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# BOUNDS FOR PROBABILISTIC INTEGER PROGRAMMING PROBLEMS

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**Abstract.** We consider stochastic integer programming problems with probabilistic constraints. The concept of a $p$-efficient point of a probability distribution is used to derive various equivalent problem formulations. Next we introduce new methods for constructing lower and upper bounds for probabilistically constrained integer programs. We also show how limited information about the distribution can be used to construct such bounds. The concepts and methods are illustrated on an example of a vehicle routing problem.

# 1 Introduction

Uncertainty is inherent in many applied discrete optimization problems. Uncertain demand occurs in network design problems, vehicle routing, scheduling, lot sizing, etc. If in the resulting integer program

$$\begin{array}{rl} \min & c^T x \\ \text{subject to} & Tx \geq \xi, \\ & Ax \geq b, \\ & x \geq 0, \quad x \text{ - integer}, \end{array}$$

the right hand side vector $\xi$ is random, it is reasonable to require that $Tx \geq \xi$ shall hold at least with some prescribed probability $p \in (0,1)$, rather than *for all* possible realizations of the right hand side. This leads to the following probabilistically constrained integer program:

$$\begin{array}{rl} \min & c^T x \\ \text{subject to} & \mathbb{P}\{Tx \geq \xi\} \geq p, \\ & Ax \geq b, \\ & x \geq 0, \quad x \text{ - integer}, \end{array} \tag{1}$$

where the symbol $\mathbb{P}$ denotes probability.

We assume throughout this paper that the matrix $T$ is integer. In this case $Tx$ is integer for all integer $x$, so there is no need to consider other random right hand side vectors than integer-valued. In fact, replacing in the constraint $\mathbb{P}\{Tx \geq \eta\}$ the random vector $\eta$ by its roundup $\xi = \lceil \eta \rceil$ strengthens the inequality without cutting off integer solutions. Therefore, with no loss of generality, we assume that $\xi$ in (1) is integer.

Linear programming models with probabilistic constraints have a long history [3, 8, 11, 12, 16]. Most of the research concentrated on the linear programming case with $\xi$ having a continuous probability distribution. A few papers handle the case of a discrete distribution [14, 19, 21, 18]. The case of integer programs with probabilistic constraints has not attracted much attention.

In section 2 we introduce the disjunctive formulation of (1) and we review its properties. In section 3 we propose a new special method, called the cone generation method, for generating lower bounds of probabilistically constrained problems. In Section 4 we consider the case of limited information about the distribution function available in form of low dimensional marginals. Section 5 is devoted to upper bounds. Finally, in section 6 we present a simple illustrating example.

We assume that in the problems above $A$ is an $m \times n$ matrix, $T$ is an $s \times n$ integer matrix; $c, x \in \mathbb{R}^n$, $b \in \mathbb{R}^m$ and $\xi$ is a random $s$-dimensional integer vector. We use $\mathbb{Z}$ and $\mathbb{Z}_+$ to denote the set of integers and nonnegative integers, respectively. The inequality '$\geq$' for vectors is always understood coordinate-wise.

## 2 Disjunctive Formulation

Let us define the sets:

$$\mathcal{D} = \{x \in \mathbb{Z}^n_+ : Ax \geq b\} \tag{2}$$

and

$$\mathcal{Z}_p = \{y \in \mathbb{Z}^s : \mathbb{P}(\xi \leq y) \geq p\}. \tag{3}$$

Clearly, problem (1) can be compactly rewritten as

$$\begin{array}{rl} \min & c^T x \\ \text{subject to} & Tx \in \mathcal{Z}_p, \\ & x \in \mathcal{D}. \end{array} \tag{4}$$

Let $F(\cdot)$ be the probability distribution function of $\xi$, $F(z) = \mathbb{P}\{\xi \leq z\}$, and let $F_i(\cdot)$ be the marginal probability distribution function of the $i$th component $\xi_i$, that is, $F_i(z_i) = \mathbb{P}\{\xi_i \leq z_i\}$.

We shall use the concept of $p$-efficient points, introduced in [14].

**Definition 2.1** *Let $p \in [0, 1]$. A point $v \in \mathbb{Z}^s$ is called a $p$-*efficient *point of the probability distribution function $F$, if $F(v) \geq p$ and there is no $y \leq v$, $y \neq v$ such that $F(y) \geq p$.*

**Theorem 2.2** *For each $p \in (0, 1)$ the set of $p$-efficient points of $F$ is nonempty and finite.*

**Proof.** For a $p$-efficient point $v$ we have

$$p \leq F(v) = \mathbb{P}\{\xi \leq v\} \leq \mathbb{P}\{\xi_i \leq v_i\} = F_i(v_i),$$

Obviously, for a scalar random variable $\xi_i$ there is exactly one $p$-efficient point: the smallest $l_i$ such that $F_i(l_i) \geq p$. Thus all $p$-efficient points satisfy the inequality $v \geq l = (l_1, \ldots, l_s)$. Our result follows now from Dickson's Lemma [2, Cor. 4.48]. □

Let $p \in (0, 1)$ and let $v^j$, $j \in J$, be the finite set of *all* $p$-efficient points of $\xi$. Defining the cones

$$K_j = v^j + \mathbb{R}^s_+, \quad j \in J,$$

we can equivalently express the set (3) as $\mathcal{Z}_p = \bigcup_{j \in J} K_j$. Thus, we obtain (for $0 < p < 1$) the following *disjunctive* formulation of (4):

$$\begin{array}{rl} \min & c^T x \\ \text{subject to} & Tx \in \bigcup_{j \in J} K_j, \\ & x \in \mathcal{D}. \end{array} \tag{5}$$

Its main advantage is an insight into the nature of the non-convexity of the feasible set.

A straightforward way to solve (1) is to enumerate all $p$-efficient points $v^j$, $j \in J$, and to process all problems of form

$$\begin{aligned} \min \quad & c^T x \\ \text{subject to} \quad & Tx \geq v^j, \\ & x \in \mathcal{D}. \end{aligned} \tag{6}$$

Simple bounding–pruning techniques can be used to avoid solving all of them.

For multi-dimensional random vectors $\xi$ the number of $p$-efficient points can be very large and their straightforward enumeration – very difficult. A better approach would be, therefore, to avoid the complete enumeration and to generate only promising $p$-efficient points. We shall discuss this issue in the next section.

# 3 Convexification and the Cone Generation Method

Let us convexify the disjunctive formulation (4). We obtain the relaxed problem

$$\min \quad c^T x \tag{7}$$
$$Ax \geq b, \tag{8}$$
$$Tx \geq \sum_{j \in J} \lambda_j v^j, \tag{9}$$
$$\sum_{j \in J} \lambda_j = 1, \tag{10}$$
$$x \geq 0, \ \ \lambda \geq 0. \tag{11}$$

Obviously, the optimal value of this problem provides a lower bound for the optimal value of (4). We should not forget, though, that the set of sll $p$-efficient points is not known.

For the solution of (7)–(11) we shall develop a special method, which separates the generation of $p$-efficient points and the solution of the approximation of the original problem using these points. It is related to column generation methods (see, e.g., [1, 4]).

### The Cone Generation Method

**Step 0:** Select a $p$-efficient point $v^0$. Set $J_0 = \{0\}$, $k = 0$.

**Step 1:** Solve the *master problem*

$$\min \quad c^T x \tag{12}$$
$$Ax \geq b, \tag{13}$$
$$Tx \geq \sum_{j \in J_k} \lambda_j v^j, \tag{14}$$
$$\sum_{j \in J_k} \lambda_j = 1, \tag{15}$$
$$x \geq 0, \ \ \lambda \geq 0. \tag{16}$$

Let $u^k$ be the vector of Lagrange multipliers associated with the constraint (14).

**Step 2:** Calculate

$$\overline{d}(u^k) = \min_{j \in J_k} (u^k)^T v^j.$$

**Step 3:** Find a $p$-efficient solution $v^{k+1}$ of the subproblem:

$$\min_{z \in \mathcal{Z}_p} (u^k)^T z$$

and let

$$d(u^k) = (v^{k+1})^T u^k.$$

**Step 4:** If $d(u^k) = \overline{d}(u^k)$ then stop; otherwise set $J_{k+1} = J_k \cup \{k+1\}$, increase $k$ by one and go to Step 1.

Few comments are in order. The first $p$-efficient point $v^0$ can be found by solving the subproblem of Step 3 for an arbitrary $u > 0$. All master problems will be solvable, if the first one is solvable, i.e., if the set $\{x \in \mathbb{R}^n_+ : Ax \geq b,\ Tx \geq v^0\}$ is nonempty.

Let us now focus our attention on solving the auxiliary problem of Step 3:

$$\min\{u^T z \mid F(z) \geq p\}, \tag{17}$$

where $F(\cdot)$ denotes the distribution function of $\xi$.

Assume that the components $\xi_i$, $i = 1, \dots, s$, are independent. Since

$$\ln(F(z)) = \sum_{i=1}^{s} \ln(F_i(z_i)),$$

we obtain a nonlinear knapsack problem:

$$\begin{aligned}
\min \quad & \sum_{i=1}^{s} u_i z_i \\
& \sum_{i=1}^{s} \ln(F_i(z_i)) \geq \ln p, \\
& z_i \geq l_i, \quad z_i \in \mathbb{Z}, \quad i = 1, \dots, s.
\end{aligned}$$

If $b_i$ is a known upper bound on $z_i$, $i = 1, \dots, s$, we can transform the above problem to a

0–1 linear programming problem:

$$\begin{aligned}
\min \quad & \sum_{i=1}^{s}\sum_{j=l_i}^{b_i} j u_i y_{ij} \\
& \sum_{i=1}^{s}\sum_{j=l_i}^{b_i} \ln(F_i(j)) y_{ij} \geq \ln p, \\
& \sum_{j=l_i}^{b_i} y_{ij} = 1, \quad i = 1,\ldots,s, \\
& y_{ij} \in \{0,1\}, \quad i = 1,\ldots,s, \quad j = l_i,\ldots,u_i.
\end{aligned} \tag{18}$$

Obviously, $z_i = \sum_{j=l_i}^{b_i} j y_{ij}$.

If the conponents of $\xi$ are dependent, bounding techniques from the next section may be employed.

# 4 Bounds via binomial moments

If the components of $\xi$ are dependent, subproblem (17) may be difficult to solve exactly. Still, some bounds on its optimal solution may prove useful. We shall develop bounds using only partial information on the distribution function of $\xi$ in the form of the marginal distributions:

$$F_{i_1\ldots i_k}(z_{i_1},\ldots,z_{i_k}) = \mathbb{P}\{\xi_{i_1} \leq z_{i_1},\ldots \xi_{i_k} \leq z_{i_k}\}, \quad 1 \leq i_1 < \ldots < i_k \leq s.$$

Since for each marginal distribution one has $F_{i_1\ldots i_k}(z_{i_1},\ldots,z_{i_k}) \geq F(z)$ the following relaxation of $\mathcal{Z}_p$ (defined by (3)) can be obtained.

**Fact 4.1** *For each $z \in \mathcal{Z}_p$ and for every $1 \leq i_1 < \ldots < i_k \leq s$ the following inequality must hold:*

$$F_{i_1\ldots i_k}(z_{i_1},\ldots,z_{i_k}) \geq p.$$

We shall base further developments on the following result of [13].

**Theorem 4.2** *For any distributon function $F : \mathbb{R}^s \to [0,1]$ and any $1 \leq k \leq s$, at every $z \in \mathbb{R}^s$ the optimal value of the following linear programming problem*

$$\begin{array}{rcl}
 & \max & v_s \\
v_0 + v_1 + v_2 + v_3 + \cdots + v_s & = & 1 \\
v_1 + 2v_2 + 3v_3 + \cdots + s v_s & = & \displaystyle\sum_{1\leq i\leq s} F_i(z_i) \\
v_2 + \binom{3}{2} v_3 + \cdots + \binom{s}{2} v_s & = & \displaystyle\sum_{1\leq i_1<i_2\leq s} F_{i_1 i_2}(z_{i_1}, z_{i_2}) \\
 & \vdots & \\
v_k + \binom{k+1}{k} v_{k+1} + \cdots + \binom{s}{k} v_s & = & \displaystyle\sum_{1\leq i_1<\ldots<i_k\leq s} F_{i_1\ldots i_k}(z_{i_1},\ldots,z_{i_k}) \\
v_0 \geq 0,\ v_1 \geq 0,\ \ldots,\ v_s \geq 0. & &
\end{array} \tag{19}$$

*provides an upper bound for* $F(z_1, \dots, z_s)$.

We can use this result to bound our auxiliary problem (17).

**Proposition 4.3** *Let* $\xi = (\xi_1, \dots, \xi_s)$ *be an integer random vector and let* $F_{i_1,\dots,i_k}$ *denote its marginal distribution functions. Then for every* $p \in (0,1)$ *and for every* $1 \le k \le s$ *the optimal value of the problem*

$$
\begin{array}{c}
\min \ u^T z \\
\begin{array}{rcl}
v_0 + v_1 + v_2 + v_3 + \cdots + v_s &=& 1 \\
v_1 + 2v_2 + 3v_3 + \cdots + sv_s &=& \displaystyle\sum_{1 \le i \le s} F_i(z_i) \\
v_2 + \binom{3}{2} v_3 + \cdots + \binom{s}{2} v_s &=& \displaystyle\sum_{1 \le i_1 < i_2 \le s} F_{i_1 i_2}(z_{i_1}, z_{i_2}) \\
&\vdots& \\
v_k + \binom{k+1}{k} v_{k+1} + \cdots + \binom{s}{k} v_s &=& \displaystyle\sum_{1 \le i_1 < \ldots < i_k \le s} F_{i_1 \ldots i_k}(z_{i_1}, \dots, z_{i_k})
\end{array} \\
v_0 \ge 0,\ v_1 \ge 0,,\ \dots,\ v_{s-1} \ge 0,\ v_s \ge p, \quad z_1 \ge l_1,\ z_2 \ge l_2,\ \dots,\ z_s \ge l_s, \\
z \in \mathbb{Z}^s
\end{array}
\tag{20}
$$

*provides a lower bound on the optimal value of (17).*

**Proof.** If $z \in \mathcal{Z}_p$, that is, $F(z) \ge p$, then the optimal value of (19) satisfies $v_s \ge p$. Thus $z$ and the solution $v$ of (19) are feasible for (20). Since the objective functions of (17) and (20) are the same, the result follows. □

Problem (20) is a nonlinear mixed-integer problem. Its advantage over the original formulation is that it uses only marginal distribution functions.

# 5 Primal feasible solution and upper bounds

Let us consider the optimal solution $x^{\text{low}}$ of the convex hull problem (7)–(11) and the corresponding multipliers $\lambda_j$. Define $J^{\text{low}} = \{j \in J : \lambda_j > 0\}$.

To generate a feasible point we consider the *restricted disjunctive* formulation:

$$
\begin{array}{rl}
\min & c^T x \\
\text{subject to} & Tx \in \bigcup_{j \in J^{\text{low}}} K_j, \\
& x \in \mathcal{D}.
\end{array}
\tag{21}
$$

It can be solved by simple enumeration of all cases for $j \in J^{\text{low}}$:

$$
\begin{array}{rl}
\min & c^T x \\
\text{subject to} & Tx \ge v^j, \\
& x \in \mathcal{D}.
\end{array}
\tag{22}
$$

An alternative strategy would be to solve the corresponding upper bounding problem (22) every time a new $p$-efficient point is generated. If $U_j$ denotes the optimal value of (22), the upper bound at iteration $k$ is

$$\bar{U}^k = \min_{0 \le j \le k} U_j. \tag{23}$$

This is computationally efficient and provides valid upper bounds at every iteration of the method.

# 6 Numerical Illustration

We have a directed graph with node set $\mathcal{N}$ and arc set $\mathcal{E}$. A set of cyclic routes $\Pi$, understood as sequences of nodes connected with arcs and such that the last node of the sequence is the same as the first one, has been selected. For each arc $e \in \mathcal{E}$ we denote by $\mathcal{R}(e)$ the set of routes containing $e$, and by $c(\pi)$ the unit cost on the route.

A random integer demand $\xi(e)$ is associated with each arc $e \in \mathcal{E}$. Our objective is to find non-negative integers $x(\pi)$, $\pi \in \Pi$, such that

$$\mathbb{P}\Big\{ \sum_{\pi \in \mathcal{R}(e)} x(\pi) \ge \xi(e),\ e \in \mathcal{E} \Big\} \ge p,$$

and the cost

$$\sum_{pi \in \Pi} c(\pi) x(\pi)$$

is minimized.

As an illustration, let us consider the graph shown in Figure 1. We assume that the demands $\xi(e)$ associated with the arcs are independent Poisson random variables with the expected values given in Table 1.

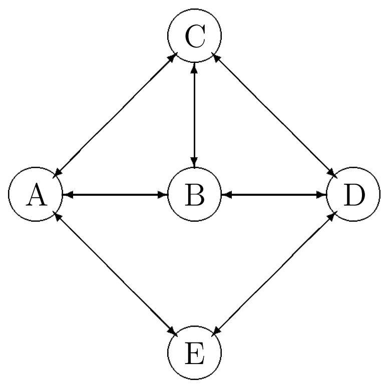


Figure 1: The graph of the example problem.

| Arc | Expected Demand |
|---|---|
| AB | 2 |
| AC | 3 |
| AE | 2 |
| BA | 1 |
| BC | 1 |
| BD | 2 |
| CA | 2 |
| CB | 1 |
| CD | 4 |
| DB | 2 |
| DC | 4 |
| DE | 3 |
| EA | 2 |
| ED | 3 |

Table 1: Expected demands

The set of routes $\Pi$ is given by the following arc–route incidence matrix $T$:

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| AB | 1 | | 1 | | 1 | | | 1 | 1 | | | | 1 | | |
| AC | | 1 | | | | 1 | 1 | | | | 1 | | | | |
| AE | | | | 1 | | | | | | 1 | | 1 | | 1 | 1 |
| BA | 1 | | 1 | | | 1 | 1 | | | 1 | | | | 1 | |
| BC | | | | | 1 | | | | | | | | 1 | | |
| BD | | | 1 | | | | | 1 | 1 | | | | | | |
| CA | | 1 | | | 1 | | | 1 | | | | 1 | | | |
| CB | | | | | | 1 | | | | | | | | 1 | |
| CD | | | | | | | 1 | | | | 1 | | 1 | | |
| DB | | | 1 | | | | 1 | | | 1 | | | | | |
| DC | | | | | | | | 1 | | | | 1 | | 1 | |
| DE | | | | | | | | | 1 | | 1 | | 1 | | 1 |
| EA | | | | 1 | | | | | 1 | | 1 | | 1 | | 1 |
| ED | | | | | | | | | | 1 | | 1 | | 1 | 1 |

For example, route 7 has the form ACDBA.

The cost coefficients associated with the routes are given by

$$c = (10\ 15\ 18\ 15\ 32\ 32\ 57\ 57\ 60\ 60\ 61\ 61\ 75\ 75\ 44).$$

Finally, the probability level is $p = 0.9$.

This problem has been solved by the cone generation method, as described in section 3. The master problem (12)–(16) was solved by the simplex method. The subproblem of Step 3 was formulated as a binary knapsack problem (18) and solved by 0–1 linear programming methods. The entire algorithm has been implemented in AMPL [7].

To generate the first $p$-efficient point we solved the subproblem of Step 3 with $u^0 = (1\ 1\ \ldots\ 1)$. This gave

$$v^0 = (6\ 7\ 6\ 4\ 5\ 6\ 6\ 4\ 9\ 4\ 7\ 7\ 5\ 7).$$

The method terminated after 27 iterations satisfying the stopping criterion of Step 4, with the objective value of the convexified problem equal to 1145.075. The solution was fractional, so this value could be considered only as a lower bound.

The upper bound (obtained already at the 14th iteration) is equal to 1152 and corresponds to the solution

$$\hat{x} = (0\ 3\ 6\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 5\ 4\ 3\ 3\ 0),$$

that is, 3 units on the route ACA, 6 on ABDBA, 5 on ACDEA , 4 on AEDCA and 3 on ABCDEA and AEDEA. The revelant $p$-efficient point is:

$$v^{14} = (7\ 8\ 7\ 5\ 3\ 6\ 6\ 3\ 8\ 5\ 7\ 8\ 7\ 7).$$

The values of the objective fuctions of the master problem and the subproblem at successive iterations are illustrated in Figure 6. We also include the upper bounds (23).

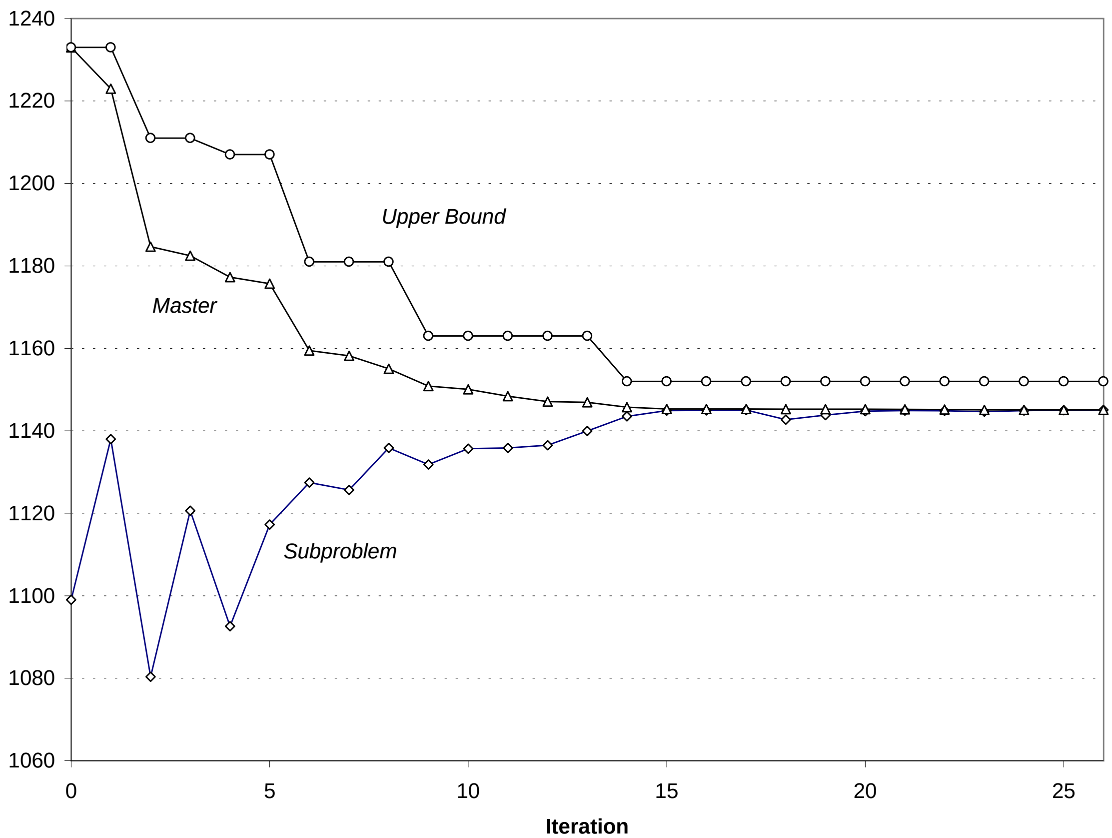


Figure 2: Objective values of the master and the subproblem in the cone generation method applied to the vehicle routing example. 'Upper Bound' is the value of the simple bound (23).

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