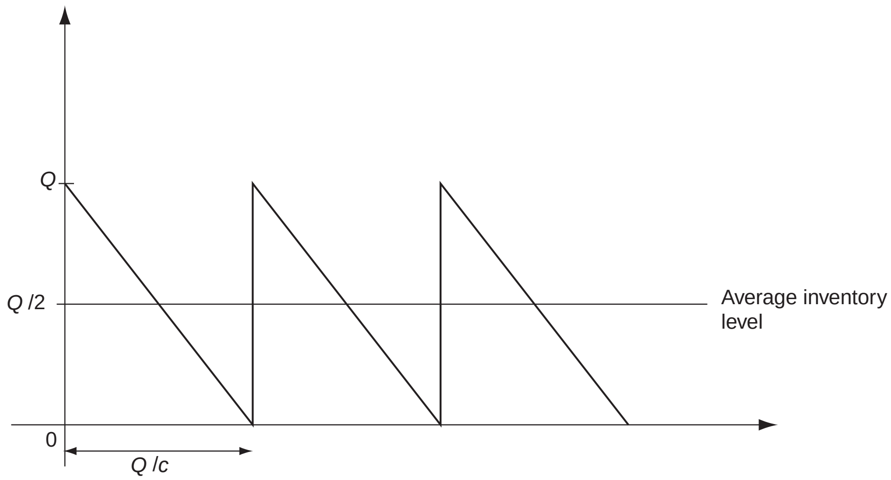

Figure 2:
Time variation of the inventory level in the simplest deterministic case.

2.2 The General Model

In this section we summarize the results obtained by Prékopa (1965, 1973a). The general form of the reliability equation is:

$$P\left(\inf_{0\le t\le T}(M + X(t) - Y(t)) \ge 0\right) = 1 - \epsilon, \tag{2.7}$$

where $X(t)$ is the total amount delivered and $Y(t)$ is the total amount consumed up to time t. In connection with these stochastic processes we introduce the following assumptions.

(a) Delivery (consumption) takes place at discrete times, the number of which is n (resp. m). We can estimate these numbers from past history.

(b) The delivery and consumption processes are stochastically independent.

(c) The delivery (consumption) process can be described by the following model (for simplicity we assume that $c = 1$, $T = 1$): whenever delivery (consumption) takes place, then there is a minimal amount delivered (consumed) equal to $\delta(\gamma)$. The n delivery amounts (m consumption amounts) can be described as the lengths of the subsequent intervals obtained by choosing a random sample of size $n - 1$ ($m - 1$), from a population uniformly distributed in the interval $[0, 1 - n\delta]$ ($[0, 1 - m\gamma]$). Let $\lambda = \delta n$ and $\mu = \gamma m$, $X_n(t, \lambda)$ the delivered amount and $Y_m(t, \mu)$ the consumed amount up to time t. (Figure 3 illustrates the delivery process).

If only the delivery process is modeled as described in (c), while the consumption process is supposed to be $Y(t) = t$, $0 \le t \le 1$, then we have

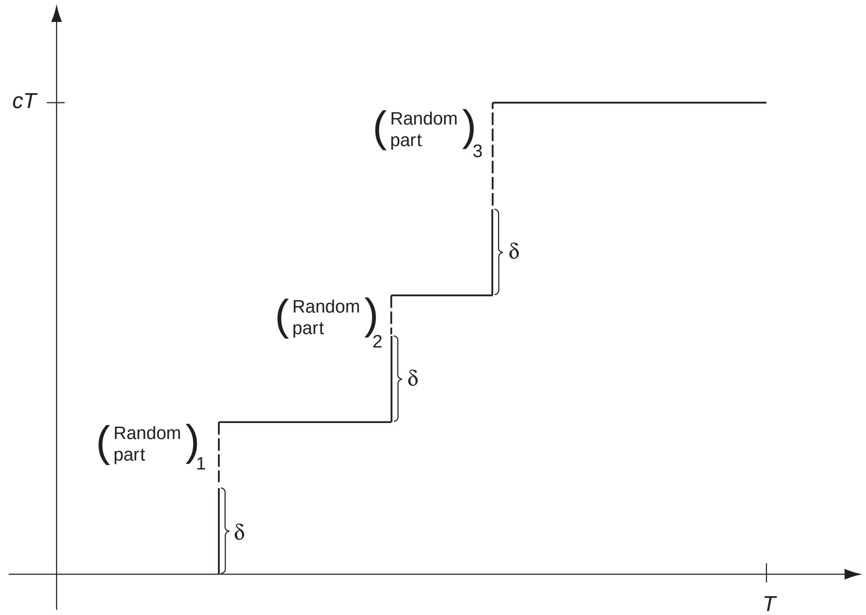

Figure 3:
Illustration of $X_n(t,\lambda)$ in case of $n = 3$ (Random part)$_1$ +(Random part)$_2$ +(Random part)$_3 = cT - 3\delta$. By assumption $c = 1$, $T = 1$.

Theorem 2.1 (**Prékopa, 1973a**). *The following relation holds true*

$$\lim_{n\to\infty} P\left(\sqrt{\frac{n}{1+(1-\lambda)^2}} \sup_{0\le t\le 1} (t - X_n(t,\lambda)) \le y\right) = \begin{cases} 1 - e^{-2y^2} & \textit{if} \quad y > 0 \\ 0 & \textit{if} \quad y \le 0. \end{cases} \tag{2.8}$$

The convergence is uniform in y.

If, on the other hand, both the delivery and consumption processes are modeled as described in (c), then we have

Theorem 2.2 (**Prékopa, 1973a**). *The following relation holds true*

$$\lim_{\substack{m\to\infty \\ n\to\infty}} P\left(\sqrt{\frac{mn}{m(1+(1-\lambda)^2)+n(1+(1-\mu)^2)}} \sup_{0\le t\le 1} (X_n(t,\lambda) - Y_m(t,\mu)) \le y\right)$$
$$= \begin{cases} 1 - e^{-2y^2} & \textit{if} \quad y > 0 \\ 0 & \textit{if} \quad y \le 0. \end{cases} \tag{2.9}$$

The convergence is uniform in y.

For $m = \infty$, $\mu = 1$ the process $Y_m(t,\mu)$ is equal to t, if $0 \le t \le 1$. Thus, if for fixed n we let $m \to \infty$ in the parentheses in formula (2.9), then we recover Theorem 2.1 from Theorem 2.2. On the other hand, we can recover Smirnov's theorem from Theorem 2.1, if we set $\lambda = 1$.

If we use the relation (2.8) for finite n, as approximate equality, replace $y = \sqrt{\frac{n}{1+(1-\lambda)^2}}M$ and set the right hand side in (2.8) equal to $1-\epsilon$, then we can obtain the approximate equality (writing $M(\lambda)$ instead of M):

$$M(\lambda) \approx \sqrt{1+(1-\lambda)^2}\sqrt{\frac{1}{2n}\log\frac{1}{\epsilon}}. \tag{2.10}$$

If c, T are not equal to 1, then to obtain the formula $M(\lambda)$ we only have to multiply the right hand side of (2.10) by cT.

In the same way, from Theorem 2.2 we can obtain the safety stock, that now we designate by $M(\lambda,\mu)$, by the approximate equality:

$$M(\lambda,\mu) \approx \sqrt{\frac{1+(1-\lambda)^2}{n} + \frac{1+(1-\mu)^2}{m}}\sqrt{\frac{1}{2}\log\frac{1}{\epsilon}}. \tag{2.11}$$

If c, T are not equal to 1, then we have to multiply the right hand side in (2.11) by cT. László (1973) and Kelle (1984) found the exact values for $M(\lambda)$ and $M(\lambda,\mu)$. The relevant formulas can be obtained by the use of a general method of Takács (1966) to find the probability distribution of the maximum of a stochastic process with interchangeable increments.

We can recover formula (2.10) from formula (2.11) if we let $m \to \infty$.

The proofs of Theorems 2.1 and 2.2 use similar ideas Doob (1949) and Donsker (1952) used to prove Kolmogorov's and Smirnov's theorems in order statistics and also ideas in connection with convergence of measures in metric probability spaces. For the latter see Prohorov (1956) or the monographs by Parthasarathy (1967).

Remark. A stochastic process $Z(t)$, $0 \le t \le 1$ is called a Brownian bridge, if its finite dimensional distributions are normal, $E(Z(t)) = 0$, $0 \le t \le 1$ and the autocovariance function is $E(Z(s)Z(t)) = \sigma^2 s(1-t)$ for $0 \le s \le t \le 1$, where σ is a positive constant. The process

$$W(t) = (t+1)Z\left(\frac{t}{t+1}\right), \qquad t \ge 0$$

is a Brownian motion process, i.e., a stochastic process with independent and normally distributed increments, $E(W(t)) = 0$, $t \ge 0$ and $E(W(s)W(t)) = \sigma^2 s$ for $0 \le s \le t$.

In the proofs of Theorems 2.1 and 2.2 the stochastic processes

$$\sqrt{\frac{n}{1+(1-\lambda)^2}}\,(X_n(t,\lambda) - t) \tag{2.12}$$

and

$$\sqrt{\frac{mn}{m(1+(1-\lambda)^2)+n(1+(1-\mu)^2)}}(X_n(t,\lambda) - Y_m(t,\lambda)) \tag{2.13}$$

are shown to converge to Brownian bridge processes. These, in turn, are transformed into Brownian motion processes. Finally, theorems in connection with maxima of Brownian motion processes are used to obtain the relations (2.8) and (2.9).

The factor $\sqrt{\frac{n}{1+(1-\lambda)^2}}$ of $X_n(t,\lambda) - t$ is obtained from the formula (see Prékopa, 1973a):

$$\mathrm{Var}\,(X_n(t,\lambda) - t) = \frac{1}{n}\left[1 + \frac{n-1}{n+1}(1-\lambda)^2\right] t(1-t), \tag{2.14}$$

as the reciprocal value of the factor of $t(1-t)$, if we take into account that $\lim_{n\to\infty} \frac{n-1}{n+1} = 1$. This implies that a slightly more accurate value of $M(\lambda)$, as compared to (2.10), is:

$$M(\lambda) \approx \sqrt{1 + \frac{n-1}{n+1}(1-\lambda)^2}\sqrt{\frac{1}{2n}\log\frac{1}{\epsilon}}. \tag{2.15}$$

Similarly, we have the equation

$$\begin{aligned}\mathrm{Var}\,(X_n(t,\lambda) - Y_m(t,\lambda)) = &\left(\frac{1}{m}\left[1 + \frac{m-1}{m+1}(1-\mu)^2\right]\right.\\ &\left.+\frac{1}{n}\left[1 + \frac{n-1}{n+1}(1-\lambda)^2\right]\right) t(1-t)\end{aligned} \tag{2.16}$$

and this implies that a slightly more accurate value of $M(\lambda,\mu)$ is given by

$$M(\lambda,\mu) \approx \sqrt{\frac{1}{m}\left[1 + \frac{m-1}{m+1}(1-\mu)^2\right] + \frac{1}{n}\left[1 + \frac{n-1}{n+1}(1-\lambda)^2\right]}\sqrt{\frac{1}{2}\log\frac{1}{\epsilon}}. \tag{2.17}$$

3 Combination with the Order up to S Inventory Control Model

We assume that the process has been going on since infinitely long time and orders are placed at times $0, \pm T, \pm 2T, \ldots$. The order up to S model uses the concept of inventory position, equal to $\alpha + \beta - \gamma$, where α = on hand inventory, β = on order inventory, γ = backlog. If $\alpha > 0$, then $\gamma = 0$ and if $\gamma > 0$, then $\alpha = 0$, so we have the relation $\alpha\gamma = 0$. Any time when an order is placed, the ordered amount is $S - (\alpha + \beta - \gamma)$.

In the conventional order up to S model the ordered amount is delivered in one quantity, and the difference between the delivery time and the time when the order is placed is called the lead time. Three types of costs are associated with the process: (1) cost of placing an order, (2) inventory holding cost, (3) cost of shortage.

The cost of placing an order usually has the form

$$f(Q) = \begin{cases} A + pQ & \text{if} \quad Q > 0 \\ 0 & \text{if} \quad Q = 0, \end{cases}$$

where Q is the ordered amount, p is the unit price of the material ordered and A is a positive constant.

The inventory holding cost is usually assumed to be proportional to time and quantity, with the proportionality factor $q^+ \geq 0$. The on hand inventory level, as a function of time, is described by the function in Figure 4, and the inventory holding cost in a time interval equals the area between that time interval and the function, multiplied by q^+.

Finally, the cost of shortage is given by the function

$$h(t, Q) = \begin{cases} q + q^- tQ & \text{if} \quad t > 0,\ Q > 0 \\ 0 & \text{if} \quad tQ = 0, \end{cases}$$

where $q \geq 0$, $q^- \geq 0$ are constants, Q is the amount of backlog and q is called stockout cost. This implies that if the magnitude of shortage is positive in an interval, then the associated cost equals q plus the area between the interval and the function, that describes the magnitude of shortage, multiplied by q^-. In conventional order up to S model the problem is to minimize long term average expected total cost, where the decision variables are T and S. For more information about the order up to S model the reader is referred to the book by Hadley and Whitin (1963).

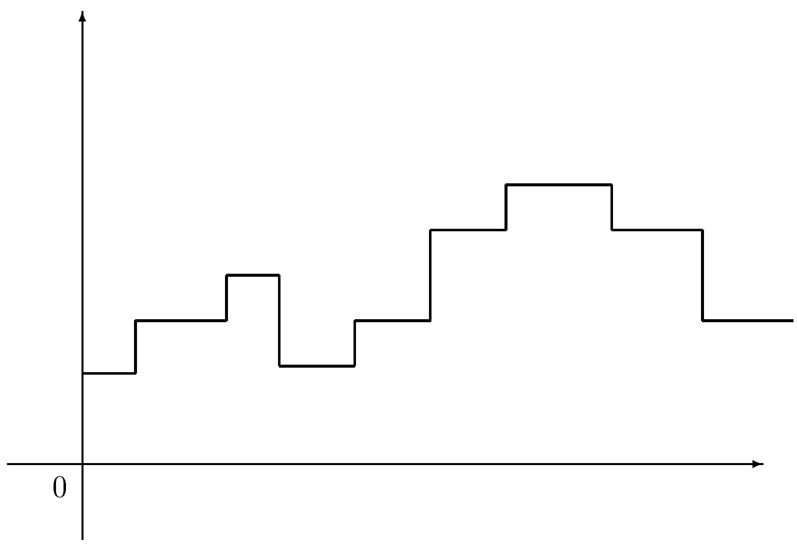

Figure 4: Time variation of on-hand inventory.

In our model we keep T fixed, thus only one decision variable remains, the level S. In view of this assumption, in the cost function we may disregard the cost of placing an order. We also assume $q = 0$, thus, both the inventory holding and stockout costs are time and quantity proportional.

The main difference between the conventional order up to S model and our model is that with us delivery takes place in an interval rather then at one time epoch. We assume that delivery begins τ time after the order is placed and has a duration of time T. Thus, if an order is placed at time 0, then the delivery takes place in the interval $(\tau, \tau + T)$.

If an order is placed at time kT, then the order up to S rule can be stated in the following alternative way: we order the amount demanded in the time interval $((k-1)T, kT)$.

Let $H(S,t)$ designate the on hand inventory and $V(S,t)$ the shortage at time t, $-\infty < t < \infty$. For later use we included S in the notations. Let further $Y(\mathcal{J})$ designate the demand in the time interval $\mathcal{J} \subset R^1$. Finally, fixing the time of an order, time 0, say, let $X(t)$ be the amount delivered in the time interval (τ, t), where $\tau \leq t \leq \tau + T$. The values of $H(S,t)$ and $V(S,t)$ will be expressed in terms of the random demands $Y(\mathcal{J})$ and the values of the stochastic process $X(t)$. Thus, we have to introduce assumptions in connection with the latter two stochastic processes.

We assume that if $\mathcal{J}_1, \ldots, \mathcal{J}_n$ are finite intervals on the real line with pairwise disjoint interiors, then $Y(\mathcal{J}_1), \ldots, Y(\mathcal{J}_n)$ are independent random variables. In addition, if $\mathcal{J}$ is an interval of lengths s, then $Y(\mathcal{J})$ has $N(\mu s, \sigma^2 s)$-distribution, where μ and σ are positive constants. (Strictly speaking, we approximate the distribution of $Y(\mathcal{J})$ by the given normal distribution.) The above assumption implies that if $\mathcal{J}_s = (0,s)$, then $Y(\mathcal{J}_s)$ is a Brownian motion process with drift μ.

The delivery process depends on the demand prior to the time when the order is placed. Without loss of generality we may assume that $T = 1$. Thus, if the order is placed at time 0, then its magnitude is equal to the amount demanded and consumed in $(-1, 0)$. If this amount is x, then $X(t)$, $\tau \leq t \leq \tau + 1$ is assumed to be a Gaussian process with $X(\tau) = 0$, $X(\tau + 1) = x$,

$$E(X(t)) = x(t - \tau), \qquad \tau \leq t \leq \tau + 1$$

$$\text{Cov}\,(X(s), X(t)) = \kappa^2 x^2 (s - \tau)(\tau + 1 - t), \qquad \tau \leq s \leq t \leq \tau + 1,$$

where

$$\kappa^2 = \frac{1}{n}\left(1 + \frac{n-1}{n+1}(1-\lambda)^2\right).$$

The assumption in connection with the stochastic process $Z(t)$ means that if we write up $X(t)$ in the form:

$$X(t) = (X(t) - x(t-\tau)) + x(t - \tau), \qquad \tau \leq t \leq \tau + 1,$$

then the first term on the right hand side can be identified with the process $X_n(t,\lambda) - t$ in Section 2. The difference is that $X(t) - x(t-\tau)$ is defined in the interval $(\tau, \tau+1)$ while $X_n(t,\lambda) - t$ is defined in $(0,1)$. In Section 2 we defined $X_n(t,\lambda) - t$ by the use of a model, and then approximated it by a Brownian bridge. Here, however, we assume that $X(t) - x(t-\tau)$ is a Brownian bridge in the time interval $(\tau, \tau+1)$. This explains the source of the above autocovariance function Cov $(X(s), X(t))$ and the assumption that $X(t)$, $\tau \leq t \leq \tau + 1$ is a Gaussian process.

4 The Optimization Problem

First we look at the objective function. This, by definition, is the expected long term average inventory holding cost plus the expected long term average cost of shortage. If we take the time averages of the two quantities in any time interval of length $T = 1$, then we obtain the long term time averages. In view of this we consider the time interval $(\tau, \tau + 1)$ in which we take the time averages of the two expected costs.

The delivery that takes place in the time interval $(\tau, \tau + 1)$ is ordered at time 0. At that time we order the amount $V(S, 0)$ plus the difference between S and the amount on hand plus on order. The on order amount will arrive before time τ. Thus, the on hand inventory at time 0 plus the total amount delivered in the time interval $(0, s)$, where $\tau < s \leq \tau + 1$, equals

$$S + V(S, 0) - (X(\tau + 1) - X(s)).$$

It follows that if $\tau < s < \tau + 1$, then the shortage at time s equals

$$V(S, s) = \begin{cases} 0, \text{ if } S + V(S, 0) - (X(\tau + 1) - X(s)) \geq Y((0, s)) + V(S, 0) \\ Y((0, s)) + V(S, 0) - (S + V(S, 0) - (X(\tau + 1) - X(s)), \text{ otherwise.} \end{cases}$$

In short, we can write

$$V(S, s) = [Y((0, s)) + X(\tau + 1) - X(s) - S]_+, \tag{4.1}$$

where the symbol $[u]_+$ means u, if $u > 0$ and 0 otherwise. Similarly,

$$H(S, s) = [S - Y((0, s)) - (X(\tau + 1) - X(s))]_+. \tag{4.2}$$

The cost function to be minimized is the following function of the variable S:

$$\begin{aligned} K(S) = q^+ &\int_\tau^{\tau+1} E([Y((0, s)) + X(\tau + 1) - X(s) - S]_+)\, ds \\ &+ q^- \int_\tau^{\tau+1} E([S - Y((0, s)) - (X(\tau + 1) - X(s))]_+) ds. \end{aligned} \tag{4.3}$$

Since we have the relation $[u]_+ - [-u]_+ = u$, for any real u, the cost function (4.3) can be written in the following form:

$$\begin{aligned} K(S) = q^- &\int_\tau^{\tau+1} E(S - Y((0, s)) - (X(\tau + 1) - X(s)) ds \\ &+ (q^+ + q^-) \int_\tau^{\tau+1} E([Y((0, s)) - (X(\tau + 1) - X(s)) - S]_+) ds. \end{aligned} \tag{4.4}$$

Let $F(y,s)$ designate the c.d.f. of the random variable $Y((0,s))+X(\tau+1)-X(s)$. By the use of integration by parts one can show that

$$\begin{aligned} E([Y((0,s))+X(\tau+1)-X(s)-S]_+) &= \int_S^\infty (y-S)d_yF(y,s) \\ &= \int_S^\infty (1-F(y,s))dy. \end{aligned} \tag{4.5}$$

In order to compute $F(y,s)$ we remark that the stochastic process $X(\tau+1)-X(s)$, $\tau \le s \le \tau+1$ is independent of the stochastic process $Y((0,s))$, $s \ge 0$. Under the condition $Y((-1,0)) = x$ the random variable $X(\tau+1)-X(s)$ has normal distribution with expectation $x(\tau+1-s)$ and variance

$$\kappa^2(s-\tau)(\tau+1-s)x^2,$$

hence under the same condition the random variable $Y((0,s))+X(\tau+1)-X(s)$ has expectation $x(\tau+1-s)+\mu s$ and variance $\kappa^2(s-\tau)(\tau+1-s)x^2+\sigma^2 s$. Thus, we have the relation

$$F(y,s) = \int_{-\infty}^\infty \Phi\left(\frac{y-x(\tau+1-s)-\mu s}{\sqrt{\delta^2(s-\tau)(\tau+1-s)x^2+\sigma^2 s}}\right)\frac{1}{\sigma}\varphi\left(\frac{x-\mu}{\sigma}\right)dx, \tag{4.6}$$

where φ is the p.d.f. and Φ the c.d.f. of the standard normal distribution.

If we take into account the equation

$$E(S-Y((0,s))-(X(\tau+1)-X(s))) = S-\mu s-\mu(\tau+1-s) = S-\mu(\tau+1)$$

and the relations (4.4), (4.5) and (4.6), we can write up the final form of the cost function:

$$K(S) = q^-(S-\mu(\tau+1)) + (q^+ + q^-)\int_S^\infty \left(1-\int_\tau^{\tau+1} F(y,s)ds\right)dy, \tag{4.7}$$

where $F(y,s)$ is given by (4.6).

The cost is minimized for S if the latter satisfies the equation

$$\int_\tau^{\tau+1} F(S,s)ds = \frac{q^+}{q^+ + q^-}. \tag{4.8}$$

The inventory control model described in Sections 3,4 has a straightforward generalization for the case of a multi-item problem.

Suppose that the number of items is r and let $K_j(S_j)$ designate the cost function (4.7) in case of the jth item, where we assume that $\tau_1 = \ldots = \tau_r = \tau$. Then we may formulate the problem in the following manner:

$$\begin{gathered} \min \sum_{j=1}^r K_j(S_j) \\ \text{subject to} \qquad S \in E, \end{gathered} \tag{4.9}$$

where $S = (S_1, \ldots, S_r)$ and E is a subset of the space R^r. For example, we may formulate

$$\begin{gathered} \min \sum_{j=1}^{r} K_j(S_j) \\ \text{subject to} \\ \sum_{j=1}^{r} d_j S_j \leq d, \quad S_j \geq 0, \quad j = 1, \ldots, r, \end{gathered} \tag{4.10}$$

where $d_1, \ldots, d_r, d$ are positive constants. Problem (4.10) is a linearly constrained nonlinear programming problem, where the objective function is convex and separable (it is the sum of r functions, where the jth term is a function of the single variable S_j).

The convexity of the objective function is the consequence of the convexity of its terms, as univariate functions. The convexity of $K_j(S_j)$ comes out immediately from (4.7) if we take into account that for each j

$$\int_{\tau}^{\tau+1} F_j(y, s) ds$$

is a c.d.f. in the variable y.

Problem (4.10) can be supplemented by some safety constraints. One reasonable way to do it is to impose upper bound on the time average of the expected shortage, i.e., the function (4.5), in connection with each item. Thus, we supplement the constraints

$$\int_{S_j}^{\infty} \left(1 - \int_{\tau}^{\tau+1} F_j(y, s) ds \right) dy \leq h_j, \qquad j = 1, \ldots, r \tag{4.11}$$

to the constraints of problem (4.10). The function on the left hand side of (4.11) is a convex function of S_j, for any $j = 1, \ldots, r$. Thus the new problem is convex.

5 Stochastic Programming Type Models

The model presented at the end of Section 4 is already of a stochastic programming type. Here we present more sophisticated ones, similar to those in Prékopa (1973b) and Prékopa and Kelle (1978).

Stochastic programming allows for the use of more realistic assumptions in connection with the nature of random phenomena and still provides us with solvable problems, at least in many cases. However, the problems are solved by algorithms rather than by formulas. In the course of the algorithms we usually need to evaluate the values of probability distribution functions along with their gradients.

First we consider the case of a single item. Let M designate the initial (or safety) inventory level that we want to determine by the use of some principle.

Time is subdivided into n periods and the sum of the lengths of the periods is equal to T. During this time an amount D of the material is delivered and an amount C is consumed. Here we assume that D and C are known constants. We formulate a model, that describes how the deliveries and consumptions in the subsequent time intervals take place, in a way that allows for the accommodation of nonhomogeneity in time of these processes.

Considering the delivery process we assume that in each time interval a minimal amount equal to $\delta \geq 0$ is delivered. Then we take a sample of size L ($L \geq n$) from the population uniformly distributed in the interval $(0, D - n\delta)$. Let $x_1^* \leq x_2^* \leq \cdots \leq x_L^*$ be the corresponding ordered sample. Then we take $n-1$ positive integers: $j_1 < j_2 < \cdots < j_{n-1} \leq L$. The delivered quantities in the n time intervals are assumed to be

$$\delta + x_{j_1}^*, \delta + x_{j_2}^* - x_{j_1}^*, \ldots, \delta + x_{j_{n-1}}^* - x_{j_{n-2}}^*, \delta + (D - n\delta) - x_{j_{n-1}}^*$$

The model for the consumption process is similar. We take a sample of size N from the population uniformly distributed in the interval $(0, C - n\gamma)$ and designate by $y_1^* \leq y_2^* \leq \cdots \leq y_N^*$ the ordered sample. Then take positive integers $k_1 < k_2 < \cdots < k_{n-1} \leq N$ and define the consumed quantities in the subsequent time intervals equal to

$$\gamma + y_{k_1}^*, \gamma + y_{k_2}^* - y_{k_1}^*, \ldots, \gamma + y_{k_{n-1}}^* - y_{k_{n-2}}^*, \gamma + (C - n\gamma) - y_{k_{n-1}}^*.$$

We assume that the delivery and consumption processes are independent. Let

$$\begin{aligned} X_1 &= x_{j_1}^*, \ X_2 = x_{j_2}^* - x_{j_1}^*, \ldots, X_{n-1} = x_{j_{n-1}}^* - x_{j_{n-2}}^*, \ X_n = (D - n\delta) - x_{j_{n-1}}^* \\ Y_1 &= y_{k_1}^*, \ Y_2 = y_{k_2}^* - y_{k_1}^*, \ldots, Y_{n-1} = y_{k_{n-1}}^* - y_{k_{n-2}}^*, \ Y_n = (C - n\gamma) - y_{k_{n-1}}^*. \end{aligned}$$

We assume that $M + D \geq C$. Then the condition of no disruption in the consumption process can be formulated in the following way:

$$\begin{array}{lcl} M + \delta + X_1 & \geq & \gamma + Y_1 \\ M + 2\delta + X_1 + X_2 & \geq & 2\gamma + Y_1 + Y_2 \\ \vdots & \vdots & \\ M + (n-1)\delta + X_1 + X_2 + \cdots + X_{n-1} & \geq & (n-1)\gamma + Y_1 + Y_2 + \cdots + Y_{n-1} \\ M + n\delta + X_1 + X_2 + X_3 \cdots + X_n & \geq & n\gamma + Y_1 + Y_2 + Y_3 + \cdots + Y_n. \end{array} \tag{5.1}$$

The last inequality can be removed because it is same as $M + D \geq C$.

The random variables $X_1, \ldots, X_{n-1}$ have the following joint p.d.f.:

$$\begin{aligned} & r(z_1, \ldots, z_{n-1}) \\ = & \frac{L!}{(j_1-1)!(j_2-j_1-1)!\ldots(j_{n-1}-j_{n-2}-1)!(L-j_{n-1}-1)!} \\ & \left(\frac{1}{D-n\delta}\right)^{n-1}\left(\frac{z_1}{D-n\delta}\right)^{j_1-1}\left(\frac{z_2}{D-n\delta}\right)^{j_2-j_1-1} \\ & \ldots\left(\frac{z_{n-1}}{D-n\delta}\right)^{j_{n-1}-j_{n-2}-1}\left(1-\frac{z_1+\cdots+z_{n-1}}{D-n\delta}\right)^{L-j_{n-1}-1} \\ & z_1+\cdots+z_{n-1} \le D-n\delta, \quad z_i \ge 0, \quad i=1,\ldots,n-1. \end{aligned} \tag{5.2}$$

Similarly, the random variables $Y_1, \ldots, Y_{n-1}$ have joint p.d.f.:

$$\begin{aligned} & s(z_1, \ldots, z_{n-1}) \\ = & \frac{N!}{(k_1-1)!(k_2-k_1-1)!\ldots(k_{n-1}-k_{n-2}-1)!(N-k_{n-1}-1)!} \\ & \left(\frac{1}{C-n\gamma}\right)^{n-1}\left(\frac{z_1}{C-n\gamma}\right)^{k_1-1}\left(\frac{z_2}{C-n\gamma}\right)^{k_2-k_1-1} \\ & \ldots\left(\frac{z_{n-1}}{C-n\gamma}\right)^{k_{n-1}-k_{n-2}-1}\left(1-\frac{z_1+\cdots+z_{n-1}}{C-n\gamma}\right)^{N-k_{n-1}-1} \\ & z_1+\cdots+z_{n-1} \le C-n\gamma, \quad z_i \ge 0, \quad i=1,\ldots,n-1. \end{aligned} \tag{5.3}$$

By (5.2) and (5.3) the random vectors $(D-n\delta)^{-1}(X_1, \ldots, X_{n-1})$ and $(C-n\gamma)^{-1}(Y_1, \ldots, Y_{n-1})$ have Dirichlet distributions.

If we are given a multi-item inventory problem, where the number of items is r, then a stochastic programming optimization model serves for the determination of the initial (or safety) inventory levels $M^{(1)}, \ldots, M^{(r)}$. Assume that the delivery and consumption processes, corresponding to the different items, are independent and let us supplement each constant and variable by a superscript, equal to l, if it belongs to item l. Then we define $P_l(M^{(l)})$ as the probability that all inequalities (5.1) are satisfied, in case of item l, and

$$\begin{aligned} W_i^{(l)} = i\gamma^{(l)} + Y_1^{(l)} + \cdots + Y_i^{(l)} - i\delta^{(l)} - X_1^{(l)} - \cdots - X_i^{(l)} \\ l=1,\ldots,r; \quad i=1,\ldots,n. \end{aligned} \tag{5.4}$$

If we use the equality $[u]_+ - [-u]_+ = u$, then the total expected (inventory holding and shortage) cost, in the n periods, in connection with item l, can be written as:

$$
\begin{aligned}
&\sum_{i=1}^{n} q_i^{+(l)} E\left(\left[W_i^{(l)} - M^{(l)}\right]_+\right) + \sum_{i=1}^{n} q_i^{-(l)} E\left(\left[M^{(l)} - W_i^{(l)}\right]_+\right) \\
&= \sum_{i=1}^{n} \left(q_i^{+(l)} + q_i^{-(l)}\right) E\left(\left[W_i^{(l)} - M^{(l)}\right]_+\right) + \sum_{i=1}^{n} q_i^{-(l)} E\left(M^{(l)} - W_i^{(l)}\right) \qquad (5.5) \\
&= \sum_{i=1}^{n} \left(q_i^{+(l)} + q_i^{-(l)}\right) E\left(\left[W_i^{(l)} - M^{(l)}\right]_+\right) + \sum_{i=1}^{n} q_i^{-(l)} M^{(l)} + \text{const.},
\end{aligned}
$$

where the const. term in the last line does not contain any of the decision variables $M^{(1)}, \ldots, M^{(r)}$ and we can remove it from the objective function.

The next step is to formulate the stochastic programming problem:

$$
\begin{gathered}
\min \left\{ \sum_{l=1}^{r} \left(c^{(l)} M^{(l)} + \sum_{i=1}^{n} \left(q_i^{+(l)} + q_i^{-(l)}\right) E\left(\left[W_i^{(l)} - M^{(l)}\right]_+\right) + \sum_{i=1}^{n} q_i^{-(l)} M^{(l)} \right) \right\} \\
\text{subject to} \\
\prod_{l=1}^{r} P_l(M^{(l)}) \geq 1 - \epsilon \\
(M^{(1)}, \ldots, M^{(r)}) \in E,
\end{gathered} \qquad (5.6)
$$

where E is some convex subset of R^n and $c^{(l)}$, $l = 1, \ldots, r$ are some nonnegative constants that can be interpreted as prices of establishing inventory capacities.

Problem (5.6) is a convex nonlinear programming problem. In fact, the objective function is clearly a convex function of $(M^{(1)}, \ldots, M^{(r)})$ and the set of feasible solutions is convex. The latter statement is the consequence of general theorems on multivariate logconcave measures that we summarize below.

In Prékopa (1971, 1973c) the notion of a logconcave measure was introduced as the probability measure defined on the Borel sets of R^n such that if $A, B \subset R^n$ are convex sets and $0 < \lambda < 1$, then

$$
P(\lambda A + (1-\lambda)B) \geq (P(A))^{\lambda}(P(B))^{1-\lambda}. \qquad (5.7)
$$

The notion of logconcavity is also used in connection with point functions. The point function $f(z) \geq 0$, $z \in R^n$ is called logconcave if for every $y, z \in R^n$ and $0 < \lambda < 1$ we have the relation

$$
f(\lambda y + (1-\lambda)z) \geq (f(y))^{\lambda}(f(z))^{1-\lambda}. \qquad (5.8)
$$

The main theorem of logconcave measures is the following.

THEOREM 5.1 (**Prékopa 1971, 1973c**). *If the probability measure P is generated by a logconcave p.d.f., then it is a logconcave measure.*

The probability density functions (5.2) and (5.3) are logconcave, hence the probability distributions, generated by these p.d.f.'s, are logconcave. The following two theorems are consequences of Theorem 5.1. For proofs see Prékopa (1995).

THEOREM 5.2. *If $F(z)$, $z \in R^n$ is the c.d.f. corresponding to a logconcave probability measure, then $F(z)$ is a logconcave point function.*

THEOREM 5.3. *If a random vector has a logconcave probability distribution, then any linear transformation of it also has logconcave distribution.*

The above theorems imply that each $P_l(M^{(l)})$ is a logconcave point function of $M^{(l)}$ which implies that

$$\prod_{l=1}^{r} P_l(M^{(l)})$$

is a logconcave function of $M = (M^{(1)}, \ldots, M^{(r)})$. The convexity of the set of $M = (M^{(1)}, \ldots, M^{(r)})$ satisfying the probabilistic constraint in problem (5.6) is a simple consequence of the last statement.

We can approximate the joint distribution of the random variables in problem (5.6) by a multivariate normal distribution. Based on the p.d.f.'s we can compute the expectation, $\mu_i^{(l)}$ and variance $(\sigma_i^{(l)})^2$ of the random variable $W_i^{(l)}$, $1 \le l \le r$, $1 \le i \le n$. The result is (see Appendix A):

$$\begin{aligned} \mu_i^{(l)} &= i\gamma^{(l)} - i\delta^{(l)} + (C^{(l)} - n\gamma^{(l)})\frac{k_i^{(l)}}{N^{(l)}+1} - (D^{(l)} - n\delta^{(l)})\frac{j_i^{(l)}}{L^{(l)}+1} \\ (\sigma_i^{(l)})^2 &= (C^{(l)} - n\gamma^{(l)})^2 \left(\frac{k_i^{(l)}}{N^{(l)}+1}\right)^2 \frac{1}{N^{(l)}+2} + (D^{(l)} - n\delta^{(l)})^2 \left(\frac{j_i^{(l)}}{L^{(l)}+1}\right)^2 \frac{1}{L^{(l)}+2}. \end{aligned} \tag{5.9}$$

Then we have the approximate value

$$E\left(\nu_i^{(l)}\right) = \int_{M^{(l)}}^{\infty} \left(1 - \Phi\left(\frac{z - \mu_i^{(l)}}{\sigma_i^{(l)}}\right)\right) dz. \tag{5.10}$$

Similarly we approximate the probabilities

$$P_l(M^{(l)}) = P(W_i^{(l)} \le M^{(l)}, \quad i = 1, \ldots, n-1),$$

$l = 1, \ldots, r$, by joint normal probability distribution function values in such a way that we replace joint normally distributed random variables for $W_1^{(l)}, \ldots, W_r^{(l)}$ with expectations and variances (5.9), further, with covariance matrix $G^{(l)} = H^{(l)} + K^{(l)}$, where $H^{(l)}$ and $K^{(l)}$ are the covariance matrices of the distributions (5.2) and (5.3), respectively, if we use them in connection with item $l = 1, \ldots, r$. For the calculation of these covariance matrices see Appendix A.

The approximation to problem (5.6) is the following

$$\min\left\{\sum_{l=1}^{r}\left(c^{(l)}M^{(l)}+\sum_{i=1}^{n}q_i^{-(l)}M^{(l)}+\sum_{i=1}^{n}\left(q_i^{+(l)}+q_i^{-(l)}\right)\int_{M^{(l)}}^{\infty}\left(1-\Phi\left(\frac{z-\mu_i^{(l)}}{\sigma_i^{(l)}}\right)\right)dz\right)\right\}$$

$$\begin{gathered}\text{subject to}\\ \prod_{l=1}^{r}P\left(W_i^{(l)}\le M^{(l)},\quad i=1,\ldots,n-1\right)\ge 1-\epsilon \\ M=(M^{(1)},\ldots,M^{(r)})\in E.\end{gathered} \tag{5.11}$$

where E is a set in $\Re^r$. There are a number of methods to solve problems (5.6) and (5.11), as described in Prékopa and Kelle (1978), Szántai (1988), Mayer (1992), Prékopa (1995) and Szántai (1997).

6 Two-stage model

Finally, we formulate a dynamic type extension of problem (5.11), a two-stage multi-item inventory control problem.

Similarly, as in Sections 5, we consider a group of r items and an inventory process that has been going on since infinitely long time. Orders are placed at times $0, \pm T, \pm 2T, \ldots$. If at time kT an order is placed, then the ordered amounts are delivered in n subintervals of the time interval $(kT+\tau, (k+1)T+\tau)$.

We distinguish between demand and consumption. The demand is an estimate of the consumption in the time interval $(kT+\tau, (k+1)T+\tau)$. At time kT we place the order equal to the demands for the r items. The demand is an r-component random vector $D=(D^{(1)},\ldots,D^{(r)})$ that we assume to be discrete with support $\{D_u,\ u\in U\}$, where U is a finite set. Let p_u be the probability corresponding to D_u, $u\in U$.

The consumption is also a random vector that we designate by $C=(C^{(1)},\ldots,C^{(r)})$. We introduce the notation:

$$E\left(C^{(l)}|D^{(l)}=D_u^{(l)}\right)=C_u^{(l)},\quad l=1,\ldots,r,\ u\in U. \tag{6.1}$$

Sometimes it is reasonable to assume that $C_u^{(l)}=D_u^{(l)},\quad l=1,\ldots,r,\ u\in U$. It is, however, not essential from our further analytical development, therefore we do not make this assumption.

The order up to S model, with r items, fits into the above described scheme. In that case the demand for item l at a time when an order is placed is equal to the consumption in the previous period, hence it is a random variable. Another random variable is the future consumption. The main point in our model formulation is that we take into account that both the delivery and consumption processes take place in a time interval, rather than at single epochs.

We assume that both the deliveries and consumptions in an interval $(kT+\tau, (k+1)T+\tau)$ follow the patterns described in Section 5. Thus we keep the notations of that Section with the exception of $M^{(l)}, \quad l = 1, \dots, r$ that designate something else in this section.

For the sake of simplicity we assume that under the condition: $D^{(l)} = D_u^{(l)}, \quad l = 1, \dots, r,$ the random vectors

$$(W_1^{(l)}, \dots, W_{n-1}^{(l)}), \quad l = 1, \dots, r \tag{6.2}$$

are independent for any given $u \in U$.

In out two-stage inventory control problem there are first stage and second stage decision variables. The first stage variables are: $M^{(l)}$, $l = 1, \dots, r$ that we interpret as storage capacities for the r items, respectively. Once we have their optimal values, they remain unchanged in time.

Before the second stage decision takes place we observe the realized demand values $D_u^{(l)}, \quad l = 1, \dots, r, \ u \in U$. The second stage variables are: $m_u^{(l)}$, $l = 1, \dots, r, \ u \in U$ which are adjustments to the safety stocks to ensure consumptions without disruption, by probability $1-\epsilon$, in the time interval $(kT+\tau, (k+1)T+\tau)$. The optimal values of the second stage variables are determined just before that time interval. If at time $kT+\tau$ the safety stock levels are m_u, $l = 1, \dots, r$, so we bring them to the levels $m^{(l)} + m_u^{(l)}, \quad l = 1, \dots, r, \ u \in U$. The adjustments incur some costs and we designate the adjustment cost function of item l by $f_l(x)$, $l = 1, \dots, r$. Thus the total adjustment cost is

$$\sum_{l=1}^{r} f_l(m_u^{(l)}), \tag{6.3}$$

if $D = D_u$. We assume that the functions $f_l(x), l = 1, \dots, r$ are convex.

Even though in practice the values of the second stage variables are determined at time kT, by the solution of a second stage problem, the discrete nature of the random vector D allows us to write both the first and second stage problems in a single large scale, decomposition type optimization problem. This problem is the following:

$$
\begin{gathered}
\min \left\{ \sum_{l=1}^{r} \left(g_{(l)}(M^{(l)}) + \frac{1}{T} \sum_{u \in U} p_u \left[f_l(m_u^{(l)}) + \sum_{i=1}^{n} q_i^{+(l)} E \left(\left[W_i^{(l)} - m^{(l)} - m_u^{(l)} \right]_+ \right) \right. \right. \right. \\
\left. \left. \left. + \sum_{i=1}^{n} q_i^{-(l)} E \left(\left[m^{(l)} + m_u^{(l)} - W_i^{(l)} \right]_+ \right) \right] \right) \right\} \\
\text{subject to} \\
\prod_{l=1}^{r} P \left(W_i^{(l)} \le m^{(l)} + m_u^{(l)}, \quad i = 1, \ldots, n-1 \right) \ge 1 - \epsilon \\
m^{(l)} + m_u^{(l)} \le M^{(l)}, m_u^{(l)} \ge 0, \qquad u \in U, \qquad l = 1, \ldots, r \\
\sum_{l=1}^{r} a^{(l)} M^{(l)} \le M.
\end{gathered} \tag{6.4}
$$

In Problem (6.4) $g_l(x)$, $l = 1, \ldots, r$ are cost functions of storage capacities; we assume that these functions are convex. The symbols $q_i^{+(l)}$, $q_i^{-(l)}$ designate nonnegative constants; these are interpreted as inventory holding and (quantity proportional) disruption costs, respectively. The values $m^{(l)}$, $l = 1, \ldots, r$ are fixed and positive. In principle we have to solve problem (6.4) for every positive $m^{(l)}$, $l = 1, \ldots, r$ up to the upper bounds $M/a^{(l)}$, $l = 1, \ldots, r$, respectively. The $a^{(l)}$, $l = 1, \ldots, r$, M and ϵ are given positive constants, $1 - \epsilon$ is called the safety level.

The objective function can be written in another form, if we take into account that

$$
\begin{aligned}
&\sum_{i=1}^{n} q_i^{+(l)} E \left(\left[W_i^{(l)} - m^{(l)} - m_u^{(l)} \right]_+ \right) + \sum_{i=1}^{n} q_i^{-(l)} E \left(\left[m^{(l)} + m_u^{(l)} - W_i^{(l)} \right]_+ \right) \\
&= \sum_{i=1}^{n} \left(q_i^{+(l)} + q_i^{-(l)} \right) E \left(\left[W_i^{(l)} - m^{(l)} - m_u^{(l)} \right]_+ \right) - \sum_{i=1}^{n} q_i^{-(l)} E \left(W_i^{(l)} - m^{(l)} - m_u^{(l)} \right)
\end{aligned} \tag{6.5}
$$

For given $u \in U$ we approximate the probability distributions of the random vectors (6.2) by normal distributions such that we keep the expectations and variances of the random variables in (6.2), calculated in Appendix B; further, we assign to them the correlation matrices $R_u^{(l)}$, $l = 1, \ldots, r$, respectively, where $R_u^{(l)}$corresponds to the covariance matrix $G_u^{(l)}$, $l = 1, \ldots, r$, presented in Appendix B. Let $\Phi(x)$, $\Phi(x; R)$ designate the univariate and multivariate standard normal distribution functions, respectively, where R is the correlation matrix.

In order to obtain the final form of problem (6.4) when normal approximation is used, we remark that the ith term in the second line of (6.5) can be written as

$$
\int_{m^{(l)} + m_u^{(l)}}^{\infty} \left(1 - \Phi \left(\frac{z - \mu_{iu}^{(l)}}{\sigma_{iu}^{(l)}} \right) \right) dz
$$

We also remark that in the third line of (6.5) the sum

$$\sum_{i=1}^{n} q_i^{-(l)} E\left(W_i^{(l)} - m^{(l)}\right)$$

does not contain decision variable. Hence it can be excluded from the objective function. The approximate problem is :

$$\begin{gathered}
\min \left\{ \sum_{l=1}^{r} \left(g_{(l)}(M^{(l)}) + \frac{1}{T} \sum_{u \in U} p_u \left[f_l(m_u^{(l)}) + \sum_{i=1}^{n} q_i^{-(l)} m_u^{(l)} \right.\right.\right. \\
\left.\left.\left. + \sum_{i=1}^{n} \left(q_i^{+(l)} + q_i^{-(l)} \right) \int_{m^{(l)}+m_u^{(l)}}^{\infty} \left(1 - \Phi\left(\frac{z - \mu_{iu}^{(l)}}{\sigma_{iu}^{(l)}} \right) \right) dz \right] \right) \right\} \\
\text{subject to} \\
\prod_{l=1}^{r} \Phi\left(\frac{m^{(l)} + m_u^{(l)} - \mu_{iu}^{(l)}}{\sigma_{iu}^{(l)}}, \quad i = 1, \ldots, n-1; R_i \right) \geq 1 - \epsilon \\
m^{(l)} + m_u^{(l)} \leq M^{(l)}, m_u^{(l)} \geq 0, \qquad u \in U, \qquad l = 1, \ldots, r \\
\sum_{l=1}^{r} a^{(l)} M^{(l)} \leq M.
\end{gathered} \tag{6.6}$$

The development of a solution method of problem (6.6) is underway.

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Appendix A

We consider the random variables $X_1, \ldots, X_{n-1}$ and $Y_1, \ldots, Y_{n-1}$ defined in Section 5 and derive the formulas for

$$\begin{array}{lll} E(X_1 + \cdots + X_i), \quad \mathrm{Var}\,(X_1 + \cdots + X_i), & 1 \le i \le n-1 \\ E(Y_1 + \cdots + Y_i), \quad \mathrm{Var}\,(Y_1 + \cdots + Y_i), & 1 \le i \le n-1 \\ \mathrm{Cov}\,(X_1 + \cdots + X_i,\ X_1 + \cdots + X_h) & 1 \le i \le h \le n-1 \\ \mathrm{Cov}\,(Y_1 + \cdots + Y_i, \quad Y_1 + \cdots + Y_h), & 1 \le i \le h \le n-1. \end{array}$$

For the derivation we use simple facts in connection with the beta-distribution, the p.d.f. of which is

$$\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} z^{\alpha-1}(1-z)^{\beta-1}, \qquad 0 < z < 1, \tag{1}$$

where $\alpha > 0$, $\beta > 0$ and the bivariate Dirichlet distribution, the p.d.f. of which is

$$\frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)} z_1^{\alpha-1} z_2^{\beta-1} (1-z_1-z_2)^{\gamma-1} \qquad (2)$$
$$z_1 + z_2 < 1, \qquad z_1 > 0, \qquad z_2 > 0,$$

where $\alpha > 0$, $\beta > 0$, $\gamma > 0$ and

$$\Gamma(p) = \int_0^\infty z^{p-1} e^{-z} dz, \qquad p > 0$$

is the gamma function. It is well-known that $\Gamma(p+1) = p\,\Gamma(p)$, for $p > 0$ and $\Gamma(n+1) = n!$ if n is a nonnegative integer.

If we use the above information then we can easily derive that the expectation and variance, corresponding to the p.d.f. given by (1) are

$$\frac{\alpha}{\alpha+\beta} \qquad \text{and} \qquad \left(\frac{\alpha}{\alpha+\beta}\right)^2 \frac{1}{\alpha+\beta+1}, \qquad (3)$$

respectively. The covariance, corresponding to the p.d.f. given by (2) is

$$\frac{-\alpha\beta}{(\alpha+\beta+\gamma)^2(\alpha+\beta+\gamma+1)}. \qquad (4)$$

Let $Z_i = (D - n\delta)^{-1} X_i$, $i = 1, \ldots, n-1$, where $\delta < D/n$. Since $(D-n\delta)^{-1}x_1^*, \ldots, (D-n\delta)^{-1}x_L^*$ is an ordered sample, taken from a population uniformly distributed in $(0,1)$, it follows that the p.d.f. of $Z_1 + \cdots + Z_i$ equals

$$\frac{L!}{(j_i-1)!(L-j_i)!} z^{j_i-1}(1-z)^{L-j_i}, \qquad 0 < z < 1.$$

In view of (3) we have the relations

$$E(Z_1 + \cdots + Z_i) = \frac{j_i}{L+1}$$
$$\mathrm{Var}\,(Z_1 + \cdots + Z_i) = \left(\frac{j_i}{L+1}\right)^2 \frac{1}{L+2}, \qquad 1 \le i \le n-2.$$

These imply that

$$E(X_1 + \cdots + X_i) = (D - n\delta)\frac{j_i}{L+1}$$
$$\mathrm{Var}\,(X_1 + \cdots + X_i) = (D-n\delta)^2 \left(\frac{j_i}{L+1}\right)^2 \frac{1}{L+2}, \qquad 1 \le i \le n-1. \qquad (5)$$

If $i < h$, then the joint p.d.f. of $Z_1 + \cdots + Z_i$ and $Z_1 + \cdots + Z_h$ is given by

$$\frac{L!}{(j_i - 1)!(j_h - j_i - 1)!(L - j_h)!} z_1^{j_i - 1} z_2^{j_h - j_i - 1} (1 - z_1 - z_2)^{L - j_h}$$
$$z_1 + z_2 < 1, \qquad z_1 > 0, \qquad z_2 > 0.$$

If we use (4) then from here we derive

$$\mathrm{Cov}\,(Z_1 + \cdots + Z_i,\ Z_1 + \cdots + Z_h) = \frac{-j_i(j_h - j_i)}{(L+1)^2(L+2)}, \qquad 1 \le i < h \le n-1$$

and then get the result:

$$\mathrm{Cov}\,(X_1 + \cdots + X_i,\ X_1 + \cdots + X_h) = (D - n\delta)^2 \frac{-j_i(j_h - j_i)}{(L+1)^2(L+2)}. \tag{6}$$

In the same way we obtain

$$E(Y_1 + \cdots + Y_i) = (C - n\gamma)\frac{k_i}{N+1} \tag{7}$$
$$\mathrm{Var}\,(Y_1 + \cdots + Y_i) = (C - n\gamma)^2 \left(\frac{k_i}{N+1}\right)^2 \frac{1}{N+2}, \quad 1 \le i \le n-1$$
$$\mathrm{Cov}\,(Y_1 + \cdots + Y_i,\ Y_1 + \cdots + Y_h) = (C - n\gamma)^2 \frac{-k_i(k_h - k_i)}{(N+1)^2(N+2)}, \quad 1 \le i < h \le n-1. \tag{8}$$

Let H and K designate the covariance matrices of the random vectors

$$(\ X_1,\ X_1 + X_2, \ldots, X_1 + \cdots + X_{n-1}) \tag{9}$$

$$(Y_1,\ Y_1 + Y_2, \ldots, Y_1 + \cdots + Y_{n-1}) \tag{10}$$

and respectively. Then for the covariance matrix G of the random variables

$$W_i = i\gamma + Y_1 + \cdots + Y_i - i\delta - X_1 - \cdots - X_i,\ i = 1, \ldots, n-1 \tag{11}$$

we have the equation $G = H + K$.

Appendix B

We present the conditional expectations and covariances of the random vectors (6.2) for given $D = D_u$. Note that we have assumed that the r random vectors (6.2) are independent.

In view of (5.9) and (6.1) we have the relation:

$$\mu_{iu}^{(l)} = E\left(W_i^{(l)} | D^{(l)} = D_u^{(l)}\right)$$

$$= i\gamma^{(l)} - i\delta^{(l)} + (C_u^{(l)} - n\gamma^{(l)})\frac{k_i^{(l)}}{N^{(l)}+1} - (D_u^{(l)} - n\delta^{(l)})\frac{j_i^{(l)}}{L^{(l)}+1}, \quad l = 1, \ldots, r, u \in U. \qquad (1)$$

Let $G_u^{(l)}$ designate the conditional covariance matrix of the random vector (6.2), given that $D^{(l)} = D_u^{(l)}$. Then $G_u^{(l)} = H_u^{(l)} + K_u^{(l)}$, where $H_u^{(l)}$ and $K_u^{(l)}$ are the conditional covariance matrices of the random vectors

$$(\ X_1^{(l)},\ X_1^{(l)} + X_2^{(l)}, \ldots, X_1^{(l)} + X_2^{(l)} \cdots + X_{n-1}^{(l)}) \qquad (2)$$

and

$$(Y_1^{(l)},\ Y_1^{(l)} + Y_2^{(l)}, \ldots, Y_1^{(l)} + Y_2^{(l)} \cdots + Y_{n-1}^{(l)}), \qquad (3)$$

respectively, given $D^{(l)} = D_u^{(l)}$.

The entries of these latter covariance matrices can be obtained by the use of the formulas in Appendix A. By the use of the equations (5) and (6) in Appendix A we immediately derive:

$$\mathrm{Var}(X_1^{(l)} + \cdots + X_i^{(l)} | D^{(l)} = D_u^{(l)}) = (D_u^{(l)} - n\delta^{(l)})^2 \left(\frac{j_i^{(l)}}{L^{(l)}+1}\right)^2 \frac{1}{L^{(l)}+2}, \qquad 1 \leq i \leq n-1, u \in U. \qquad (4)$$

$$\mathrm{Cov}\,(X_1^{(l)} + \cdots + X_i^{(l)},\ X_1^{(l)} + \cdots + X_h^{(l)} | D^{(l)} = D_u^{(l)}) = (D_u^{(l)} - n\delta^{(l)})^2 \frac{-j_i^{(l)}(j_h^{(l)} - j_i^{(l)})}{(L_i^{(l)}+1)^2(L_i^{(l)}+2)} \qquad (5)$$

$$1 \leq i < h \leq n-1,\ u \in U.$$

Slightly more complicated is the calculation of the matrices and covariances of the random vector (3). If, in addition to the conditioning on $D^{(l)}$ we condition on C too, then equations (8) provide us with the formulas:

$$\mathrm{Var}\,(Y_1^{(l)} + \cdots + Y_i^{(l)} | D^{(l)} = D_u^{(l)}, C^{(l)}) = (C^{(l)} - n\gamma^{(l)})^2 \left(\frac{k_i^{(l)}}{N^{(l)}+1}\right)^2 \frac{1}{N^{(l)}+2} \qquad (6)$$

$$1 \leq i \leq n-1,\ 1 \leq l \leq r,\ u \in U.$$

$$\mathrm{Cov}\,(Y_1^{(l)} + \cdots + Y_i^{(l)},\ Y_1^{(l)} + \cdots + Y_h^{(l)} | D^{(l)} = D_u^{(l)}, C^{(l)}) = (C^{(l)} - n\gamma^{(l)})^2 \frac{-k_i^{(l)}(k_h^{(l)} - k_i^{(l)})}{(N^{(l)}+1)^2(N^{(l)}+2)} \qquad (7)$$

$$1 \leq i < h \leq n-1,\ 1 \leq l \leq r,\ u \in U.$$

Using (6) and (7), simple calculation shows that

$$\mathrm{Var}\,(Y_1^{(l)} + \cdots + Y_i^{(l)} | D^{(l)} = D_u^{(l)}) = \mathrm{Var}\,(C^{(l)} | D^{(l)} = D_u^{(l)}) \left(\frac{k_i^{(l)}}{N^{(l)}+1} \right)^2 \frac{N^{(l)}+3}{N^{(l)}+2}$$

$$+(C_u^{(l)} - n\gamma^{(l)})^2 \left(\frac{k_i^{(l)}}{N^{(l)}+1} \right)^2 \frac{1}{N^{(l)}+2}, \quad 1 \le i \le n-1,\ 1 \le l \le r,\ u \in U. \tag{8}$$

and

$$\mathrm{Cov}(Y_1^{(l)} + \cdots + Y_i^{(l)},\ Y_1^{(l)} + \cdots + Y_h^{(l)} | D^{(l)} = D_u^{(l)}) = \mathrm{Var}(C^{(l)} | D^{(l)} = D_u^{(l)}) \frac{k_i^{(l)}(k_i^{(l)} + (N^{(l)}+1)k_h^{(l)})}{(N^{(l)}+1)^2(N^{(l)}+2)}$$

$$-(C_u^{(l)} - n\gamma^{(l)})^2 \frac{k_i^{(l)} \left(k_h^{(l)} - k_i^{(l)} \right)}{(N^{(l)}+1)^2(N^{(l)}+2)}, \quad 1 \le i < h \le n-1,\ 1 \le l \le r,\ u \in U. \tag{9}$$

By the use of (4) and (8) we obtain the variance of the random variable $W_i^{(l)}$, conditioned on $D^{(l)} = D_u^{(l)}$:

$$\left(\sigma_{iu}^{(l)} \right)^2 = \mathrm{Var}\, \left(W_i^{(l)} | D^{(l)} = D_u^{(l)} \right)$$

$$= (C_u^{(l)} - n\gamma^{(l)})^2 \left(\frac{k_i^{(l)}}{N^{(l)}+1} \right)^2 \frac{1}{N^{(l)}+2} + \left(D_u^{(l)} - n\delta^{(l)} \right)^2 \left(\frac{j_i^{(l)}}{L^{(l)}+1} \right)^2 \frac{1}{L^{(l)}+2} \tag{10}$$

$$1 \le i \le n-1,\ 1 \le l \le r,\ u \in U.$$