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Abstract. Prékopa (1973, 1995) has proved that if T is an $r \times n$ random matrix with independent, normally distributed rows such that their covariance matrices are constant multiples of each other, then the function $h(x) = P(Tx \geq \mathbf{b})$ is quasi-concave in R^n, where $\mathbf{b}$ is a constant vector. We prove that, under same condition, the converse is also true, a special quasi-concavity of $h(x)$ implies the above-mentioned property of the covariance matrices.

1 Introduction

In the theory of programming under probabilistic constraints an important problem is the convexity of the set

$$D = \{x | h(x) \geq p\} ,$$

where

$$h(x) = P(Tx \geq \xi) ,$$

p is a fixed probability ($0 < p < 1$), T a random matrix and ξ a random vector.

For constant T and continuously distributed random ξ with logconcave p.d.f. Prékopa (1971, 1973) has proved that $h(x)$ is also a logconcave function. This fact clearly implies the convexity of the set D.

For the case of a random technology matrix a few convexity theorems are also known. One of them is the following.

Theorem 1 *Let ξ be constant and*

$$T = \begin{pmatrix} T_1 \\ \vdots \\ T_r \end{pmatrix}$$

a random matrix with independent, normally distributed rows such that their covariance matrices are constant multiples of each other. Then $h(x)$ is a quasi-concave function in R^n.

The quasi-concavity of $h(x)$ implies the convexity of the set D. Similar theorem holds if the columns of T satisfy the condition that their covariance matrices are constant multiples of each other. The right hand side vector ξ may also be random. In this case we assume that it is independent of T and its covariance matrix is a constant multiple of any of the other covariance matrices.

Recently Henrion (2006) proved the convexity of the set D for independent $T_1, ..., T_r$ under same condition.

The purpose of the paper is to show that under same conditions the converse of Theorem 1 is also true.

While the sum of concave functions is also concave, the same is not true, in general, for quasi-concave functions. However, we can define a special class of quasi-concave functions such that the sums and products, within the class, are also quasi-concave.

Definition. Let $h_1(x), ..., h_r(x)$ be quasi-concave functions in a convex set E. We say that they are uniformly quasi-concave if for any $x, y \in E$ either

$$min(h_i(x), h_i(y)) = h_i(x) , \quad i = 1, ..., r$$

or

$$min(h_i(x), h_i(y)) = h_i(y) , \quad i = 1, ..., r .$$

Obviously, the sum of uniformly quasi-concave functions, on the same set, is also quasi-concave and if the functions are also nonnegative, then the same holds for their product as well. The latter property is used in the next section, where we prove our main result.

2 The Main Theorem

First let $r = 1$ and consider the function

$$h(x) = P(Tx \leq b) \ ,$$

where T is a random vector and b is a constant. The following lemma was first proved by Kataoka and van de Panne and Popp (1963). See also Prékopa (1965).

Lemma. If T has normal distribution, then the function $h(x)$ is quasi-concave on the set

$$\left\{ x \mid \ P(Tx \leq b) \geq \frac{1}{2} \right\} \ .$$

Proof. We prove the equivalent statement: for any $p \geq \frac{1}{2}$ the set

$$\{ x \mid \ P(Tx \leq b) \geq p \ \} \tag{2.1}$$

is convex.

Let Φ designate the c.d.f. of the $N(0, 1)$-distribution and let $\mu = E(T)$. For any x that satisfies $x^T Cx > 0$,

$$\begin{aligned} h(x) &= P(Tx \leq b) \\ &= P\left(\frac{(T - \mu)x}{\sqrt{x^T Cx}} \leq \frac{b - \mu x}{\sqrt{x^T Cx}} \right) \\ &= \Phi \left(\frac{b - \mu x}{\sqrt{x^T Cx}} \right) \geq p \end{aligned}$$

is equivalent to

$$\mu^T x + \Phi^{-1}(p)\sqrt{x^T Cx} \leq b \ . \tag{2.2}$$

Since $\Phi^{-1}(p) \geq 0$ and $\sqrt{x^T Cx}$ is a convex function in R^n, it follows that the inequality (2.2) determines a convex set. $\square$

Let r be an arbitrary positive integer and introduce the function:

$$h_i(x) = P(T_i x \leq b_i) \ , \quad i = 1, ..., r \ .$$

If $\mu_i = E(T_i) = 0$, $i = 1, ..., r$ and $p \geq 1/2$, then the inequality

$$P(T_i x \leq b_i) \geq P(Tx \leq b) \geq p$$

shows that we have to assume $b_i \geq 0$, $i = 1, ..., r$, otherwise the set (2.1) is empty. So, let $b_i \geq 0$, $i = 1, ..., r$. We also see that, under the same condition,

$$E = \bigcap_{i=1}^{r} \{ x \mid P(T_i x \leq b_i) \geq p \} \tag{2.3}$$

$$= \bigcap_{i=1}^{r} \left\{ x \mid \sqrt{x^T C x} \le \frac{b_i}{\Phi^{-1}(p)} \right\} .$$

Each function h_i is quasi-concave on the set (2.3).

Theorem 2 *Let $b_i > 0$, $i = 1, ..., r$ and $E(T_i) = 0$, $i = 1, ..., r$. If the functions $h_1, ...h_r$ are uniformly quasi-concave on the set E, defined by (2.3), and $C_i \neq 0$, $i = 1, ..., r$, then each C_i is a constant multiple of a covariance matrix C.*

Proof. We already know that the functions $h_1, ..., h_r$ are all quasi-concave on the set (2.3) and that

$$h_i(x) = \Phi\left(\frac{b_i}{\sqrt{x^T C x}}\right) , \quad i = 1, ..., r .$$

It is enough to show that if we take two functions h_i, h_j, $i \neq j$, then the corresponding covariance matrices C_i, C_j are constant multiples of each other. Let h_1 and h_2 be the two functions.

Since h_1 and h_2 are uniformly quasi-concave on E, it follows that for any two vectors $y, z \in E$, the inequality

$$\Phi\left(\frac{b_1}{\sqrt{y^T C_1 y}}\right) \ge \Phi\left(\frac{b_1}{\sqrt{z^T C_1 z}}\right)$$

implies that

$$\Phi\left(\frac{b_2}{\sqrt{y^T C_2 y}}\right) \ge \Phi\left(\frac{b_2}{\sqrt{z^T C_2 z}}\right) .$$

An equivalent form of the statement is that the inequality

$$z^T C_1 z \ge y^T C_1 y \tag{2.4}$$

implies that

$$z^T C_2 z \ge y^T C_2 y . \tag{2.5}$$

From linear algebra we know that for any two quadratic forms, in the same variables, there exists a basis such that both quadratic forms are sums of squares if the variables are expressed in that basis. In our case this means that there exist linearly independent vectors $a_1, ..., a_n$ such that the transformation

$$x = a_1 u_1 + ... + a_n u_n = Au ,$$

applied to the quadratic forms $x^T C_1 x$ and $x^T C_2 x$, takes them to the forms

$$x^T C_1 x = u^T A^T C_1 A u = \lambda_1 u_1^2 + ... + \lambda_n u_n^2$$

$$x^T C_2 x = u^T A^T C_2 A u = \gamma_1 u_1^2 + ... + \gamma_n u_n^2 ,$$

where $\lambda_i, \gamma_i \ge 0$, $i = 1, ..., n$ and $\lambda_1 + ... + \lambda_n > 0$, $\gamma_1 + ... + \gamma_n > 0$.

The transformation $x = Au$ transforms the set E into a set H that is the intersection of ellipsoids with centers in the origin and main axes lying in the coordinate axes. If $u = A^{-1}y$ and $v = A^{-1}z$, then the statement that (2.4) implies (2.5) can be formulated in such a way that if $u, v \in H$, then

$$\lambda_1 v_1^2 + ... + \lambda_n v_n^2 \geq \lambda_1 u_1^2 + ... + \lambda_n u_n^2$$

implies that

$$\gamma_1 v_1^2 + ... + \gamma_n v_n^2 \geq \gamma_1 u_1^2 + ... + \gamma_n u_n^2 \ .$$

This, in turn, is the same as the statement:

$$\lambda_1(v_1^2 - u_1^2) + ... + \lambda_n(v_n^2 - u_n^2) \geq 0$$

implies that

$$\gamma_1(v_1^2 - u_1^2) + ... + \gamma_n(v_n^2 - u_n^2) \geq 0 \ .$$

Let us introduce the notation $w_i = v_i^2 - u_i^2 \ , \quad i = 1, ..., n$. Then a further form of the statement is:

$$\lambda_1 w_1 + ... + \lambda_n w_n \geq 0 \tag{2.6}$$

implies that

$$\gamma_1 w_1 + ... + \gamma_n w_n \geq 0 \ . \tag{2.7}$$

The above implication is true for any $w = (w_1, ..., w_n)$ in an open convex set around the origin in R^n. It follows that it is also true without any limitation for the variables $w_1, ..., w_n$.

By Farkas' theorem there exists a nonnegative number α such that

$$\gamma = (\gamma_1, ..., \gamma_n) = \alpha(\lambda_1, ..., \lambda_n) = \alpha\lambda \ . \tag{2.8}$$

Since $\gamma \neq 0$, $\lambda \neq 0$, the number α must be positive. The relation can be written in matrix form:

$$\begin{pmatrix} \gamma_1 & & 0 \\ & \ddots & \\ 0 & & \gamma_n \end{pmatrix} = \alpha \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} \ . \tag{2.9}$$

If we take into account that

$$\begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} = A^T C_1 A$$

$$\begin{pmatrix} \gamma_1 & & 0 \\ & \ddots & \\ 0 & & \gamma_n \end{pmatrix} = A^T C_2 A \ ,$$

and combine it with (2.9), we can derive the equation

$$A^T C_2 A = \alpha \ A^T C_1 A \ .$$

Since A is a nonsingular matrix, we conclude that

$$C_2 = \alpha\, C_1 \ .$$

This proves the theorem. □

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