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András Prékopa Gergely Mádi-Nagy

Abstract. A function $u(z)$ is a utility function if $u'(z) \geq 0$. It is called risk averse if we also have $u''(z) \leq 0$. Some authors, however, require that $u^{(i)}(z) \geq 0$ if i is odd and $u^{(i)}(z) \leq 0$ if i is even. The notion of a multiattribute utility function can be defined by requiring that it is increasing in each variable and concave as an s-variate function. A stronger condition, similar to the one in case of a univariate utility function, requires that, in addition, all partial derivatives of total order m should be nonnegative if m is odd and nonpositive if m is even. In this paper we present a class of functions in analytic form such that each of them satisfies this stronger condition. We also give sharp lower and upper bounds for $E[u(X_1, \ldots, X_s)]$ under moment information with respect to the joint probability distribution of the random variables $X_1, \ldots, X_s$ assumed to be discrete and representing wealths.

1 Introduction

The most general definition of a utility function $u(z),\ z \geq 0$ only requires that it should be an increasing function, i.e., $u'(z) > 0$. It is called risk averse, if we also have $u''(z) < 0$ which means that the function is also concave.

Pratt (1964) and Arrow (1970) stress the importance of utility functions with decreasing risk aversion. If we take the Arrow-Pratt measure of absolute risk aversion:

$$-\frac{u''(z)}{u'(z)}, \tag{1.1}$$

then the above requirement implies that $u'''(z) > 0$. More generally, we may require:

$$u^{(m+1)}(z) \text{ does not change sign} \tag{1.2}$$

or

$$(-1)^{n-1}u^{(n)}(z) > 0,\ n = 1, \ldots, m+1 \tag{1.3}$$

or

$$(-1)^{n-1}u^{(n)}(z) > 0,\ n = 1, 2, \ldots. \tag{1.4}$$

Utility functions satisfying (1.4) are called *mixed* by Caballe and Pomansky (1996). For economic justification see Ingersoll (1987). Relation (1.4) means that $u(-z)$ is a *completely monotone function*. In view of this, a utility function satisfying (1.3) will be called *monotone of order* $m+1$.

If $u(z)$ is a mixed utility function with $u(0) = 1$, then, by a well known theorem (see Feller, 1971, p.439) it admits the representation

$$u(z) = \int_0^\infty e^{-sz} dF(s), \tag{1.5}$$

where $F(s)$ is a c.d.f. on $[0, \infty)$.

Examples of mixed utility function are:

$$u(z) = a \log\left(1 + \frac{z}{b}\right),\ u(z) = ae^{-bz},$$

where $a > 0$, $b > 0$. The next example is, on the other hand, a utility function that satisfies (1.3):

$$u(z) = -\int_z^\infty \int_{x_m}^\infty \cdots \int_{x_2}^\infty [1 - F(x_1)]\, dx_1 \cdots dx_m, \tag{1.6}$$

where F is a c.d.f. The integral in (1.6) is finite if and only if

$$\int_0^\infty x^{m+1} dF(x) < \infty.$$

The value $-u(z)$ has a simple economic interpretation. First we remark that, as it is easy to show, the following equation holds true:

$$\int_z^\infty \int_{x_m}^\infty \cdots \int_{x_2}^\infty [1 - F(x_1)]\, dx_1 \cdots dx_m = \int_z^\infty (x - z)^{m+1} dF(x). \tag{1.7}$$

If F is the c.d.f. of a random demand and the supply is equal to z, then the value in (1.7) is the expected penalty of the unserved demand with penalty function

$$q(x) = \begin{cases} 0 \text{ if } x \le z \\ (x-z)^{m+1} \text{ if } x > z. \end{cases}$$

If $m = 0$ then it is the expected unserved demand.

The value (1.7) also appears in stochastic ordering theory. The random variable X dominates the random variable Y in order m, in symbols: $X \succ_m Y$, iff

$$\int_z^\infty (x-z)^m dF(x) \ge \int_z^\infty (x-z)^m dG(x), \tag{1.8}$$

where F, G are the c.d.f's of X and Y, respectively. By equation (1.7), the inequality (1.8) can be expressed in terms of the m-fold integrals.

Brochet, Cox and Witt (1986) apply a utility function $u(z)$ with $u^{(4)} < 0$ in connection with an insurance problem. They assume the existence of the first three moments of the random loss L, and apply the Markov-Krein theorem concerning Chebyshev systems. The result is that if we take

$$maxE(u(L)) \text{ (or } minE(u(L)))$$

on those set of c.d.f.'s that have the prescribed three moments, then the extremal distribution does not depend on the special utility function. It is uniquely determined by the requirement that $u^{(4)} < 0$.

More generally, if (1.2) holds true, and the first m moments of a random variable X are known, the, by the use of the methodology of the Markov moment problem (see Krein, Nudelmann, 1977) we can obtain extremal distributions to represent $maxE(u(X))$ and $minE(u(X))$. Those are called upper and lower principal representations of the moments $\mu_1, \ldots, \mu_m$. Again, under the mentioned condition, these extremal distributions do not depend on the utility function $u(z)$.

Thus, in order to create upper and lower bounds for the expectation $E(u(X))$ taken with the true distribution, we only have to assess the values of the utility function $u(z)$ at the supports of the extremal distributions.

More elegant is the bounding procedure if X has an unknown discrete distribution with known support. In this case the methodology of the discrete moment problem (see Prékopa, 1990) can be applied to obtained bounds for $E(u(X)))$. That possibility obviously carries over to utility functions satisfying (1.3) or (1.4).

Multiattribute utility functions have also been extensively researched, see, e.g. Keeney and Raiffa (1976), Dyer and Sarin (1979), Dyer, Fishburn, Steuer, Wallenius and S. Zionts (1992). In most cases relatively simple sums of products of single attribute utility functions provide us with multiattribute ones. However, large collection of analytic formulas that can serve for applications does not exist. In this paper our purpose is to improve on the situation and introduce a class of multiattribute utility functions in such a way that we assume the knowledge of s single attribute utility functions, each satisfying relations (1.4) and then couple them into one s-attribute utility function.

The risk averse multiattribute utility function may be defined in such a way that $u(z_1, \ldots, z_s)$ is increasing in each variable and concave as an s-variate function. In addition, we may require

$$\frac{\partial^{i_1+\cdots+i_s} u(z_1, \ldots, z_s)}{\partial z_1^{i_1} \cdots \partial z_s^{i_s}} \text{ does nor change sign if } i_1 + \cdots + i_s = m + 1 \tag{1.9}$$

or

$$(-1)^{i_1+\cdots+i_s-1} \frac{\partial^{i_1+\cdots+i_s} u(z_1, \ldots, z_s)}{\partial z_1^{i_1} \cdots \partial z_s^{i_s}} > 0, \ 1 \le i_1 + \cdots + i_s \le m + 1 \tag{1.10}$$

or

$$(-1)^{i_1+\cdots+i_s-1} \frac{\partial^{i_1+\cdots+i_s} u(z_1, \ldots, z_s)}{\partial z_1^{i_1} \cdots \partial z_s^{i_s}} > 0, \ 1 \le i_1 + \cdots + i_s. \tag{1.11}$$

These are multivariate counterparts of relations (1.2), (1.3) and (1.4), respectively.

Our class of multiattribute utility functions is given by

Definition 1.1 *Let $k \ge 1$ and D an open convex set. We define the utility function u as:*

$$u(z_1, \ldots, z_s) := log \left[k(e^{g_1(z_1)} - 1) \cdots (e^{g_s(z_s)} - 1) - 1 \right], \tag{1.12}$$

where for every $(z_1, \ldots, z_n) \in D$ the following conditions hold:

$$e^{g_j(z_j)} > 2, \ j = 1, \ldots, s, \tag{1.13}$$

$$\begin{array}{rcl} g_j'(z_j) & > & 0 \\ g_j^{(i)}(z_j) & \ge & 0, \ \textit{if } i > 1 \textit{ and is odd} \\ g_j^{(i)}(z_j) & \le & 0, \ \textit{if } i \textit{ is even} \\ & & j = 1, \ldots, s. \end{array} \tag{1.14}$$

The $g_j(z_j)$ functions can be chosen eg. from the following type of functions:

$$\begin{array}{rl} a \log\left(1 + \frac{z}{b}\right), & \text{where } a > 0, \ b > 0, \\ ae^{-bz}, & \text{where } a > 0, \ b > 0, \\ a_n z^n + \cdots + a_1 z + a_0, & \text{with suitably chosen coefficients.} \end{array}$$

It is obvious that functions (1.12) are strictly increasing in each variable in their domain. It is also true, but no longer obvious, that the functions are concave. We prove it in Section 2, together with other properties of the functions (1.12). In Section 3 we look at the maximization and minimization problems of $E(u(X_1, \ldots, X_s))$, where $X_1, \ldots, X_s$ are discrete random variables with known supports and with some known univariate and multivariate moments. The extremal distributions serve for bounding the above expectation. In Section 4 we present numerical examples and, finally, in Section 5 we draw some conclusions.

2 Properties of the multiattribute utility functions (1.12)

First we need the notion of a logconcave function.

Definition 2.1 *Let $E \subset \mathbb{R}^s$ be a convex set and $f \geq 0$ a function defined on E. The function f is said to be logconcave if for any $\boldsymbol{x}, \boldsymbol{y} \in E$ and $0 \leq \lambda \leq 1$ we have the relation*

$$f\left(\lambda \boldsymbol{x} + (1-\lambda)\boldsymbol{y}\right) \geq \left[f\left(\boldsymbol{x}\right)\right]^{\lambda} \left[f\left(\boldsymbol{y}\right)\right]^{1-\lambda}. \tag{2.1}$$

If in (2.1) the opposite inequality holds, we call the function logconvex on E.

If f is positive valued on E, then its logconcavity is equivalent to the concavity of $log f$.
If f is logconcave on E, then its definition can be extended to the entire space $\mathbb{R}^s$ by setting $f(\boldsymbol{z}) = 0,\ \boldsymbol{z} \notin E$.

The product of any number of logconcave functions is logconcave. The logconcavity property, however, does not carry over for sums. Product and sums of logconvex functions, defined on the same convex set, are also logconvex. However, the definition of a logconvex function, on a convex set E, cannot be extended, in general, to the entire space $\mathbb{R}^s$.

The following statement holds true:

Theorem 2.1 *(Prékopa, 1995, p.324, Lemma 11.2.2). If f is logconcave on $E \subset \mathbb{R}^s$ and $p > 0$, then $f(\boldsymbol{z}) - p$ is logconcave on the set*

$$\left\{\boldsymbol{z} | f\left(\boldsymbol{z}\right) \geq p\right\}. \tag{2.2}$$

The proof is simple, we only have to use the arithmetic mean - geometric mean inequality. It turns out from the proof of the theorem that if f is strictly logconcave, i.e., $f\left(\lambda \boldsymbol{x} + (1-\lambda)\boldsymbol{y}\right) > \left[f\left(\boldsymbol{x}\right)\right]^{\lambda} \left[f\left(\boldsymbol{y}\right)\right]^{1-\lambda}$ whenever $\boldsymbol{x} \neq \boldsymbol{y}$ and $0 < \lambda < 1$, then the same holds for $f(\boldsymbol{z}) - p$ on the set (2.2).

Based on Theorem 2.1 we prove

Theorem 2.2 *The functions (1.12) are concave on D.*

Proof. In view of (1.14), it follows that each function

$$e^{g_j(z_j)},\ j = 1, \ldots, s$$

is logconcave on D. By Theorem 2.1 this holds for the functions

$$e^{g_j(z_j)} - 1$$

as well. This implies that

$$k(e^{g_1(z_1)} - 1) \cdots (e^{g_s(z_s)} - 1)$$

is logconcave on D. For every $z \in D$ its value is greater than 1, hence the repeated application of Theorem 2.1 proves the assertion. □

The next theorem presents our central result in connection with functions (1.12).

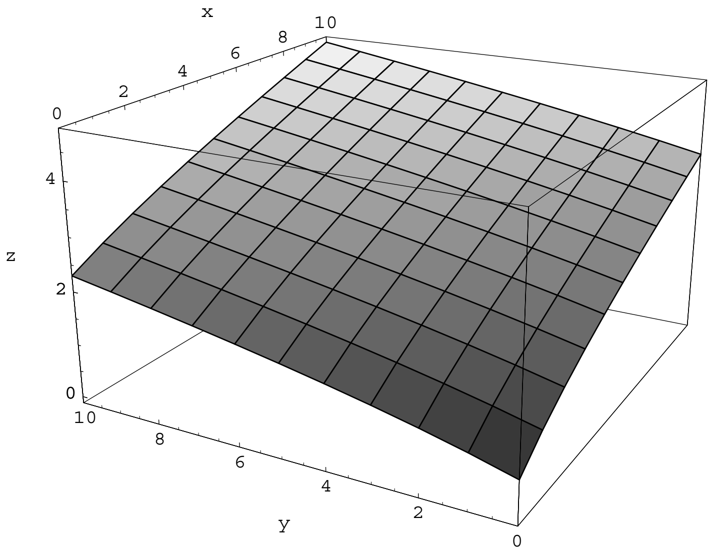


Figure 1: Graph of the utility function (1.12), in case of $s=2$, $g_1(z_1)=\frac{1}{5}z_1+1$, $g_2(z_2)=\frac{1}{10}z_2+1$.

Theorem 2.3 *For every $\boldsymbol{z}=(z_1,\ldots,z_s)\in D$ we have*

$$\begin{aligned} &\frac{\partial^{i_1+\cdots+i_s}u}{\partial z_1^{i_1}\cdots\partial z_s^{i_s}}>0,\ \textit{if } i_1+\cdots+i_s \textit{ is odd,}\\ \textit{and}\quad & \\ &\frac{\partial^{i_1+\cdots+i_s}u}{\partial z_1^{i_1}\cdots\partial z_s^{i_s}}<0,\ \textit{if } i_1+\cdots+i_s \textit{ is even.}\end{aligned} \tag{2.3}$$

First we prove the assertion for the case of $g_j(z_j)=z_j$, $j=1,\ldots,s$.

Lemma 2.4 *Property (2.3) holds for*

$$u(\boldsymbol{z}):=u_0(\boldsymbol{z}):=log\left[k(e^{z_1}-1)\cdots(e^{z_s}-1)-1\right]. \tag{2.4}$$

We need the following

Assertion 2.5 *Consider the function*

$$f(z)=log\left(k(e^z-1)-1\right),$$

where $k\geq 1$ constant and $z>log2$. We assert that

$$\begin{aligned}\frac{df}{dz}&=1+\frac{k+1}{ke^z-(k+1)}\\ \frac{d^if}{dz^i}&=(-1)^{i-1}\sum_{h=1}^{i}a_h^{(i)}\left(\frac{k+1}{ke^z-(k+1)}\right)^h,\ i=2,3,\ldots,\end{aligned} \tag{2.5}$$

where the numbers $a_h^{(i)}$ are defined by

$$
\begin{aligned}
a_1^{(i)} &= 1,\ \ i = 2, 3, \ldots \\
a_1^{(2)} &= a_2^{(2)} = 1 \\
a_h^{(i)} &= a_{h-1}^{(i-1)}(h-1) + a_h^{(i-1)}h,\ \ h = 2, \ldots, i-1 \\
a_i^{(i)} &= a_{i-1}^{(i-1)}(i-1).
\end{aligned}
\tag{2.6}
$$

Proof of Assertion 2.5. The assertion is trivial for $i = 1$ and can easily be checked for $i = 2$. We use induction and assume that the equation in the second line of (2.5) holds for some $i \geq 2$. It follows that

$$
\begin{aligned}
\frac{d^{(i+1)}f}{dz^{i+1}} &= (-1)^i \sum_{h=1}^{i} a_h^{(i)} h \left(\frac{k+1}{ke^z - (k+1)} \right)^{h-1} \frac{(k+1)ke^z}{(ke^z - (k+1))^2} \\
&= (-1)^i \sum_{h=1}^{i} a_h^{(i)} h \left(\frac{k+1}{ke^z - (k+1)} \right)^{h-1} \left(\frac{k+1}{ke^z - (k+1)} + \left(\frac{k+1}{ke^z - (k+1)} \right)^2 \right) \\
&= (-1)^i \left[\sum_{h=1}^{i} a_h^{(i)} h \left(\frac{k+1}{ke^z - (k+1)} \right)^{h} + \sum_{h=1}^{i} a_h^{(i)} h \left(\frac{k+1}{ke^z - (k+1)} \right)^{h+1} \right] \\
&= (-1)^i \sum_{h=1}^{i} \left(a_h^{(i)} h + a_{h-1}^{(i)}(h-1) \right) \left(\frac{k+1}{ke^z - (k+1)} \right)^{h} \\
&\quad + (-1)^i a_i^{(i)} i \left(\frac{k+1}{ke^z - (k+1)} \right)^{i+1} \\
&= (-1)^i \sum_{h=1}^{i+1} a_h^{(i+1)} \left(\frac{k+1}{ke^z - (k+1)} \right)^{h}.
\end{aligned}
$$

This proves the assertion. □

Assertion 2.5 implies that the odd order derivatives of f are positive and the even order derivatives of f are negative. It is worth to mention that if we apply the equation in the second line of (2.5) for the case of $i = 1$, then we obtain the formula in the first line without the 1.

Proof of Lemma 2.4. Let

$$
k_1 = k(e^{z_2} - 1) \cdots (e^{z_s} - 1)
$$

and hold the values of $z_2, \ldots, z_s$ fixed. Then $u_0(z_1, \ldots, z_s)$ is a function of the single variable z_1 and has the form:

$$
u_0(z_1, \ldots, z_s) = log\left(k_1(e^{z_1} - 1) - 1 \right). \tag{2.7}
$$

By Assertion 2.5 we have

$$
\begin{aligned}
\frac{\partial u_0}{\partial z_1} &= 1 + \left(\frac{k_1 + 1}{k_1 e^{z_1} - (k_1 + 1)} \right) \\
\frac{\partial^{i_1} u_0}{\partial z_1^{i_1}} &= (-1)^{i_1 - 1} \sum_{h=1}^{i_1} a_h^{(i_1)} \left(\frac{k_1 + 1}{k_1 e^{z_1} - (k_1 + 1)} \right)^{h},\ \ i_1 \geq 2.
\end{aligned}
\tag{2.8}
$$

Similar formulas can be written up for the derivatives of $u_0(z_1,\dots,z_s)$ with respect to any of its variables. This implies that the assertion of the lemma holds for that special case where we take derivatives only with respect to a single variable.

To prove the assertion for mixed derivatives we rewrite the equations in (2.8) and take the derivatives of order i_2 with respect to z_2 etc. Let us introduce the notation

$$k_2 = k(e^{z_1}-1)(e^{z_3}-1)\cdots(e^{z_s}-1).$$

We have the equation

$$\begin{aligned} k_1e^{z_1}-(k_1+1) &= k_1(e^{z_1}-1)-1 \\ &= k(e^{z_1}-1)(e^{z_2}-1)\cdots(e^{z_s}-1)-1 \\ &= k_2(e^{z_2}-1)-1 = k_2e^{z_2}-(k_2+1). \end{aligned}$$

Using this, (2.8) can be written as:

$$\begin{aligned} \frac{\partial u_0}{\partial z_1} &= 1+\frac{k_1+1}{k_2+1}\frac{k_2+1}{k_2e^{z_2}-(k_2+1)} \\ \frac{\partial^{i_1} u_0}{\partial z_1^{i_1}} &= (-1)^{i_1-1}\sum_{h=1}^{i_1} a_h^{(i_1)}\left(\frac{k_1+1}{k_2+1}\right)^h\left(\frac{k_2+1}{k_2e^{k_2}-(k_2+1)}\right)^h,\ i\ge 2. \end{aligned} \tag{2.9}$$

Now we fix the values $z_1,z_3,\dots,z_s$ and consider the functions (2.9) as functions of the single variable z_2. If we take the first derivatives with respect to z_2, then for the term corresponding to the subscript h we obtain

$$\begin{aligned} &\frac{\partial}{\partial z_2}\left(\frac{k_2+1}{k_2e^{z_2}-(k_2+1)}\right)^h \\ &= -h\left(\frac{k_2+1}{k_2e^{z_2}-(k_2+1)}\right)^{h-1}\frac{(k_2+1)k_2e^{z_2}}{(k_2e^{z_2}-(k_2+1))^2} \\ &= -h\left(\frac{k_2+1}{k_2e^{z_2}-(k_2+1)}\right)^{h-1}\left(\frac{k_2+1}{k_2e^{z_2}-(k_2+1)}+\left(\frac{k_2+1}{k_2e^{z_2}-(k_2+1)}\right)^2\right) \\ &= -h\left[\left(\frac{k_2+1}{k_2e^{z_2}-(k_2+1)}\right)^h+\left(\frac{k_2+1}{k_2e^{z_2}-(k_2+1)}\right)^{h+1}\right]. \end{aligned}$$

If we take the further derivative until of order i_2, we can see that

$$\frac{\partial^{i_1+i_2}u_0}{\partial z_1^{i_1}\partial z_2^{i_2}} = (-1)^{i_1+i_2-1}\times(\text{positive value}).$$

Proceeding this way along the derivatives with respect to $z_3,\dots,z_s$, the lemma follows. □

Regarding the general case, we use the following

Assertion 2.6 *If the univariate function $g(z)$ has the property*

$$\begin{aligned} g'(z) &> 0 \\ g^{(i)}(z) &\ge 0,\ \text{if } i>1 \text{ and is odd} \\ g^{(i)}(z) &\le 0,\ \text{if } i \text{ is even} \end{aligned} \tag{2.10}$$

on the set D and the function $f(z)$ has the property

$$\begin{array}{rcl} f^{(i)}(z) & > & 0, \text{ if } i \text{ is odd} \\ f^{(i)}(z) & < & 0, \text{ if } i \text{ is even} \end{array} \tag{2.11}$$

on the set $\{g(z)|z \in D\}$, then (2.11) is also true for their composition, i.e.

$$\begin{array}{rcl} [f(g(z))]^{(i)} & > & 0, \text{ if } i \text{ is odd} \\ [f(g(z))]^{(i)} & < & 0, \text{ if } i \text{ is even} \end{array} \tag{2.12}$$

on the set D.

Proof of Assertion 2.6. It is easy to see (by induction) that

$$[f\,(g\,(z))]^{(i)} = \sum_{l=1}^{i} \sum_{l+k_1+\cdots+k_l=i} f^{(l)}(g(z))g^{(k_1+1)}(z)\cdots g^{(k_l+1)}(z). \tag{2.13}$$

The condition in the second sum is equvivalent with

$$(l-1) + k_1 + \cdots + k_s = i - 1. \tag{2.14}$$

If i is odd (even) $\Longrightarrow$ $i-1$ is even (odd) $\Longrightarrow$ the number of odd terms in the sum (2.14) is always even (odd) $\Longrightarrow$ by (2.10) and (2.11) the number of nonpositive terms is even (odd) in each product of (2.13) $\Longrightarrow$ each product of (2.13) is nonnegative (nonpositive). Since $f^{(i)}(g(z))g'(z)\cdots g'(z) > 0(< 0)$ is always in the sum (2.13) then $[f(g(z))]^{(i)} > 0(< 0)$ follows. □

Proof of Theorem 2.3. Introduce the following functions

$$u_k(z_1, \ldots, z_s) := u_0(g_1(z_1), \ldots, g_k(z_k), z_{k+1}, \ldots, z_s),\ k = 0, 1, \ldots, s. \tag{2.15}$$

We assert that the above functions have property (2.3) and since $u(z_1, \ldots, z_s) = u_s(z_1, \ldots, z_s)$ this proves the theorem. We prove it by induction. For $k = 0$ relations (2.3) hold, by Lemma 2.4. Assume that (2.3) holds for k, i.e.,

$$\frac{\partial^{i_1+\cdots+i_s} u_k}{\partial z_1^{i_1} \cdots \partial z_s^{i_s}} = (-1)^{i_1+\cdots+i_s-1} \times (\text{positive value}). \tag{2.16}$$

Then for $k+1 \leftarrow k$, if we consider the univariate function

$$f(z_{k+1}) := (-1)^{i_1+\cdots+i_k+i_{k+2}+\cdots+i_s} \frac{\partial^{i_1+\cdots+i_k+i_{k+2}+\cdots+i_s} u_k}{\partial z_1^{i_1} \cdots \partial z_k^{i_k} \partial z_{k+2}^{i_{k+2}} \cdots \partial z_s^{i_s}}(z_1, \ldots, z_s) \tag{2.17}$$

for fixed $z_1, \ldots, z_k, z_{k+2}, \ldots, z_s$ values, then $f(z_{k+1})$ satisfies property (2.11) because of (2.16). Also, $g_{k+1}(z_{k+1})$ satisfies (2.10), and by Assertion 2.6 $f(g_{k+1}(z_{k+1}))$ has property (2.12). From this we derive

$$\begin{aligned} \frac{\partial^{i_1+\cdots+i_s} u_{k+1}}{\partial z_1^{i_1} \cdots \partial z_s^{i_s}} &= (-1)^{i_1+\cdots+i_k+i_{k+2}+\cdots+i_s} f^{(i_{k+1})}(g_{k+1}(z_{k+1})) = \\ &(-1)^{i_1+\cdots+i_s-1} \times (\text{positive value}). \end{aligned} \tag{2.18}$$

This proves the theorem. □

3 Discrete moment problems

Let $Z = \{z_0, \ldots, z_n\}$ be the support of a discrete random variable X. Suppose that the probability distribution of X is unknown, but known are the moments $\mu_k = E[X^k],\ k = 0, \ldots, m$, where $m < n$ and also known is the support set Z. Let $f(z),\ z \in Z$ be a known discrete function. Our aim is to find minimum and maximum of $E[f(X)]$.

The univariate discrete moment problem is defined as the LP:

$$\begin{array}{ll} & \min(\max) \displaystyle\sum_{i=0}^{n} f_i p_i \\ \text{subject to} & \\ & \displaystyle\sum_{i=0}^{n} z_i^{\alpha} p_i = \mu_\alpha,\ \alpha = 0, \ldots, m \\ & p_i \geq 0,\ i = 0, \ldots, n, \end{array} \tag{3.1}$$

where $f_i = f(z_i)$, $i = 0, \ldots, n$.

Prékopa (1990) has given a simple characterization of the dual feasible bases if the function $f(z),\ z \in Z$ has a higher order convexity property. This is formulated in terms of higher order divided differences.

The first order divided difference of f, corresponding to the points z_{i_1}, z_{i_2}, is designated and defined as:

$$[z_{i_1}, z_{i_2}; f] = \frac{f(z_{i_2}) - f(z_{i_1})}{z_{i_2} - z_{i_1}}.$$

The k^{th} order divided difference, corresponding to the points $z_{i_1}, \ldots, z_{i_{k+1}}$, is defined recursively as

$$[z_{i_1}, \ldots, z_{i_{k+1}}; f] = \frac{[z_{i_2}, \ldots, z_{i_{k+1}}; f] - [z_{i_1}, \ldots, z_{i_k}; f]}{z_{i_{k+1}} - z_{i_1}}.$$

A function $f(z),\ z \in Z$ is said to be *convex of order* k if all of its k^{th} order divided differences are nonnegative. If all k^{th} order divided differences are positive, then the function is said to be strictly convex of order k.

If the dual feasible bases are known we can find the solutions of the LP (3.1) much simpler by the use of the dual algorithm.

The exact method solving (3.1) can be found in Appendix A.

The discrete moment problem can be extended for the expected value of multivariate functions, acting on random vectors. Assume that the support of the random variable X_j is the set Z_j. Then the support of the random vector $\boldsymbol{X} = (X_1, \ldots, X_s)^T$ is a subset of the set $Z = Z_1 \times \cdots \times Z_s$. Let

$$p_{i_1 \ldots i_s} = P(X_1 = z_{1i_1}, \ldots, X_s = z_{si_s}),\ 0 \leq i_j \leq n_j,\ j = 1, \ldots, s,$$

$$\mu_{\alpha_1 \ldots \alpha_s} = E[X_1^{\alpha_1} \cdots X_s^{\alpha_s}] = \sum_{i_1=0}^{n_1} \cdots \sum_{i_s=0}^{n_s} z_{1i_1}^{\alpha_1} \cdots z_{si_s}^{\alpha_s} p_{i_1 \ldots i_s},$$

where $\alpha_1, \ldots, \alpha_s$ are nonnegative integers. The number $\mu_{\alpha_1 \ldots \alpha_s}$ will be called the $(\alpha_1, \ldots, \alpha_s)$-order moment of the random vector $\boldsymbol{X}$. The sum $\alpha_1 + \cdots + \alpha_s$ is called the total order of the moments.

Suppose now that the probability distribution of $\boldsymbol{X}$ is unknown but known are all moment of total order at most m and further univariate moments of the marginal distributions. More precisely, we assume that the following moments are known:

$$\begin{array}{ll} & E[X_1^{\alpha_1} \cdots X_s^{\alpha_s}],\ \alpha_1 + \cdots \alpha_s \le m \\ \text{and} & \\ & E[X_k^{\alpha_k}],\ m \le \alpha_k \le m_k,\ k = 1, \ldots, s. \end{array} \tag{3.2}$$

This means, in terms of the symbols, $\mu_{\alpha_1 \ldots \alpha_s}$ the known moments are:

$$\begin{array}{ll} & \mu_{\alpha_1 \ldots \alpha_s},\ \alpha_1 + \cdots \alpha_s \le m \\ \text{and} & \\ & \mu_{\alpha_1 \ldots \alpha_s},\ \alpha_j = 0,\ j = 1, \ldots, k-1, k+1, \ldots, s,\ m \le \alpha_k \le m_k,\ k = 1, \ldots, s. \end{array} \tag{3.3}$$

Let $f(\boldsymbol{z})$, $\boldsymbol{z} \in Z$ be a discrete function and introduce the notation $f_{i_1 \ldots i_s} = f(z_{i_1}, \ldots, z_{i_s})$. Our multivariate discrete moment problem (MDMP) is the following LP:

$$\begin{array}{ll} & \min(\max) \displaystyle\sum_{i_1=0}^{n_1} \cdots \sum_{i_s=0}^{n_s} f_{i_1 \ldots i_s} p_{i_1 \ldots i_s} \\ \text{subject to} & \\ & \displaystyle\sum_{i_1=0}^{n_1} \cdots \sum_{i_s=0}^{n_s} z_{1i_1}^{\alpha_1} \cdots z_{si_s}^{\alpha_s} p_{i_1 \ldots i_s} = \mu_{\alpha_1 \ldots \alpha_s} \\ & \text{ for } \alpha_j \ge 0,\ j = 1, \ldots, s;\ \alpha_1 + \cdots \alpha_s \le m \text{ and} \\ & \text{ for } \alpha_j = 0,\ j = 1, \ldots, k-1, k+1, \ldots, s,\ m \le \alpha_k \le m_k,\ k = 1, \ldots, s; \\ & p_{i_1 \ldots i_s} \ge 0,\ \text{all } i_1, \ldots, i_s. \end{array} \tag{3.4}$$

Here the $p_{i_1 \ldots i_s}$ are the decision variables, everything else in the LP is given.

The optimum value of the minimization (maximization) problem (3.4) is a lower (upper) bound for $E[f(X_1, \ldots, X_s)]$. The bounds are also sharp in the sense that no better bounds can be given based on the moments (3.3).

Mádi-Nagy and Prékopa (2004) gave an efficient method for bounding the expectation of a multivariate function of discrete random variables under moment information. The central results in this respect are the theorems which characterize the structures of the dual feasible bases under some assumptions that concern the divided differences of the function. This structures are written precisely in Appendix B.

If we consider a multivariate discrete function $f(\boldsymbol{z})$, $\boldsymbol{z} \in Z = Z_1 \times \cdots \times Z_s$, where $Z_j = \{z_{j0}, \ldots, z_{jn_j}\}$, $j = 1, \ldots, s$, and take the subset

$$\begin{aligned} Z_{I_1 \ldots I_s} &= \{z_{1i},\ i \in I_1\} \times \cdots \times \{z_{si},\ i \in I_s\} \\ &= Z_{1I_1} \times \cdots \times Z_{sI_s}, \end{aligned} \tag{3.5}$$

where $|I_j| = k_j + 1,\ j = 1, \ldots, s$, then we can define the $(k_1, \ldots, k_s)$-order divided difference of f on the set (3.5) in an iterative way. For the sake of simplicity we assume that $z_{j0} < z_{j1} < \cdots < z_{jn_j},\ j = 1, \ldots, s$. First we take the k_1^{th} divided difference with respect to the first variable, then the k_2^{th} divided difference with respect to the second variable etc. This operations can be executed in any order even in a mixed manner, the result is always the same. Let

$$[z_{1i},\ i \in I_1; \cdots ; z_{si},\ i \in I_s; f] \tag{3.6}$$

designate the $(k_1, \ldots, k_s)$-order divided difference. The sum $k_1 + \cdots + k_s$ is called the total order of the divided difference.

A multivariate discrete function is said to be (strictly) convex of order m if all of its divided differences of total order m are nonnegative (positive).

It is known and easy to see, that if a certain order partial derivative of a function defined on an open convex set of $\mathbb{R}^s$ is a continuous and nonnegative (positive) function, then the same order divided differences corresponding to some (distinct) points of the domain are also nonnegative (positive).

By Theorem 2.3 functions (1.12) are strictly convex of order m if m is odd and their negatives are strictly convex of order m if m is even.

Given the dual feasible bases above, we may look at it as an initial basis and carry out the dual algorithm of linear programming to obtain the sharp bounds. The knowledge of an initial dual feasible basis has two main advantages. First it saves roughly half of the running time of the entire dual algorithm. Second, it improves on the numerical accuracy of the computation that we carry out in connection with our LP's. In Section 4 we follow this way for bounding the expected value of utility functions (1.12).

4 Numerical examples for bounding $E\left[u(X_1, X_2, X_3)\right]$

In this section we consider utility functions (1.12) for the case where $s = 3$ and $g_j(z_j)$ is linear $j = 1, 2, 3$, i.e.

$$\begin{gathered} u(z_1, z_2, z_3) = log\left[(e^{\alpha_1 z_1 + a_1} - 1)(e^{\alpha_2 z_2 + a_2} - 1)(e^{\alpha_3 z_3 + a_3} - 1) - 1\right] \\ (z_1, z_2, z_3) \in Z, \end{gathered} \tag{4.1}$$

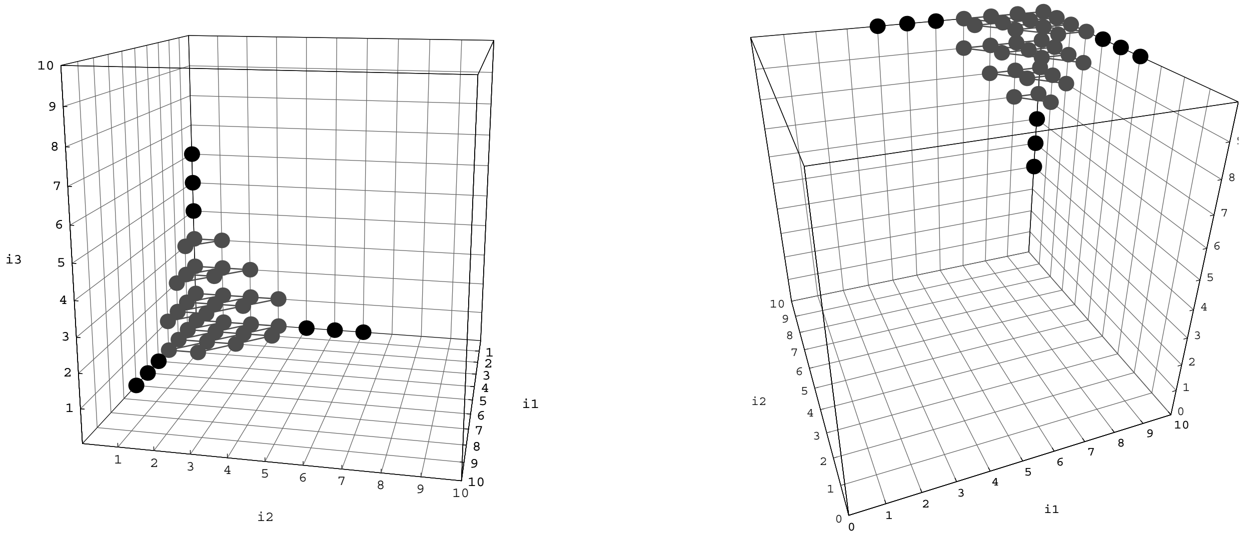
where Z is specialized as follows:

$$Z = (0, 1, 2, 3, 4, 5, 6, 7, 8, 9) \times (0, 1, 2, 3, 4, 5, 6, 7, 8, 9) \times (0, 1, 2, 3, 4, 5, 6, 7, 8, 9).$$

Assume that

$$e^{\alpha_j z_j + a_j} > 2,\ j = 1, 2, 3,\ \text{for } (z_1, z_2, z_3) \in Z. \tag{4.2}$$

We know from Theorem 2.3 that the odd order partial derivatives of (4.1) are positive, while the even order derivatives of it are negative at any point that satisfies (4.2). This means that the odd (even) order divided differences of (4.1) on Z are positive (negative).



(a) (b)

Figure 2: The initial bases of (4.3) and (4.4) on figures (a) and (b) respectively

In the following numerical examples we consider the MDMP (3.4) with the objective function (4.1), where $m,\ m_j,\ j = 1,2,3$ are even numbers. We give sharp lower and upper bounds for the expected value of the utility function (4.1) by the use of the dual algorithm.

Considering the results of Mádi-Nagy and Prékopa (2004) (the related theorems are in Appendix B), the collection of vectors corresponding to the subscripts in

$$\{(i_1,i_2,i_3)|i_1+i_2+i_3 \le m \text{ or } i_k = 0, k \ne j, m \le i_j \le m_j, j = 1,2,3\}, \tag{4.3}$$

is a dual feasible basis in problem (3.4) (by Theorem B.1 of Appendix B). Similarly, the collection of vectors corresponding to the subscripts in

$$\begin{gathered}\{(i_1,i_2,i_3)|(9-i_1)+(9-i_2)+(9-i_3) \le m \text{ or } i_k = 9, k \ne j, m \le 9-i_j \le m_j,\\ j = 1,2,3\},\end{gathered} \tag{4.4}$$

is dual feasible basis in problem (3.4) (by Theorem B.2 of Appendix B). Both bases provide us with bounds, the first are a lower while the second are an upper bound. The bases, on the other hand, can serve as initial bases in the dual algorithms that we carry out to obtain the best bounds.

Example 4.1 *Consider the function (4.1) with parameters $\alpha_1 = \alpha_2 = \alpha_3 = a_1 = a_2 = a_3 = 1$. Assume that X_1, X_2, X_3 are independent and each one has uniform distribution on $\{0,1,2,3,4,5,6,7,8,9\}$. If μ_α designates the α^{th} moment of this distribution then we have the following numerical values:*

$$\begin{gathered}\mu_1 = 4.5,\ \ \mu_2 = 28.5,\ \ \mu_3 = 202.5,\ \ \mu_4 = 1533.3,\ \ \mu_5 = 12082.5,\ \ \mu_6 = 97840.5,\\ \mu_7 = 808043,\ \ \mu_8 = 6773133.3\end{gathered}$$

In view of the independence assumption, we have the equation

$$\mu_{\alpha_1\alpha_2\alpha_3} = \mu_{\alpha_1}\mu_{\alpha_2}\mu_{\alpha_3}$$

that can be used to compute the mixed moments, by the use of the individual ones. We consider two subcases.

(a) *Out of the mixed moments we take into account* μ_{110}, μ_{101}, μ_{011}. *As regards the individual moments, we look at four instances: (i)* μ_1, μ_2, *(ii)* μ_1, μ_2, μ_3, μ_4, *(iii)* μ_1, μ_2, μ_3, μ_4, μ_5, μ_6, *(iv)* μ_1, μ_2, μ_3, μ_4, μ_5, μ_6, μ_7, μ_8. *The results are summarized in Table 1.*

Table 1:

m	m_1	m_2	m_3	Minimum	Iteration	Maximum	Iteration
2	2	2	2	16.083862403	93	16.439400518	69
2	4	4	4	16.236742070	124	16.337970820	128
2	6	6	6	16.265375750	211	16.297838921	123
2	8	8	8	16.272378408	309	16.294804990	294

(b) *We take into account the mixed moments* $\mu_{\alpha_1\alpha_2\alpha_3}$, $\alpha_1+\alpha_2+\alpha_3 \le 4$ *and the following instances out of the individual moments: (i)* μ_1, μ_2, μ_3, μ_4, *(ii)* μ_1, μ_2, μ_3, μ_4, μ_5, μ_6, *(iii)* μ_1, μ_2, μ_3, μ_4, μ_5, μ_6, μ_7, μ_8. *We present the results in Table 2.*

Table 2:

m	m_1	m_2	m_3	Minimum	Iteration	Maximum	Iteration
4	4	4	4	16.256237098	331	16.337929898	364
4	6	6	6	16.284878189	466	16.297815868	686
4	8	8	8	16.288316597	802	16.294784936	669

It is interesting to note that if we compute the bounds obtained for $m = 2$, $m_1 = m_2 = m_3 = 6$ *and* $m = 4$, $m_1 = m_2 = m_3 = 4$, *then both the lower and upper bounds are better in the first case, where in addition to the mixed moments, individual moments of higher order are also taken into account. On the other hand, the better bounds are obtained in a considerably smaller number of iterations.*

Example 4.2 *Consider the function (4.1) with parameters* $\alpha_1 = 1.75$, $\alpha_2 = 1.25$, $\alpha_3 = 0.75$, $a_1 = 3$, $a_2 = 2$, $a_3 = 1$. *In this case we consider the random variables*

$$X_1 = min\,(X + Y_1, 9)$$

$$X_2 = min\,(X + Y_2, 9)$$

Table 3:

The univariate moments:

$$\begin{array}{lll}
\mu_{100} = 2.998513417 & \mu_{010} = 3.4952766952 & \mu_{001} = 3.9877364472 \\
\mu_{200} = 11.970752028 & \mu_{020} = 15.656400887 & \mu_{002} = 19.755120076 \\
\mu_{300} = 56.566152813 & \mu_{030} = 81.725154019 & \mu_{003} = 112.30540309 \\
\mu_{400} = 303.24629588 & \mu_{040} = 477.8221998 & \mu_{004} = 706.03779766 \\
\mu_{500} = 1793.9994786 & \mu_{050} = 3047.9142598 & \mu_{005} = 4788.6277304 \\
\mu_{600} = 11479.787496 & \mu_{060} = 20808.527145 & \mu_{006} = 34417.875979 \\
\mu_{700} = 78231.253035 & \mu_{070} = 149838.80505 & \mu_{007} = 258672.80968 \\
\mu_{800} = 560760.43405 & \mu_{080} = 1125170.3103 & \mu_{008} = 2012508.6375
\end{array}$$

The mixed moments:

$$\begin{array}{lll}
\mu_{110} = 11.467123056 & \mu_{101} = 12.932392935 & \mu_{011} = 14.907585494 \\
\mu_{210} = 48.683197524 & \mu_{201} = 54.487215932 & \mu_{021} = 70.047590573 \\
\mu_{310} = 241.9635138 & \mu_{301} = 269.16222018 & \mu_{031} = 380.16979536 \\
\mu_{120} = 54.713725033 & \mu_{102} = 67.768956274 & \mu_{012} = 77.525335998 \\
\mu_{220} = 243.00350342 & \mu_{202} = 297.13805008 & \mu_{022} = 377.36866108 \\
\mu_{130} = 300.53883351 & \mu_{103} = 402.86511308 & \mu_{013} = 458.16671305 \\
\mu_{111} = 52.95734708 & & \\
\mu_{211} = 237.4121764 & \mu_{121} = 265.14890738 & \mu_{112} = 291.35952474
\end{array}$$

$$X_3 = min\,(X + Y_3, 9)$$

where X, Y_1, Y_2, Y_3 *have Poisson distributions with parameters* $1, 2, 2.5, 3$*, respectively. Note that* X_1, X_2, X_3 *are stochastically dependent. The moments that we take into account are presented in Table 3.*

The results are contained in Tables 4 and 5.

Table 4:

m	m_1	m_2	m_3	Minimum	Iteration	Maximum	Iteration
2	2	2	2	18.466954935	62	18.572924791	46
2	4	4	4	18.532630264	111	18.550298509	126
2	6	6	6	18.541879509	178	18.544391959	148
2	8	8	8	18.543136443	263	18.543344110	191

Table 5:

m	m_1	m_2	m_3	Minimum	Iteration	Maximum	Iteration
4	4	4	4	18.532852070	254	18.550297658	325
4	6	6	6	18.541926465	742	18.544391052	658
4	8	8	8	18.543148260	542	18.543343503	736

Looking at the tables we may say that if the lower and upper bounds are not close to the each other in one instance, i.e., we do not have satisfactory approximation to the value $E[u(\boldsymbol{X})]$*, then first it is advisable to increase the number of individual moments rather then the number of mixed moments. This way we may obtain better results in a shorter time.*

5 Conclusions

We have presented, in analytic form, a class of multiattribute functions $u(z_1, \ldots, z_s)$ that have the property that all their odd order derivatives are positive and all their even order derivatives are negative. We also gave theoretical and numerical methods to obtain best bounds for the expectation of $E[u(X_1, \ldots, X_s)]$, where $X_1, \ldots, X_s$ are discrete random variables with finite supports, under moment information. The latter means that we take into account multivariate moments of order up to a given number of the entire distribution and additional moments of the univariate marginal distributions.

Numerical examples are presented. They show that very good bounds can be obtained if we take into account relatively low order (e.g. 2) multivariate moments but higher order (e.g. 6, 8) moments of the marginal distributions.

A Univariate discrete moment problem

In what follows the reader is supposed to be familiar with the basic concepts and methods of linear programming. A brief introduction to it is presented in Prékopa (1996).

If we introduce the notations

$$\boldsymbol{f} = \begin{pmatrix} f_0 \\ \vdots \\ f_n \end{pmatrix}, \ \boldsymbol{p} = \begin{pmatrix} p_0 \\ \vdots \\ p_n \end{pmatrix}, \ \boldsymbol{\mu} = \begin{pmatrix} \mu_0 \\ \vdots \\ \mu_m \end{pmatrix},$$

$$A = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ z_0 & z_1 & \cdots & z_n \\ \vdots & \vdots & \ddots & \vdots \\ z_0^m & z_1^m & \cdots & z_n^m \end{pmatrix},$$

then problem (3.1) can be written in the concise form

$$\begin{array}{ll} & \min(\max)\ \boldsymbol{f}^T\boldsymbol{p} \\ \text{subject to} & \\ & A\boldsymbol{p} = \boldsymbol{\mu} \\ & \boldsymbol{p} \geq \mathbf{0}. \end{array} \tag{A.1}$$

If $\boldsymbol{a}_0, \boldsymbol{a}_1, \ldots, \boldsymbol{a}_n$ designate the columns of A, the another form of the LP in (3.1) is

$$\begin{array}{ll} & \min(\max)\ \sum_{i=0}^{n} f_i p_i \\ \text{subject to} & \\ & \sum_{i=0}^{n} \boldsymbol{a}_i p_i = \boldsymbol{\mu} \\ & p_i \geq 0,\ i = 0, \ldots, n. \end{array} \tag{A.2}$$

Since A is a Vandermonde matrix, it has rank $m+1$ and all $m+1$ columns of A are linearly independent, i.e., form a basis of the collection of columns $\boldsymbol{a}_0, \boldsymbol{a}_1, \ldots, \boldsymbol{a}_n$. As usual, we designate by B a basis of the columns of A and also the matrix, formed by the basic vectors, in increasing order of the subscripts.

If $B^{-1}\boldsymbol{\mu} \geq \mathbf{0}$, then B is a (primal) feasible basis and if

$$\boldsymbol{f}_B^T B^{-1} \boldsymbol{a}_i \leq (\geq) f_i, \text{ for all } i \tag{A.3}$$

then B is a dual feasible basis in the minimization (maximization) problem. In (A.3) equality holds if i is the subscript of a basic vector. It is shown in Prékopa (1990) that if $L_B(z)$ is the Lagrange interpolation polynomial determined by the base points $\{z_i, i \in I_B\}$, where I_B is the set of subscripts of the basic vectors, then

$$\boldsymbol{f}_B^T B^{-1} \boldsymbol{a}_i = L_B(z_i),\ i = 0, \ldots, n. \tag{A.4}$$

Figure 3: Subscripts of a dual feasible basis of a minimum problem in case of $m = 4$

Thus, the dual feasibility of the basis means that $L_B(z)$ never goes above (below) the function $f(z)$ if the objective function is to be minimized (maximized).

It follows that if B_1, B_2 are the dual feasible bases in the minimization and maximization problems, respectively, then we have the relation

$$L_{B_1}(z) \leq f(z) \leq L_{B_2}(z), \ z \in Z \tag{A.5}$$

This, in turn, implies that

$$E\left[L_{B_1}(X)\right] \leq E\left[f(X)\right] \leq E\left[L_{B_2}(X)\right]. \tag{A.6}$$

Any of the two inequalities in (A.6) is best possible, in other word, sharp, if the basis involved is optimal in the LP (A.2), i.e., it is primal feasible as well.

The dual feasible bases have a simple characterization if the function $f(z), \ z \in Z$ has a higher order convexity property. This is formulated in terms of higher order divided differences.

The following theorem is proved in Prékopa (1990):

Theorem A.1 *Suppose that the function $f(z), \ z \in Z$ is convex of order $m+1$. Then if, the subscripts of B have following structure:*

$$\begin{array}{lcc} & m+1 \textit{ even} & m+1 \textit{ odd} \\ \textit{min problem} & u, u+1, \ldots, v, v+1 & 0, u, u+1, \ldots, v, v+1 \\ \textit{max problem} & 0, u, u+1, \ldots, v, v+1, n & u, u+1, \ldots, v, v+1, n \end{array} \tag{A.7}$$

then B is a dual feasible basis in the corresponding problem. If the function is strictly convex of order $m+1$, then B is a dual feasible basis iff it has the structure (A.7).

Theorem A.1 provides us with a simple way to find dual feasible bases in problem (A.2). Any dual feasible basis, on the other hand, can serve as the initial basis in the dual algorithm to solve the problem. If $f(z), \ z \in Z$ is strictly convex of order $m+1$, then the dual algorithm takes the following simple form:

Step 0: Find an initial dual feasible basis, by the use of Theorem A.1.

Step 1: Check for primal feasibility, i.e., $B^{-1}\boldsymbol{\mu} \geq 0$. If yes, then stop, B is an optimal basis. If no, then go to Step 2.

Step 2: Choose any i satisfying $(B^{-1}\boldsymbol{\mu})_i < 0$. Remove the i^{th} vector from the basis B and include that unique vector into the basis which makes the new basis dual feasible. Go to Step 1.

B Multivariate discrete moment problem

In the two theorems below we recall some facts from the paper by Mádi-Nagy and Prékopa (2004), which give dual feasible basis structures. Let us use the following notation (compatible with the notations used in Mádi-Nagy and Prékopa, 2004) for the coefficient matrix and vectors of (3.4):

$$\begin{array}{rrcl} & \min(\max) & \boldsymbol{f}^T\boldsymbol{p} & \\ \text{subject to} & & & \\ & \widehat{A}\boldsymbol{p} & = & \widehat{\boldsymbol{b}} \\ & \boldsymbol{p} & \geq & \mathbf{0}. \end{array} \tag{B.1}$$

If $\widehat{B}_1$ ($\widehat{B}_2$) is a dual feasible basis in the minimization (maximization) problem, $V_{\widehat{B}_1}$ ($V_{\widehat{B}_2}$) is the corresponding objective function value and V_{min} (V_{max}) is the optimum value in the same problem, then we have the relations

$$V_{\widehat{B}_1} \leq V_{min} \leq E[f(X_1, \dots, X_s)] \leq V_{max} \leq V_{\widehat{B}_2}. \tag{B.2}$$

We will use the following subscript sets:

$$I = I_0 \cup \left(\cup_{j=1}^s I_j\right), \tag{B.3}$$

where

$$I_0 = \{(i_1, \dots, i_s) |\ 0 \leq i_j \leq m-1,\ \text{integers},\ j = 1, \dots, s,\ i_1 + \cdots + i_s \leq m\} \tag{B.4}$$

and

$$\begin{array}{l} I_j = \{(i_1, \dots, i_s) |\ i_j \in K_j,\ i_l = 0\ l \neq j\} \\ K_j = \{k_j^{(1)}, \dots, k_j^{(|K_j|)}\} \subset \{m, m+1, \dots, n_j\},\ j = 1, \dots, s. \end{array} \tag{B.5}$$

We assume that the cardinality of each K_j, $j = 1, \dots, s$ is even.

We will also use some further subscript sets, called subscript structures, labelled them by *min* and *max*.

$$\begin{array}{cc} min & u^{(j)}, u^{(j)} + 1, \dots, v^{(j)}, v^{(j)} + 1 \\ max & m, u^{(j)}, u^{(j)} + 1, \dots, v^{(j)}, v^{(j)} + 1, n_j. \end{array} \tag{B.6}$$

Theorem B.1 *Suppose that the function $f(\boldsymbol{z})$, $\boldsymbol{z} \in Z$ has nonnegative divided differences of total order $m+1$, and, in addition, in each variable z_j it has nonnegative divided differences of order $m + |K_j|$, where the set K_j has one of the min structures in (B.6), $j = 1, \dots, s$.*

Under these conditions, if $m_k + 1 = m + |K_k|$, $k = 1, \dots, s$, then the set of columns $\widehat{B}$ of $\widehat{A}$ in problem (B.1), with the subscript set I, is a dual feasible basis in the minimization problem.

Theorem B.2 *Suppose that the function $f(\boldsymbol{z})$, $\boldsymbol{z} \in Z$ has nonnegative divided differences of total order $m+1$, where $m+1$ is odd and, in addition, in each variable z_j it has nonnegative divided differences of order $m + |K_j|$, where K_j has one of the max structures in (B.6). Under these conditions, if $m_k + 1 = m + |K_k|$, $k = 1, \dots, s$, then the set of columns $\widehat{B}$ of $\widehat{A}$ in problem (B.1), with the subscript set $(n_1, \dots, n_s) - I$, is a dual feasible basis in the maximization problem.*

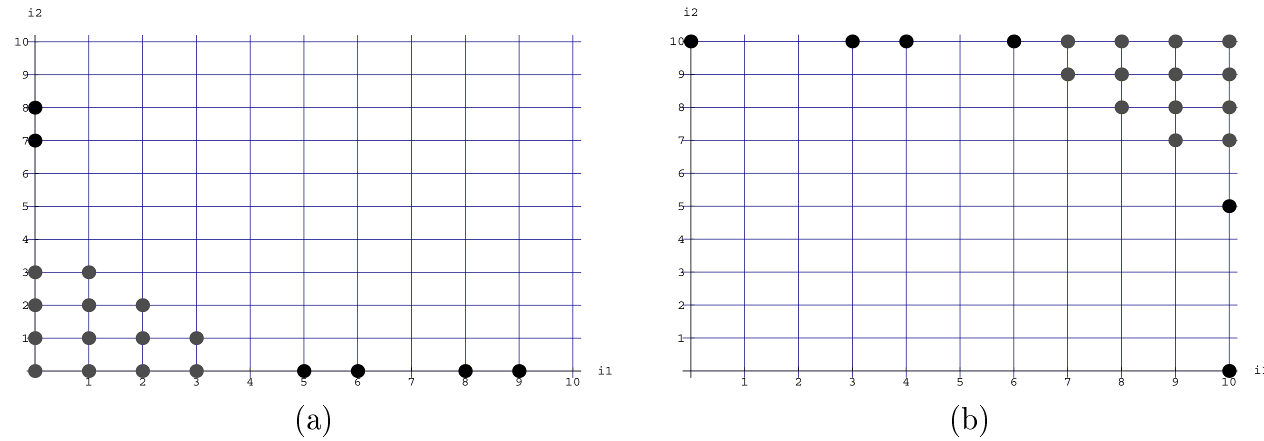


Figure 4: Dual feasible subscript structures corresponding to Theorems B.1 (on (a)) and B.2 (on (b)). Elements of I_0 are colored by gray while elements of I_j's are black.

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