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Abstract. Compound Poisson distributions play important role in many applications (telecommunication, hydrology, insurance, etc.). In this paper, we prove that some of the compound Poisson distributions have the logconcavity property that makes them applicable in stochastic programming problems. The proofs are based on classical Turan types theorem and orthogonal polynomials.

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1 Introduction

Let $X_1, X_2, ...,$ be a sequence of nonnegative valued i.i.d random variables and consider the sum

$$S = X_1 + X_2 + ... + X_N,$$

where N has Poisson distribution with parameter $\lambda > 0$:

$$p_n = \frac{\lambda^n}{n!} e^{-\lambda}, n = 0, 1, 2...,$$

and $N, X_1, X_2...,$ are mutually independent. The probability distribution of S is called compound Poisson distribution. If the $X_i, i = 1, 2, ...,$ are integer valued then so is S, and for this case we introduce the notation $g_n = P(S = n), n = 0, 1, 2, ...$

Compound Poisson distributions play important role in many applied areas: acturial mathematics, physics, engineering, operations research, etc., see, e.g, Bowers et al. ([4]), Takács ([14]), Prékopa ([9]). The logconcavity property in connection with a compound Poisson distribution comes up primarily in stochastic optimization, where frequently the convexity of the optimization problem depends on that property (see Prékopa, 1995). One example is the bond portfolio construction problem with probabilistic constraints. In that problem we suppose that an insurance company keeps its wealth in bonds and wants to be able to meet the claims in subsequent periods with high probability (see Prékopa, 2003).

The notion of a logconcave sequence was first introduced by Fekete (1912) under the name of *2*-times or twice positive sequence as a special case of an r-times positive sequence, when $r = 2$. The sequence of nonnegative elements... $a_{-2}, a_{-1}, a_0,$... is said to be r-times positive if the matrix

$$A = \begin{pmatrix} \ddots & \ddots & \ddots & & \\ \ddots & a_0 & a_1 & a_2 & \\ \ddots & a_{-1} & a_0 & a_1 & \ddots \\ & a_{-2} & a_{-1} & a_0 & \ddots \\ & & \ddots & \ddots & \ddots \end{pmatrix}.$$

has no negative minor of order smaller than or equal to r (a minor is the determinant of a finite square part of the matrix traced out by the same number of rows as columns).

The twice-positive sequences are those for which we have

$$\begin{vmatrix} a_i & a_j \\ a_{i-t} & a_{j-t} \end{vmatrix} = a_i a_{j-t} - a_j a_{i-t} \geq 0,$$

for every $i \leq j$ and $t \geq 1$. It is easy to see that the above inequality holds if and only if for every i we have $a_i^2 \geq a_{i-1}a_{i+1}$.

Fekete (1912) proved the following important theorem.

Theorem 1.1 *The convolution of two r-times positive sequences is at least r-times positive.*

Twice-positive sequences are also called *logconcave* sequences. Theorem 1.1 states that the convolution of two logconcave sequences is logconcave.

A univariate discrete probability distribution, defined on the integers, is said to be logconcave if the sequence of the corresponding probabilities is logconcave.

Various applications of logconcave sequences are known in probability theory, combinatorics, etc. Surprisingly, logconcavity property came up in connection with orthogonal polynomials. The first theorem in this respect was proved by Turán (1950). It states that if $P_n(x)$ is the nth Legendre's polynomial, $-1 \leq x \leq 1$, then we have the inequality

$$P_n(x)^2 \geq P_{n-1}(x)P_{n+1}(x). \tag{1.1}$$

Inequalities of the type (1.1), valid for orthogonal polynomials, are called Turán type inequalities. In recent years, many Turán type inequalities have been established for Laguerre polynomials, Hermite polynomials, Bessel functions, Tschebychef polynomials, etc. Some of them will be used in this paper to prove logconcavity of special compound Poisson distributions.

The organization of the paper is as follows. In Section 2, we prove that the sequence $\{g_n\}_{n=1}^{\infty}$ is logconcave for the case of a compound Poisson random variable with geometrically distributed terms. In Section 3, we prove the logconcavity of the compound Poisson distribution for the case of Poisson distributed terms. Finally, in Section 4, we use the notion of a logconcave function $f(x), x \in \mathbb{R}$, meaning that $f(\lambda x + (1-\lambda)y) \geq (f(x))^{\lambda}(f(y))^{1-\lambda}$ for any $x, y \in \mathbb{R}, 0 < \lambda < 1$, and for the reader's convenience, we reproduce Oschwald's proof of the logconcavity of the continuous part of the compound Poisson distribution with exponential distributed terms, then generalize the proof for a class of compound distributions.

2 Logconcavity of the compound Poisson distribution with geometrically distributed terms

If $X_1, X_2, ...$ has geometric distributions with support $\{1, 2, 3..\}$ and $P(X_i = n) = pq^{n-1}(n = 1, 2, ..)$ one can easily verify that

$$P(S = n) = \sum_{k=1}^{n} \binom{n-1}{k-1} p^k (1-p)^{n-k} \lambda^k \frac{e^{-\lambda}}{k!}. \tag{2.1}$$

Theorem 2.1 $g_1, g_2, g_3, ...,$ *forms a log-concave sequence.*

Proof Simple calculation shows that for n = 1,2,...,

$$\begin{aligned} g_n &= \sum_{k=1}^{n} \binom{n-1}{k-1} p^k (1-p)^{n-k} (\lambda)^k \frac{e^{-\lambda}}{k!} \\ &= \sum_{k=1}^{n} \binom{n-1}{k-1} \left(\frac{p\lambda}{1-p}\right)^k (1-p)^n \frac{e^{-\lambda}}{k!}. \end{aligned} \tag{2.2}$$

Let $\frac{p\lambda}{1-p} = x$. Then we can write

$$g_n = \sum_{k=1}^{n} \binom{n-1}{k-1} x^k (1-p)^n \frac{e^{-\lambda}}{k!}. \tag{2.3}$$

If we use (2.3), the relation ${g_n}^2 \geq g_{n-1}g_{n+1}$, $n = 2, 3, ...$ can be rewritten as:

$$\left(\sum_{k=1}^{n} \binom{n-1}{k-1} \frac{x^k}{k!}(1-p)^n e^{-\lambda}\right)^2 \geq \left(\sum_{k=1}^{n-1} \binom{n-2}{k-1} \frac{x^k}{k!}(1-p)^{n-1} e^{-\lambda}\right) \times \left(\sum_{k=1}^{n+1} \binom{n}{k-1} \frac{x^k}{k!}(1-p)^{n+1} e^{-\lambda}\right). \tag{2.4}$$

If we divide by $(1-p)^{2n}e^{2\lambda}$ on both sides, then we obtain

$$\left(\sum_{k=1}^{n} \binom{n-1}{k-1} \frac{x^k}{k!}\right)^2 \geq \left(\sum_{k=1}^{n-1} \binom{n-2}{k-1} \frac{x^k}{k!}\right) \left(\sum_{k=1}^{n+1} \binom{n}{k-1} \frac{x^k}{k!}\right). \tag{2.5}$$

Let $B_n = \sum_{k=1}^{n} \binom{n-1}{k-1} \frac{x^k}{k!}$ for $n = 1, 2, ...$ It suffices to show that $\{B_n\}_{n=1}^{\infty}$ is a logconcave sequence.

We have the following identity known as Pascal's rule:

$$\binom{n}{n-k} = \binom{n-1}{k-1} + \binom{n-1}{n-k-1}. \tag{2.6}$$

If we apply it in the formula for B_n, then we get:

$$\begin{aligned} B_n &= \sum_{k=1}^{n} \frac{x^k}{k!} \left[\binom{n}{n-k} - \binom{n-1}{n-k-1}\right] \\ &= \sum_{k=1}^{n} \frac{x^k}{k!} \binom{n}{n-k} - \sum_{k=1}^{n-1} \frac{x^k}{k!} \binom{n-1}{n-k-1} \\ &= \sum_{k=0}^{n} \frac{x^k}{k!} \binom{n}{n-k} - \sum_{k=0}^{n-1} \frac{x^k}{k!} \binom{n-1}{n-k-1} \\ &= L_n(-x) - L_{n-1}(-x), \end{aligned} \tag{2.7}$$

where

$$L_n^{(\alpha)}(x) = \sum_{k=0}^{n} \binom{n+\alpha}{n-k} \frac{(-x)^k}{k!},$$

are the Laguerre polynomials [10]. For the case of $\alpha = 0$, we have $L_n^{(0)}(x) = L_n(x)$. It is well-known (see,e.g.,Riordan, 1968) that the Laguerre polynomials satisfy the following recurrence equation:

$$x\frac{d}{dx}L_n(x) = nL_n(x) - nL_{n-1}(x). \tag{2.8}$$

This implies that $B_n = -x\dfrac{L_n'(-x)}{n}$. In order to prove the logconcavity of $B_n(x)$, it is enough to prove the logconcavity of $b_n(x) = -\dfrac{L_n'(-x)}{n}$. We use another recurrence formula for Laguerre polynomials from Riordan (1968):

$$D^p L_n^{(\alpha)}(x) = (-1)^p L_{n-p}^{(\alpha+p)}(x). \tag{2.9}$$

For the case of $p = 1$ and $\alpha = 0$, we have $L_n'(x) = -L_{n-1}^{(1)}(x)$. This implies that $b_n = \dfrac{L_{n-1}^{(1)}(-x)}{n}$.

To prove the logconcavity of $\{b_n\}$, we make use of the result derived in Simic (2003) that the sequence $\left\{L_n^{(a)}x/\binom{n+a}{n}\right\}$ is log-concave for $a > -1$ and $x \in \mathbb{R}$. It follows that $\{b_n\}$ is a logconcave sequence. $\square$

Based on Theorem 2.1, an interesting inequality on the confluent hypergeometric function can be derived.

Corollary 2.2 *For $k > 0, x > 0$: ${}_1F_1(1+k;2;x)^2 \geq {}_1F_1(k;2;x){}_1F_1(2+k;2;x)$.*

Proof Let $X_1, X_2, ...,$ be i.i.d geometrically distributed random variables with support $\{0, 1, 2, ..\}$, i.e., $P(X_i = n) = pq^n (n = 0, 1, 2, ...)$. Then the following formula holds for the probability mass function of the compound Poisson distributions:

$$P(S = n) = \sum_{k=1}^{\infty} \binom{n+k-1}{n} p^k (1-p)^n \lambda^k \frac{e^{-\lambda}}{k!}.$$

Let $x = p\lambda$ and $g_n = P(S = n)$, then we have

$$g_n = \sum_{k=1}^{\infty} \binom{n+k-1}{n} x^k (1-p)^n \frac{e^{-\lambda}}{k!}. \tag{2.10}$$

This distribution has connection with the confluent hypergeometric function:

$${}_1F_1(a;c;x) = \sum_{k=0}^{\infty} \frac{(a)_k}{(c)_k}\frac{x^k}{k!}, \tag{2.11}$$

where $(a)_k = a(a+1)...(a+k-1)$ is the Pochhammer's symbol. It can easily be shown that

$$\sum_{k=1}^{\infty} \binom{n+k-1}{n} \frac{x^k}{k!} = {}_1F_1(1+k;2;x). \tag{2.12}$$

Theorem 2.1 and equation (2.12) imply the statement. $\square$

3 Logconcavity of the compound Poisson distribution with Poisson distributed terms

If $X_1, X_2, ...$ are i.i.d Poisson distributed random variables with parameter $\mu > 0$, one can easily verify that

$$P(S = n) = \sum_{k=1}^{\infty} \frac{(k\mu)^n e^{-k\mu}}{n!} \frac{\lambda^k e^{-\lambda}}{k!}, \qquad n = 0, 1, 2, ... \tag{3.1}$$

Let $x = e^{-\mu}\lambda$. Then we can write:

$$g_n = P(S = n) = \frac{\mu^n e^{-\lambda}}{n!} \sum_{k=1}^{\infty} k^n \frac{x^k}{k!}. \tag{3.2}$$

We have the following

Theorem 3.1 *The sequence $\{g(n)\}_{n=1}^{\infty}$ is logconcave.*

Proof Equation (3.2) can be rewritten as

$$g_n = \mu^n e^{-\lambda} e^x \frac{B_n(x)}{n!}, \qquad n = 1, 2, ...,$$

where the $B_n(k)$ are the Bell polynomials. It suffices to show that the sequence $\{B_n(x)/n!\}_{n=1}^{\infty}$ is logconcave.

It is well-known that (Bender-Canfield's Theorem) if $\{1, Z_1, Z_2, ...\}$ is a logconcave sequence of non-negative real numbers and the sequence $\{a(n)\}_{n=0}^{\infty}$ is defined by

$$\sum_{n=0}^{\infty} \frac{a(n)}{n!} y^n = exp\left(\sum_{j=1}^{\infty} \frac{Z_j}{j} y^j\right), \tag{3.3}$$

then the sequence $\{a(n)/n!\}_{n=0}^{\infty}$ is logconcave and the sequence $\{a(n)\}_{n=0}^{\infty}$ is logconvex.

Note that $e^{(e^y-1)x} = exp\left(\sum_{j=1}^{\infty} \frac{x}{j!} y^j\right)$. In addition, we have

$$e^{(e^y-1)x} = \sum_{n=0}^{\infty} \frac{B_n(x)}{n!} y^n, \tag{3.4}$$

thus,

$$exp\left(\sum_{j=1}^{\infty} \frac{x}{j!} y^j\right) = \sum_{n=0}^{\infty} \frac{B_n(x)}{n!} y^n. \tag{3.5}$$

Let $Z_j = \dfrac{x}{(j-1)!}$ for $j \geq 1$. It is easy to check that the sequence $1, Z_1, Z_2...$ is logconcave for $x \geq 1$ or $x \leq 0$. Thus, according to the Bender-Canfield's Theorem, $\{B_n(x)/n!\}_{n=1}^{\infty}$ is logconcave sequence for $x \geq 1$ or $x \leq 0$.

If $0 < x < 1$, we can always find $u \geq 1$ and $v \leq 0$ such that $x = u + v$. Furthermore, it is proved in [3] that we have the following identity:

$$B_n(u+v) = \sum_{k=0}^{n} \binom{n}{k} B_k(u) B_{n-k}(v), \tag{3.6}$$

hence, equation (3.6) can be rewritten as

$$\frac{B_n(u+v)}{n!} = \sum_{k=0}^{n} \frac{B_k(u)}{k!} \frac{B_{n-k}(v)}{(n-k)!}. \tag{3.7}$$

In other words, $B_n(x)/n!$ is the convolution of two logconcave sequences $B_n(u)/n!$ and $B_n(v)/n!$. Thus, the sequence $\{B_n(x)/n!\}_{n=1}^{\infty}$ is logconcave for any $x \in \mathbb{R}$. $\square$

4 Logconcavity of the compound Poisson distribution with exponentially distributed terms

If the terms in a compound Poisson random variable are continuously distributed, then the probability distribution of S is of a mixed type. It has positive probability mass at 0 and has a continous part with a p.d.f. M.Oschwald (1985) showed that the p.d.f of the continuous part of the probability distribution of S is logconcave, if the terms X_i are exponentially distributed. Oschwald published his result only in his thesis, that he wrote under the supervision of the second author of this paper. Here we reproduce his proof, for the reader's convenience, in a slightly more general form: we assume that N has logconcave distribution that is not necessarily Poisson. Suppose that the terms have the exponential p.d.f:

$$\mu e^{-\mu x}. \tag{4.1}$$

Then the p.d.f of the continuous part of S is

$$f(x) = \sum_{i=1}^{\infty} p_i \frac{\mu^i}{(i-1)!} e^{-\mu x} x^{i-1}, \qquad x > 0, \tag{4.2}$$

and $P(S = 0) = p_0$, where $p_0 = P(N = 0)$.

Theorem 4.1 *If N has logconcave distribution on the set of nonnegative integers and the terms X_i, $i = 1, 2, ...$ are exponentially distributed, then the continuous part of the distribution of S is logconcave.*

Proof It suffices to prove that

$$(lnf(x))'' \leq 0.$$

Simple calculation shows that

$$(lnf(x))'' = \left(-\mu + \frac{\displaystyle\sum_{i=1}^{\infty} p_i \frac{\mu^i}{(i-1)!}(i-1)x^{i-2}}{\displaystyle\sum_{i=1}^{\infty} p_i \frac{\mu^i}{(i-1)!}x^{i-1}} \right)'. \tag{4.3}$$

The derivative of the left-hand-side, with respect to x, equals

$$\begin{aligned}(lnf(x))'' &= [\sum_{i=1}^{\infty} p_i \frac{\mu^i}{(i-1)!}x^{i-1} \sum_{i=1}^{\infty} p_i \frac{\mu^i}{(i-1)!}(i-2)(i-1)x^{i-3} \\ &\quad - \left(\sum_{i=1}^{\infty} p_i \frac{\mu^i}{(i-1)!}(i-1)x^{i-2} \right)^2] : \left(\sum_{i=1}^{\infty} p_i \frac{\mu^i}{(i-1)!}x^{i-1} \right)^2. \end{aligned} \tag{4.4}$$

If we start the summation from $i = 0$ and use the Cauchy product formula, then the right-hand-side of (4.4) becomes

$$\sum_{j=0}^{\infty} \sum_{i=0}^{j} [p_{i+1}p_{j+1-i}x^{j-2} \frac{\mu^{(j+2)}}{i!(j-i)!} i(i-1) - p_{i+1}p_{j+1-i}x^{j-2} \frac{\mu^{(j+2)}}{i!(j-i)!} i(j-i)] =$$

$$\sum_{j=0}^{\infty} x^{j-2}\mu^{j+2} \sum_{i=0}^{j} p_{i+1}p_{j+1-i} \frac{(i-1)i - (j-i)i}{i!(j-i)!}.$$

Since x and μ are positive, it suffices to prove that the inner sum is nonpositive. Thus, it is enough to consider two cases as follows.

Case 1: j is even. We combine the first term with the last term, the second term with the second to the last term, etc. In this case we have

$$\sum_{i=1}^{j} p_{i+1}p_{j+1-i} \frac{(i-1)i - (j-i)i}{i!(j-i)!}$$

$$\begin{aligned} &= \sum_{i=1}^{j/2} (p_{i+1}p_{j+1-i} \frac{2i-j-1}{i!(j-i)!} + p_{j+2-i}p_i \frac{j-2i+1}{i!(j-i)!}) \\ &= \sum_{i=1}^{j/2} (p_{i+1}p_{j+1-i} - p_i p_{j+2-i}) \frac{2i-j-1}{i!(j-i)!}. \end{aligned} \tag{4.5}$$

Case 2: j is odd. We combine the first term with the last term, the second term with the second to the last term, etc. In this case we have

$$\sum_{i=1}^{j} p_{i+1}p_{j+1-i}\frac{(i-1)i-(j-i)i}{i!(j-i)!}$$

$$\begin{aligned}
&= \sum_{i=0}^{(j-1)/2} p_{i+1}p_{j+1-i}\left(\frac{i(2i-j-1)}{i!(j-i)!}+\frac{(j-i)(j-2i-1)}{i(j-i)!}\right)\\
&= \sum_{i=1}^{(j-1)/2} p_{i+1}p_{j+1-i}\frac{i(2i-j-1)}{i!(j-i)!}+\sum_{i=0}^{(j-3)/2} p_{i+1}p_{j+1-i}\frac{(j-i)(j-2i-1)}{i!(j-i)!}\\
&= \sum_{i=1}^{j/2}\left(p_{i+1}p_{j+1-i}\frac{2i-j-1}{(i-1)!(j-i)!}+p_ip_{j+i-2}\frac{j-2i-1}{(i-1)!(j-i)!}\right). \qquad (4.6)
\end{aligned}$$

Since in both (4.5) and (4.6), the factor $(2i-j-1)$ is negative and the logconcavity of $\{p_n\}_{n=0}^{\infty}$ implies that $p_{i+1}p_{j+i-1}-p_ip_{j+2-i}$ is nonnegative for $i=1,...,[j/2]$, the assertion follows. □

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