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A Variant of the Hungarian Inventory Control Model

Nilay Noyan András Prékopa

Abstract. The term 'Hungarian inventory control model' refers to a model system initiated by Prékopa (1965) and Ziermann (1964), where the ordered amount is delivered in an interval, rather than at a designated time epoch according to some stochastic process and consumption takes place in the same interval according to some other stochastic process. The problem is to determine the minimum level of initial safety stock that ensures continuous consumption, without disruption, in the whole time interval with a prescribed high probability. The models in this system have been primarily of static (single-stage) type. Recently Prékopa (2004) has shown that the interval type delivery and consumption processes can be combined with classical inventory models and also formulated a dynamic type (two-stage) model with such interval type processes and probabilistic constraints. In this paper we modify the assumptions of those models and formulate simpler, numerically more tractable models. We also discuss the computational aspects of our problems and present numerical examples.

Keywords: *Inventory Control, Stochastic Programming, Two-Stage Stochastic Programming Problem, Probabilistic Constraints.*

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1 Introduction

The term 'Hungarian inventory control model' refers to a model system, where both the deliveries of the ordered amounts and consumption take place in an interval according to some random processes, rather than at one time epoch. The problem is to determine the minimum level of initial safety stock that ensures continuous consumption, without disruption, in the whole time interval with a prescribed high probability.

The Hungarian inventory control model was initiated by Prékopa (1965) and Ziermann (1964). The original model is of a static and single item type, where the delivered and consumed amounts are assumed to be the same and the mathematical tool used to solve the problem comes from order statistics. In Prékopa (1965) already more general models have been presented and some theorems proved in Prékopa (1973a) have been used to numerically solve the problems. From the later literature in connection with the Hungarian inventory control models we mention the papers by Prékopa and Kelle (1978), Kelle (1984) and the summarizing paper of Prékopa (1980).

Recently Prékopa (2004) has shown that interval type delivery and consumption processes can be combined with classical inventory models. The 'order up to S' model is taken as an example. He also presented dynamic type (two-stage) inventory models using interval type delivery and consumption stochastic processes which appear in the Hungarian models. However, the solutions of the obtained nonlinear programming problems are computationally intensive. They involve the solutions of nonlinear decomposition type problems, along with the calculation of the multivariate Dirichlet distribution function and gradient values. A normal approximation to the Dirichlet distribution alleviates the numerical difficulties but still further research is needed to come up with efficient numerical solutions for the problems. Those models are hybrid type stochastic programming models, i.e. both probabilistic constraints and penalties for unsatisfied demands are used.

In the present paper we keep some of the main characteristics of the new models in Prékopa (2004) but introduce simpler, numerically more tractable formulations. We also discuss the numerical solution methods to our problems and present numerical examples.

In Section 2 we recall some earlier results and develop mathematical tools for our model constructions. In Section 3 we formulate a probabilistically constrained multi-item inventory control model with interval type delivery and consumption processes. The consumption process is assumed to be linear with random, normally distributed slope while the expectation of total consumption minus the delivery process is approximated by a Brownian bridge. In Section 4 a two-stage model combined with probabilistic constraints is formulated. We assume that the consumption and delivery processes in connection with the different items are stochastically independent. Research is underway to take stochastic dependence into consideration. Finally, in Section 5 the computational aspects are discussed and numerical examples are presented.

2 Assumptions of the Proposed Model and its Approximation

For the details of the original Hungarian inventory control model and some new variants of it see Prékopa (2004). In connection with the delivery and consumption processes we make the following assumptions:

(a) We assume that delivery takes place during an interval rather than at a single time epoch. We also assume that the delivery process begins τ time after the order is placed and has a duration of time T. Thus, if an order is placed at time 0 then the delivery takes place in the interval $[\tau, \tau + T]$.

(b) Deliveries take place at discrete times the number of which is fixed and designated by n; it can be obtained from past history.
The n delivery times are random and their joint probability distribution is the same as that of n random points chosen independently from the interval $[\tau, \tau+T]$ according to a uniform distribution.

(c) The delivery and consumption processes are stochastically independent.

(d) The consumption of the material is linear with random intensity c. Thus, the total consumption in the time interval $[\tau, \tau + T]$ is cT. We assume that c is normally distributed with mean value μ and variance σ_c^2.

(e) The amount delivered in $[\tau, \tau + T]$ is equal to the expected total consumption in that interval. Let $c_0 \stackrel{\text{def}}{=} E(cT)$.

(f) The delivery process can be described by the following model: whenever delivery takes place there is a minimal amount delivered equal to δ. The remaining parts of the n delivery amounts can be described as the lengths of the subsequent intervals obtained by choosing a random sample of size $n - 1$ from a population uniformly distributed in the interval $[0, c_0 - n\delta]$.

Assumptions $(a), (b), (c)$ and (f) have already been introduced in Prékopa (1965) and Kelle (1984) has investigated the case where (d) holds true.

Let $\lambda = \delta n/c_0$ and let $X_n(t, \lambda)$ denote the amount delivered in the time interval (τ, t) where $\tau \le t \le \tau+T$. An amount M of safety stock ensures consumption without disruption, in the same time interval, if and only if $M + X_n(t, \lambda) - c(t-\tau) \ge 0$ for any $\tau \le t \le \tau + T$. If we want this to happen with probability at least $1 - \epsilon$ then M has to satisfy the following probabilistic constraint:

$$P\left(\sup_{\tau \le t \le \tau+T} \{c(t-\tau) - X_n(t, \lambda)\} \le M\right) \ge 1 - \epsilon, \tag{2.1}$$

where $0 < \epsilon < 1$.

Without loss of generality we may assume $T = 1$, therefore $c_0 = E(c) = \mu$. If T is not equal to 1 we have to multiply the safety stock level obtained from the model by T to find the actual safety stock level.

For the case of a constant c we have the following

Theorem 1 (Prékopa, 1973a). *The probability distribution of the stochastic process*

$$\sqrt{\frac{n}{1+(1-\lambda)^2}}\,(X_n(t,\lambda)-t) \tag{2.2}$$

converges weakly to the probability distribution of the standard Brownian bridge, i.e. the Gaussian process $U(t)$ *defined in* $[0,1]$ *having continuous sample functions with probability* 1 *and satisfying*

$$E\left(U(t)\right)=0, \qquad 0\le t\le 1$$
$$E\left(U(s)U(t)\right)=s(1-t), \qquad 0\le s\le t\le 1.$$

If we apply the results in Prékopa (1973a) for the stochastic process $X_n(t,\lambda)-c_0(t-\tau)$ then we easily obtain the following relations:

$$E(X_n(t,\lambda)-c_0(t-\tau))=0, \qquad \tau\le t\le \tau+1$$

$$\mathrm{E}\left[(X_n(s,\lambda)-c_0(t-\tau))(X_n(t,\lambda)-c_0(t-\tau))\right]=\kappa^2c_0^2(s-\tau)(\tau+1-t), \qquad \tau\le s\le t\le\tau+1,$$

where

$$\kappa^2=\frac{1}{n}\left(1+\frac{n-1}{n+1}(1-\lambda)^2\right).$$

On the other hand, **Theorem 1** tells us that the stochastic process

$$\frac{1}{c_0}\sqrt{\frac{n}{1+\frac{n-1}{n+1}(1-\lambda)^2}}\,(c_0(t-\tau)-X_n(t,\lambda)) \tag{2.3}$$

converges to the standard Brownian bridge process in the interval $[\tau,\tau+1]$.

In view of the results mentioned above we approximate the stochastic process $X_n(t,\lambda))-c_0(t-\tau)$ defined in the interval $[\tau,\tau+1]$ by a Brownian bridge process with $\sigma^2=\kappa^2c_0^2$.

Let us introduce the notation

$$Y(t,\lambda)=c(t-\tau)-X_n(t,\lambda)=(c_0(t-\tau)-X_n(t,\lambda))+(c-c_0)(t-\tau)$$

Taking into account our Brownian bridge approximation, the stochastic process $Y(t,\lambda)$ can be approximated as follows:

$$Y(t,\lambda)\approx\tilde{Y}(t,\lambda)\overset{\text{def}}{=}Z(t-\tau,\lambda)c_0\sqrt{\frac{1+\frac{n-1}{n+1}(1-\lambda)^2}{n}}+(c-c_0)(t-\tau), \qquad \tau\le t\le\tau+1, \tag{2.4}$$

or, introducing $\bar{t} = (t - \tau)$,

$$Y(t,\lambda) \approx \tilde{Y}(t,\lambda) = \bar{Y}(\bar{t},\lambda) \stackrel{\text{def}}{=} Z(\bar{t},\lambda)\; c_0 \sqrt{\frac{1 + \frac{n-1}{n+1}(1-\lambda)^2}{n}} + (c - c_0)\bar{t}, \qquad 0 \le \bar{t} \le 1, \quad (2.5)$$

where $Z(\bar{t},\lambda)$ is a standard Brownian bridge process in the interval $[0,1]$. From now on we will be dealing with the process $\bar{Y}(\bar{t},\lambda)$ rather than with $Y(t,\lambda)$.

Remark 1. It is shown in Doob (1949) that if $Z(t)$, $0 \le t \le 1$, is a standard Brownian bridge process then the process

$$W(t) = (t+1) Z\left(\frac{t}{t+1}\right), \qquad t \ge 0$$

is a standard Brownian motion process, i.e. a stochastic process with independent increments, $E(W(t)) = 0$ $(t \ge 0)$ and $E(W(s)W(t)) = s$ for $0 \le s \le t$. This result was used by Doob to derive the limiting distributions of the Smirnov's statistic. We will use it for a similar purpose later in our paper.

3 A Stochastic Programming Type Inventory Control Model

Here we present a stochastic programming type problem in connection with the model described in Section 2.

Throughout this section we approximate the stochastic process $Y(t,\lambda)$ by the Gaussian process given in (2.4).

3.1 Formulation of the Objective Function

The objective function, by definition, equals the expected long term average inventory holding cost plus the expected long term average shortage cost. If we take the time averages of the two quantities in the time interval $[\tau, \tau+1]$, we obtain the long term time averages. We assume that both the inventory holding and stockout costs are time and quantity proportional with proportionality factors q^+ and q^-, respectively.

First we elaborate on the single item case. The amount ordered at time 0 will be delivered in the time interval $(\tau, \tau+1)$ and in that interval no delivery arising from any other order takes place. This implies that the initial safety stock M for the period $[\tau, \tau+1]$ equals the on hand inventory at time τ.

If $\tau < s < \tau + 1$ then the on hand inventory at time s equals $[M - Y(s,\lambda)]_+$ and the shortage at time s equals $[Y(s,\lambda) - M]_+$, where $[u]_+ = \max\{u, 0\}$.

The approximate cost function to be minimized is the following function of the variable M:

$$\tilde{C}(M) = q^- \int_\tau^{\tau+1} E([\tilde{Y}(s,\lambda) - M]_+)\, ds + \; q^+ \int_\tau^{\tau+1} E([M - \tilde{Y}(s,\lambda)]_+) ds \tag{3.1}$$

Since we have the relation $[u]_+ - [-u]_+ = u$ for any real u, (3.1) can be written in the following form:

$$\tilde{C}(M) = q^+ \int_\tau^{\tau+1} E\left(M - \tilde{Y}(s,\lambda)\right) ds + \; (q^+ + q^-) \int_\tau^{\tau+1} E\left([\tilde{Y}(s,\lambda) - M]_+\right) ds \tag{3.2}$$

Let $\tilde{F}(y,s)$ designate the c.d.f. of the random variable $\tilde{Y}(s,\lambda)$. Using integration by parts it can be shown that

$$E\left([\tilde{Y}(s,\lambda) - M]_+\right) = \int_M^\infty (1 - \tilde{F}(y,s)) dy. \tag{3.3}$$

According to (2.4) and (2.5) we have the following equation:

$$\tilde{F}(y,s) = P\left(\tilde{Y}(s,\lambda) \le a\right) \tag{3.4}$$

$$= P\left(Z(\bar{s},\lambda)\; c_0 \sqrt{\frac{1 + \frac{n-1}{n+1}(1-\lambda)^2}{n}} + (c - c_0)\bar{s} \le a\right), \quad \bar{s} = s - \tau \qquad \tau \le s \le \tau + 1,$$

where $Z(\bar{s},\lambda)$ is a standard Brownian bridge process in the interval $[0,1]$.

Since it is assumed that the delivery and consumption processes are stochastically independent and $c \sim N(c_0, \sigma_c^2)$, we can obtain $\tilde{F}(y,s)$ as follows:

$$\tilde{F}(y,s) = \Phi\left(\frac{y}{\sqrt{(s-\tau)(1+\tau-s)c_0^2 \frac{(1+\frac{n-1}{n+1}(1-\lambda)^2)}{n} + \sigma_c^2 (s-\tau)^2}}\right). \tag{3.5}$$

Since $E\left(c(s-\tau) - X_n(s,\lambda)\right)$ is approximated by $E\left(Z(s,\lambda)\; c_0 \sqrt{\frac{1+\frac{n-1}{n+1}(1-\lambda)^2}{n}} + (c - c_0)(s-\tau)\right)$, $E[Z(s,\lambda)] = 0$ and $E(c - c_0) = 0$, we have the following:

$$E\left(M + X_n(s,\lambda) - c(s-\tau)\right) \approx M. \tag{3.6}$$

Using (3.6) we can write the final form of the approximate cost function:

$$\tilde{C}(M) = q^+ M + (q^- + q^+) \int_M^\infty \left(1 - \int_\tau^{\tau+1} \tilde{F}(y,s) ds\right) dy, \tag{3.7}$$

where $\tilde{F}(y,s)$ is given by (3.5).

The cost function of the inventory control model described above has a straightforward generalization for the case of a multi-item problem.

Suppose that the number of items is r and let $\tilde{C}_j(M_j)$ designate the cost function (3.7) for the j^{th} item. Then the cost function to be minimized is $\sum_{j=1}^{r} \tilde{C}_j(M_j)$. It is a convex function since its terms are univariate convex functions. The convexity of $\tilde{C}_j(M_j)$ follows immediately from (3.7) if we take into account the fact that for each j

$$\int_{\tau_j}^{\tau_j+1} \tilde{F}_j(y,s)ds$$

is a c.d.f. in the variable y.

3.2 Probabilistic Constraint Formulation

First we consider the case of a single item and designate by M the safety stock that we want to determine.

For a random delivery process $X_n(t,\lambda)$ the safety stock must satisfy the probabilistic constraint (2.1), therefore in order to formulate our model we have to find the corresponding probability distribution function.

3.2.1 Maximum of a Brownian motion process

Let τ_a denote the first time when the Brownian motion process $\sigma B(t,\lambda)+\mu t$ hits the level $a>0$, where $\sigma>0$ and μ are real constants and $B(t,\lambda)$ is a standard Brownian motion process. Then (see Bachelier, 1900, Baxter and Donsker, 1957, Takács, 1967) we have:

$$P(\tau_a \le t) = \Phi\left(\frac{-a+\mu t}{\sigma\sqrt{t}}\right) + e^{\frac{2a\mu}{\sigma^2}}\ \Phi\left(\frac{-a-\mu t}{\sigma\sqrt{t}}\right), \tag{3.8}$$

where Φ is the c.d.f. of the standard normal distribution.

Remark 2. *The event $\tau_a > t$ is equivalent to the event $\sup\limits_{0\le u\le t} \sigma B(u)+\mu u < a$.*

3.2.2 Probability distribution function of the supremum of $\bar{Y}$ for some fixed consumption rate c

Using (2.4) and the transformation $\bar{t} = \frac{s}{s+1}$ we obtain the following equations (see also Remark 1):

$$P(\sup_{0\le\bar{t}\le 1} \bar{Y}(\bar{t},\lambda) \le a)$$

$$= P\left(\sup_{0\le s}\left\{ Z(\frac{s}{s+1},\lambda) + \frac{(c-c_0)}{c_0}\sqrt{\frac{n}{1+\frac{n-1}{n+1}(1-\lambda)^2}}\frac{s}{s+1}\right\} \le a\frac{1}{c_0}\sqrt{\frac{n}{1+\frac{n-1}{n+1}(1-\lambda)^2}}\right)$$

$$= P\left(\sup_{0\le s}\left\{ \frac{W(s,\lambda)}{s+1} + \frac{(c-c_0)}{c_0}\sqrt{\frac{n}{1+\frac{n-1}{n+1}(1-\lambda)^2}}\frac{s}{s+1}\right\} \le a\frac{1}{c_0}\sqrt{\frac{n}{1+\frac{n-1}{n+1}(1-\lambda)^2}}\right)$$

$$= P\left(\sup_{0\le s}\left\{W(s,\lambda)+\frac{1}{c_0}\sqrt{\frac{n}{1+\frac{n-1}{n+1}(1-\lambda)^2}}(c-c_0-a)s\right\}\le a\frac{1}{c_0}\sqrt{\frac{n}{1+\frac{n-1}{n+1}(1-\lambda)^2}}\right),$$

where $W(s,\lambda)$ is a Brownian motion process with $s \ge 0$ and $\sigma = 1$.

Let us introduce the following notations:
$d \overset{\text{def}}{=} \frac{c-c_0-a}{c_0}\sqrt{\frac{n}{1+\frac{n-1}{n+1}(1-\lambda)^2}}$ and $a_0 \overset{\text{def}}{=} \frac{a}{c_0}\sqrt{\frac{n}{1+\frac{n-1}{n+1}(1-\lambda)^2}}$. Then we have

$$P\left(\sup_{0\le\bar{t}\le 1}\bar{Y}(\bar{t},\lambda)<a\right)=P\left(\sup_{0\le s}(\sigma B(s,\lambda)+ds)<a_0\right), \tag{3.9}$$

where $B(s,\lambda)$ is the standard Brownian motion process.

According to Remark 2 and (3.9) we have

$$P\left(\sup_{0\le\bar{t}\le 1}\bar{Y}(\bar{t},\lambda)<a\right)=\lim_{s_0\to\infty}P(\tau_{a_0}>s_0)=1-\lim_{s_0\to\infty}P(\tau_{a_0}\le s_0). \tag{3.10}$$

By (3.8) and (3.10) we obtain

$$P\left(\sup_{0\le s}(\sigma B(s,\lambda)+ds)<a_0\right)=1-\lim_{s_0\to\infty}\left[\Phi\left(\frac{-a_0+ds_0}{\sigma\sqrt{s_0}}\right)+e^{\frac{2a_0d}{\sigma^2}}\,\Phi\left(\frac{-a_0-ds_0}{\sigma\sqrt{s_0}}\right)\right]. \tag{3.11}$$

$$=\begin{cases}1-e^{\frac{2a_0d}{\sigma^2}} & \text{if } d<0\\ 0 & \text{if } d\ge 0.\end{cases}$$

3.2.3 Probability distribution function of the supremum of $\tilde{Y}$ for a random consumption rate c

According to (2.5) , Remark 2 and (3.11) we have the equality

$$P\left(\sup_{\tau\le t\le\tau+1}\tilde{Y}(t,\lambda)<a\right)=\int_{-\infty}^{c_0+a}\left(1-e^{2a_0\left(\frac{x-c_0-a}{c_0}\sqrt{\frac{n}{1+\frac{n-1}{n+1}(1-\lambda)^2}}\right)}\right)g(x)dx, \tag{3.12}$$

where $g(x)$ is the p.d.f. of the random variable c.

Since it is assumed that $c\sim N(c_0,\sigma_c^2)$, we further derive

$$\int_{-\infty}^{c_0+a}\left(1-e^{2a_0\left(\frac{x-c_0-a}{c_0}\sqrt{\frac{n}{1+\frac{n-1}{n+1}(1-\lambda)^2}}\right)}\right)g(x)dx=\Phi\left(\frac{a}{\sigma_c}\right)-\int_{-\infty}^{c_0+a}e^{-K}\,\frac{e^{-\frac{\left(x-\left(2{\sigma_c}^2\frac{a_0}{c_0}\sqrt{\frac{n}{1+\frac{n-1}{n+1}(1-\lambda)^2}}+c_0\right)\right)^2}{2{\sigma_c}^2}}}{\sqrt{2\pi}\sigma_c}dx$$

$$= \Phi\left(\frac{a}{\sigma_c}\right) - e^{-K}\Phi\left(\frac{a - 2\sigma_c^2\frac{a_0}{c_0}\sqrt{\frac{n}{1+\frac{n-1}{n+1}(1-\lambda)^2}}}{\sigma_c}\right),$$

where

$$K \stackrel{\text{def}}{=} \frac{2a^2}{c_0^2}\left(\frac{n}{1+\frac{n-1}{n+1}(1-\lambda)^2}\right)\left(1 - \frac{\sigma_c^2}{c_0^2}\left(\frac{n}{1+\frac{n-1}{n+1}(1-\lambda)^2}\right)\right).$$

Finally we conclude

$$P\left(\sup_{\tau\le t\le\tau+1}\tilde{Y}(t,\lambda) < a\right)$$

$$= \Phi\left(\frac{a}{\sigma_c}\right) - e^{-K}\Phi\left(\frac{a}{\sigma_c}\left(1 - \frac{2\sigma_c^2}{c_0^2}\left(\frac{n}{1+\frac{n-1}{n+1}(1-\lambda)^2}\right)\right)\right). \quad (3.13)$$

3.3 Formulation of the Multi-item Stochastic Programming Model

Let r be the number of items. We want to determine the initial (or safety) inventory levels $M^{(1)},\dots,M^{(r)}$. Assume that the delivery and consumption processes corresponding to the different items are independent. The constants and variables associated with item l will be given the superscript l.

Let $\lambda^{(l)} = \delta^{(l)}n^{(l)}/c_0^{(l)}$ and designate by $X_n^{(l)}(t,\lambda^{(l)})$ the amount of item l delivered in the time interval $(\tau^{(l)},t)$ where $\tau^{(l)} \le t \le \tau^{(l)}+1$.

Let $Y^{(l)}(t,\lambda^{(l)}) \stackrel{\text{def}}{=} c^{(l)}(t-\tau^{(l)}) - X_n^{(l)}(t,\lambda^{(l)})$ and $c_0^{(l)} \stackrel{\text{def}}{=} E(c^{(l)})$. Similarly to (2.4) the stochastic process $Y^{(l)}(t,\lambda^{(l)})$ is approximated as follows:

$$Y^{(l)}(t,\lambda^{(l)}) \approx \tilde{Y}^{(l)}(t,\lambda^{(l)}) \stackrel{\text{def}}{=} Z(t-\tau^{(l)},\lambda^{(l)})c_0^{(l)}\sqrt{\frac{1+\frac{n^{(l)}-1}{n^{(l)}+1}(1-\lambda^{(l)})^2}{n^{(l)}}} + (c^{(l)}-c_0^{(l)})(t-\tau^{(l)}). \quad (3.14)$$

Let us introduce the notation

$$\tilde{P}_l(M^{(l)}) \stackrel{\text{def}}{=} P\left(\sup_{\tau^{(l)}\le t\le\tau^{(l)}+1}\tilde{Y}^{(l)}(t,\lambda^{(l)}) \le M^{(l)}\right).$$

By the use of (3.13) we obtain the formula

$$\tilde{P}_l(M^{(l)}) = \Phi\left(\frac{M^{(l)}}{\sigma_c^{(l)}}\right) - e^{-K^{(l)}}\Phi\left(\frac{M^{(l)}}{\sigma_c^{(l)}}\left(1 - \frac{2(\sigma_c^{(l)})^2}{(c_0^{(l)})^2}\left(\frac{n^{(l)}}{1+\frac{n^{(l)}-1}{n^{(l)}+1}(1-\lambda^{(l)})^2}\right)\right)\right), \quad (3.15)$$

where

$$K^{(l)} \stackrel{\text{def}}{=} \frac{2(M^{(l)})^2}{(c_0^{(l)})^2}\left(\frac{n^{(l)}}{1+\frac{n^{(l)}-1}{n^{(l)}+1}(1-\lambda^{(l)})^2}\right)\left(1 - \frac{(\sigma_c^{(l)})^2}{(c_0^{(l)})^2}\left(\frac{n^{(l)}}{1+\frac{n^{(l)}-1}{n^{(l)}+1}(1-\lambda^{(l)})^2}\right)\right).$$

The approximate expected cost in connection with item l can be written as:

$$q^{-(l)} \int_{\tau^{(l)}}^{\tau^{(l)}+1} E\left(\left[\tilde{Y}^{(l)}(t,\lambda^{(l)}) - M^{(l)}\right]_+\right) + q^{+(l)} \int_{\tau^{(l)}}^{\tau^{(l)}+1} E\left(\left[M^{(l)} - \tilde{Y}^{(l)}(t,\lambda^{(l)})\right]_+\right).$$

Let $\tilde{F}^{(l)}(y,s)$ designate the c.d.f. of the random variable $\tilde{Y}^{(l)}(s,\lambda^{(l)})$. In view of (3.5),

$$\tilde{F}^{(l)}(y,s) = \Phi\left(\frac{y}{\sqrt{(s-\tau^{(l)})(1+\tau^{(l)}-s)(c_0^{(l)})^2 \frac{\left(1+\frac{n^{(l)}-1}{n^{(l)}+1}(1-\lambda^{(l)})^2\right)}{n} + \sigma_c^2(s-\tau^{(l)})^2}}\right). \tag{3.16}$$

The next step is to formulate the stochastic programming problem, where we have terms similar to those in Section 3.1 and Section 3.2 (instead of a single item now we have r items) and additional terms that are costs of storage capacities. Assume that these cost functions are linear. Then our problem is:

$$\begin{gathered}
\min\left\{\sum_{l=1}^{r}\left[a^{(l)}M^{(l)} + q^{-(l)}\int_{\tau^{(l)}}^{\tau^{(l)}+1} E\left(\left[\tilde{Y}^{(l)}(t,\lambda^{(l)}) - M^{(l)}\right]_+\right)\right.\right. \\
\left.\left. +q^{+(l)}\int_{\tau^{(l)}}^{\tau^{(l)}+1} E\left(\left[M^{(l)} - \tilde{Y}^{(l)}(t,\lambda^{(l)})\right]_+\right)\right]\right\} \\
\text{subject to} \\
\prod_{l=1}^{r}\tilde{P}_l(M^{(l)}) \geq 1-\epsilon \\
(M^{(1)},\dots,M^{(r)}) \in E,
\end{gathered} \tag{3.17}$$

where E is some convex subset of $\mathbb{R}^r$ and $a^{(l)}$, $l = 1,\dots,r$ are some nonnegative constants that can be interpreted as prices of establishing unit inventory capacities.

By the use of equations (3.7) and (3.15) we obtain the following approximate problem:

$$\begin{gathered}
\min\left\{\sum_{l=1}^{r}\left[a^{(l)}M^{(l)} + q^{+(l)}M^{(l)}\right.\right. \\
\left.\left. + \left(q^{+(l)}+q^{-(l)}\right)\left(\int_{M^{(l)}}^{\infty}\left(1-\int_{\tau^{(l)}}^{\tau^{(l)}+1}\tilde{F}^{(l)}(y,s)ds\right)dy\right)\right]\right\} \\
\text{subject to} \\
\prod_{l=1}^{r}\left[\Phi\left(\frac{M^{(l)}}{\sigma_c^{(l)}}\right) - e^{-K^{(l)}}\Phi\left(\frac{M^{(l)}}{\sigma_c^{(l)}}\left(1-\frac{2(\sigma_c^{(l)})^2}{(c_0^{(l)})^2}\left(\frac{n^{(l)}}{1+\frac{n^{(l)}-1}{n^{(l)}+1}(1-\lambda^{(l)})^2}\right)\right)\right)\right] \geq 1-\epsilon \\
(M^{(1)},\dots,M^{(r)}) \in E,
\end{gathered} \tag{3.18}$$

where $\tilde{F}^{(l)}(y,s)$ and $K^{(l)}$ are given by (3.16) and (3.15), respectively; $c_0^{(l)}$ and $(\sigma_c^{(l)})^2$ are the mean and variance of the random consumption rate $c^{(l)}$ for item l, $l=1,\dots,r$.

Theorem 2. *For every $l=1,\dots,r$ the probability $\tilde{P}_l(M^{(l)})$ is a logconcave point function of $M^{(l)}$.*

Proof. By definition

$$\tilde{P}_l(M^{(l)}) = P\left(\sup_{\tau^{(l)}\le t\le \tau^{(l)}+1} \tilde{Y}^{(l)}(t,\lambda^{(l)}) \le M^{(l)}\right).$$

The process $\tilde{Y}^{(l)}(t,\lambda^{(l)})$ is assumed to have continuous sample functions (see Doob, 1953). Hence if we take a sequence $t_m,\ m=1,2,\dots,$ dense in the interval $[\tau^{(l)},\tau^{(l)}+1]$, then the following relation holds true:

$$\tilde{P}_l(M^{(l)}) = \lim_{N\to\infty} P\left(c^{(l)}(t_m-\tau^{(l)}) - X_n^{(l)}(t_m,\lambda^{(l)}) \le M^{(l)}, m=1,\dots,N\right). \quad (3.19)$$

Since the multivariate normal probability distribution function is logconcave (see Prékopa, 1995) and logconcavity is preserved while passing to the limit, the assertion follows from (3.19). □

Theorem 2 implies that the set of $\left(M^{(1)},\dots,M^{(r)}\right)$ satisfying the probabilistic constraints in problem (3.18) is convex. Since (in view of Section 3.1) the objective function is also convex, it follows that (3.18) is a convex nonlinear programming problem.

4 A Two-stage Model

In this section we consider a group of r items and an inventory process that has been going on since infinitely long time. Orders are placed at times $0, \pm T, \pm 2T, \dots$. If at time kT an order is placed for item l, the ordered amount is delivered at $n^{(l)}$ discrete time epochs during the time interval $(kT+\tau^{(l)}, (k+1)T+\tau^{(l)})$.

The demand for item l is based on a forecast of the total consumption in the time interval $(kT+\tau^{(l)}, (k+1)T+\tau^{(l)})$. As in Prékopa (2004) the demand is an r-component random vector $D=(D^{(1)},\dots,D^{(r)})$ which is assumed to be discrete with support $\{D_u, u\in U\}$ where U is a finite set. Let p_u be the probability corresponding to D_u.

As in Prékopa (2004) the consumption is also assumed to be a random vector, denoted by $C=(C^{(1)},\dots,C^{(r)})$ and we introduce the following notations:

$$c_u^{(l)} = (C^{(l)}|D^{(l)}=D_u^{(l)}), \qquad u\in U, \qquad l=1,\dots,r,$$

$$c_{0u}^{(l)} \stackrel{\text{def}}{=} E(C^{(l)}|D^{(l)}=D_u^{(l)}), \qquad u\in U, \qquad l=1,\dots,r.$$

Unlike in Prékopa (2004) at time kT we place orders equal to the expected consumptions for the r items. Thus in our model we have $c_{0u}^{(l)} = D_u^{(l)}$.

Let $Y_u^{(l)}(t, \lambda^{(l)}) \stackrel{\text{def}}{=} c_u^{(l)}(t - \tau^{(l)}) - X_n^{(l)}(t, \lambda^{(l)})$. Suppose that the safety stock of item l at the beginning of the delivery period $(\tau^{(l)}, \tau^{(l)} + T)$ is $m^{(l)}$. Then we adjust the safety stock by an additional amount $m_u^{(l)}$ to satisfy the probabilistic constraint

$$P\left(\sup_{\tau^{(l)} \le t \le \tau^{(l)} + T} \tilde{Y}_u^{(l)}(t, \lambda^{(l)}) \le m^{(l)} + m_u^{(l)} \right) \ge 1 - \epsilon.$$

In the two-stage inventory control problem there are first and second stage decision variables; we give a subscript u to each second stage variable. The first stage variables are $W^{(l)}$, $l = 1, \ldots, r$ and the second stage variables are $m_u^{(l)}$, $u \in U$, $l = 1, \ldots, r$.

In the first stage we decide the values $W^{(l)}$, $l = 1, \ldots, r$ which can be interpreted as storage capacities corresponding to the r items. The $s^{(l)}(x)$, $l = 1, \ldots, r$ are the cost functions of storage capacities $W^{(l)}$, $l = 1, \ldots, r$ and we assume that they are convex. The second stage problem comes up after the demand values $D = (D^{(1)}, \ldots, D^{(r)})$, $l = 1, \ldots, r$ have been observed. We prescribe that no disruption occurs in any of the r consumptions in the time intervals $(kT + \tau^{(l)}, (k+1)T + \tau^{(l)})$, $l = 1, \ldots, r$, with probability $1 - \epsilon$. The optimal values of the second stage variables $m_u^{(l)}$, $u \in U$, $l = 1, \ldots, r$, which are the adjustment values of the safety stocks to make the probabilistic constraints feasible, are determined immediately before the time intervals $(\tau^{(l)}, \tau^{(l)} + T)$, $l = 1, \ldots, r$. If at time $kT + \tau^{(l)}$ the safety stock levels are $m^{(l)}, l = 1, \ldots, r$ the obtained new stock levels are $m^{(l)} + m_u^{(l)}, l = 1, \ldots, r,\ u \in U$. The adjustments incur some costs; the adjustment cost function of item l is denoted by $f^{(l)}(x)$, $l = 1, \ldots, r$. We assume that these functions are convex. Thus the total adjustment cost is $\sum_{l=1}^{r} f^{(l)}(m_u^{(l)})$ if the observed value of D is D_u. Finally, the objective function is the sum of the costs of the capacities plus the long term average inventory holding and shortage costs for the r items.

Even though in practice the values of the second stage variables are determined at time kT, by solving the second stage problem the discrete nature of the random vector D allows us to formulate both the first and second stage problems by the use of a single optimization problem. Thus we have the following nonlinear two-stage programming problem under uncertainty:

$$\min\left\{\sum_{l=1}^{r}\left(s^{(l)}(W^{(l)})+\sum_{u\in U}p_u\left[q^{+(l)}m_u^{(l)}\right.\right.\right.$$
$$\left.\left.\left.+\left(q^{+(l)}+q^{-(l)}\right)\left(\int_{m^{(l)}+m_u^{(l)}}^{\infty}\left(1-\int_{\tau^{(l)}}^{\tau^{(l)}+1}\tilde{F}^{(l)}(y,s)ds\right)dy\right)+f^{(l)}(m_u^{(l)})\right]\right)\right\}$$
$$\text{subject to}$$
$$\prod_{l=1}^{r}P\left(\sup_{\tau^{(l)}\le t\le\tau^{(l)}+T}\tilde{Y}_u^{(l)}(t,\lambda^{(l)})\le m^{(l)}+m_u^{(l)}\right)\ge 1-\epsilon,\qquad u\in U \tag{4.1}$$
$$m^{(l)}+m_u^{(l)}\le W^{(l)},m_u^{(l)}\ge 0\qquad u\in U,\qquad l=1,\dots,r$$
$$\sum_{l=1}^{r}a^{(l)}W^{(l)}\le W.$$

The $q^{+(l)}$, $q^{-(l)}$ are given proportionality factors for the inventory holding and shortage costs, respectively, while W, $a^{(l)}$ $(l=1,\dots,r)$, and ϵ are given positive constants. Note that $a^{(l)}$, $l=1,\dots,r$ can be interpreted as unit sizes, whereas W is the total capacity of the storage space.

If we rewrite the problem (4.1) by using the results obtained in Section 3 then we have:

$$\min\left\{\sum_{l=1}^{r}\left(s^{(l)}(W^{(l)})+\sum_{u\in U}p_u\left[q^{+(l)}m_u^{(l)}\right.\right.\right.$$
$$\left.\left.\left.+\left(q^{+(l)}+q^{-(l)}\right)\left(\int_{m^{(l)}+m_u^{(l)}}^{\infty}\left(1-\int_{\tau^{(l)}}^{\tau^{(l)}+1}\tilde{F}^{(l)}(y,s)ds\right)dy\right)+f^{(l)}(m_u^{(l)})\right]\right)\right\}$$
$$\text{subject to}$$
$$\prod_{l=1}^{r}\left[\Phi\left(\frac{m^{(l)}+m_u^{(l)}}{\sigma_{c_u}^{(l)}}\right)\right.$$
$$\left.-e^{-K_u^{(l)}}\Phi\left(\frac{m^{(l)}+m_u^{(l)}}{\sigma_{c_u}^{(l)}}\left(1-\frac{2(\sigma_{c_u}^{(l)})^2}{(c_{0u}^{(l)})^2}\left(\frac{n^{(l)}}{1+\frac{n^{(l)}-1}{n^{(l)}+1}(1-\lambda^{(l)})^2}\right)\right)\right)\right]\ge 1-\epsilon,\ u\in U \tag{4.2}$$
$$m^{(l)}+m_u^{(l)}\le W^{(l)},m_u^{(l)}\ge 0,\qquad u\in U,\qquad l=1,\dots,r$$
$$\sum_{l=1}^{r}a^{(l)}W^{(l)}\le W.$$

Here $\tilde{F}(y,s)$ is given by (3.16),

$$K_u^{(l)}\stackrel{\text{def}}{=}\frac{2(m^{(l)}+m_u^{(l)})^2}{(c_{0u}^{(l)})^2}\left(\frac{n^{(l)}}{1+\frac{n^{(l)}-1}{n^{(l)}+1}(1-\lambda^{(l)})^2}\right)\left(1-\frac{(\sigma_{c_u}^{(l)})^2}{(c_{0u}^{(l)})^2}\left(\frac{n^{(l)}}{1+\frac{n^{(l)}-1}{n^{(l)}+1}(1-\lambda^{(l)})^2}\right)\right),$$

$c_{0u}^{(l)}$ and $(\sigma_{c_u}^{(l)})^2$ are the mean and variance of the random consumption rate for item l, $l = 1, \ldots, r$, if the observed value of D is D_u.

The solution of problem (4.2) provides the optimal solutions of the first stage problem and the optimal solutions of all possible second stage problems simultaneously. Similarly to (3.18), Problem (4.2) can also be shown to be a convex nonlinear programming problem.

5 Computational Aspects

The two-stage model (4.2) is a convex nonlinear programming problem with $r(|U| + 1)$ variables, $r|U| + 1$ linear constraints and $|U|$ nonlinear constraints, where r is the number of different items while U is the set of demand scenarios.

There are a number of methods available to solve this type of problem, for an overview see Prékopa (2003). Here we have used a MATLAB implementation of a nonlinear optimization algorithm based on the Sequential Quadratic Programming (SQP) method (Powell, 1983). In this method a general nonlinear problem is solved by constructing a sequence of quadratic subproblems, i.e., optimization problems with a quadratic objective function and linear constraints. An estimate of the Hessian of the Lagrangian is updated at each iteration using the BFGS formula (see Shanno, 1970). A line search is performed using a merit function similar to that proposed in Han (1977) and Powell (1978a, b). The QP subproblem is solved using an active set strategy similar to that described in Coleman and Li (1996). To calculate the objective function values we used a recursive adaptive Simpson quadrature technique.

For problems with a large number of items and/or scenarios large scale nonlinear optimization techniques are needed; for a comprehensive list of references see Boggs et al. (1994), Conn et al. (1994), Gould and Toint (2000).

5.1 Approximating the solution by using pLEPs

A variety of techniques, such as proximal-type algorithms (e.g. Auslender and Haddon, 1995) and reduced gradient methods exist to solve linearly constrained convex nonlinear problems. Using the concept of p-level efficient points (pLEPs), introduced in Prékopa (1990), we can formulate an approximate (discretized) version of our problem in this form.

Definition 1. *Let $p \in [0, 1]$. A point $\mathbf{v} \in \mathbb{Z}^s$ is called a p-level efficient point of the probability distribution function F, if $F(\mathbf{v}) \geq p$ and there is no $\mathbf{y} \leq \mathbf{v}, \mathbf{y} \neq \mathbf{v}$ such that $F(\mathbf{y}) \geq p$.*

Consider a problem of the form

$$\begin{array}{ll} \min & f(\mathbf{x}) \\ \text{s.t.} & P(T\mathbf{x} \geq \xi) \geq p \\ & A\mathbf{x} = b, \quad \mathbf{x} \geq 0 \end{array} \tag{5.3}$$

where the random vector ξ has discrete distribution with the distribution function F.

Let $\mathbf{z}^{(1)}, \ldots, \mathbf{z}^{(N)}$ designate the p-level efficient points of the distribution function F (an algorithm for enumerating all pLEPs is described in Prékopa, (2003)). Problem (5.3) is a disjunctive programming problem:

$$\begin{array}{ll} \min & f(\mathbf{x}) \\ \text{s.t.} & T\mathbf{x} \geq \mathbf{z}^{(i)} \text{ for at least one } i = 1, \ldots, N \\ & A\mathbf{x} = b, \quad \mathbf{x} \geq 0 \end{array} \tag{5.4}$$

Problem (5.4) is relaxed as the linearly constrained problem:

$$\begin{array}{ll} \min & f(\mathbf{x}) \\ \text{s.t.} & T\mathbf{x} \geq \sum_i^N \lambda_i \mathbf{z}^{(i)} \\ & A\mathbf{x} = b, \quad \mathbf{x} \geq 0 \\ & \sum_i^N \lambda_i = 1, \lambda \geq 0 \end{array} \tag{5.5}$$

In order to be able to apply this approach to our model (4.2) we approximate the random variable $\zeta_u^{(l)} \stackrel{\text{def}}{=} \sup_{\tau \leq t \leq \tau + T} \tilde{Y}_u^{(l)}(t, \lambda)$ by a discrete random variable $\xi_u^{(l)}$ with possible values $z_{u1}^{(l)} < z_{u2}^{(l)} < \ldots < z_{uN}^{(l)}$ and distribution function

$$F_{\xi_u^{(l)}}(z_{ui}^{(l)}) \stackrel{\text{def}}{=} \begin{cases} F_{\zeta_u^{(l)}}(z_{ui}^{(l)}) & \text{for } i = 1, \ldots, N-1 \\ 1 & \text{for } i = N. \end{cases}.$$

The values $z_{u1}^{(l)}, \ldots, z_{uN}^{(l)}$ can be chosen to be equidistant on some interval $[0, B_u^{(l)}]$, where $F_{\zeta_u^{(l)}}(B_u^{(l)}) \geq 1 - \bar{\epsilon}$ for some prescribed small tolerance $\bar{\epsilon}$.

For the base case of our multi-item examples (see Section 6.2) we have found that by setting $\bar{\epsilon} = 0.01$ and $N = 100$ the above method approximates the optimal objective function value of (4.2) within an error of 0.2% and the optimal solution within an error of 4% using about 28 pLEPs for each scenario.

However, for high-dimensional random vectors ξ the number of p-efficient points can be very large and enumerating all of them does not provide us with an efficient method. Therefore if we have a large number of different items and/or scenarios a different approach not requiring the enumeration of all pLEPs, similar to the cone generation method described in Dentcheva et al. (1999), is recommended.

6 Numerical Examples

We present two small numerical examples for problem (4.2), one is single item, the other one is two-item. The running time of our MATLAB implementation is 5 seconds (wallclock time) for the single item case and 15 seconds (wallclock time) for the two-item case on a 2.40 GHz Pentium 4 PC.

6.1 Single-item problem

First we establish a base case, then change various parameters in it and observe the behavior of the model.

6.1.1 Parameters for the base case

We want to ensure consumption without disruption with probability $1-\epsilon$, where $\epsilon = 0.2$.

The number of deliveries in the time interval is $n = 10$. The parameter λ, defined in Section 2, is equal to 0.6.

The safety stock of the item at the beginning of the delivery period is $m = 1.9$.

The total storage capacity is $W = 20$ and the unit size of the item is $a = 1$.

The inventory holding cost and the shortage cost factors are $q^+ = 0.1$ and $q^- = 100$, respectively.

Let $f(x) = 2x$ be the cost of adjusting the safety stock by an amount of x and $s(y) = y$ be the cost of storage capacity y.

The demand that equals the expected total consumption is a discrete random variable with possible values $c_{01} = 3.2$, $c_{02} = 3.4$, $c_{03} = 3.6$, $c_{04} = 3.8$, $c_{05} = 4$, $c_{06} = 4.2$ and corresponding probabilities $p_1 = 0.1257, p_2 = 0.1724, p_3 = 0.2019, p_4 = 0.2019, p_5 = 0.1724$, $p_6 = 0.1257$ (these values have been obtained from a truncated and discretized normal distribution). The standard deviation of the consumption rate in first three cases and in last three cases is given by $\sigma_{c_u} = 0.375c_{0u}$ and $\sigma_{c_u} = 0.45c_{0u}$, respectively.

6.1.2 Numerical Results

The results for the base case are given in **Table 1**.

Given initial stock	1.9	1.9	1.9	1.9	1.9	1.9
Optimal additional stock amount	m_1 =0	m_2 =0.0025	m_3 =0.1144	m_4 =0.2074	m_5 =0.2074	m_6 =0.2074
Total initial safety stock	1.9	1.9025	2.0144	2.1074	2.1074	2.1074
Total expected consumption	3.2	3.4	3.6	3.8	4	4.2
Proportion of initial safety stock to total expected consumption	59.38%	55.96%	55.96%	55.46%	52.69%	50.18%
Probability of ensuring continuous production	0.901	0.8811	0.8811	0.837	0.8133	0.8
	The optimal objective function value is 4.2014					

Table 1

In the following examples every parameter value is the same as in the base case, except where mentioned.

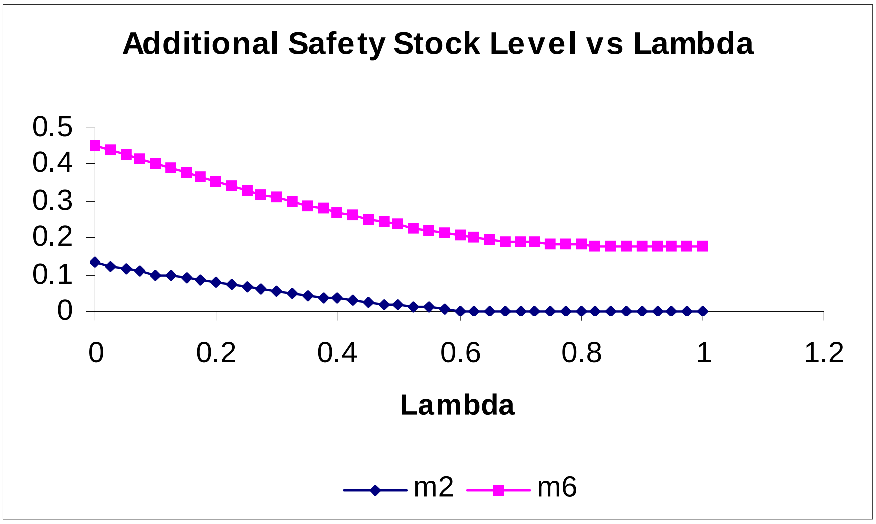

Figure 1

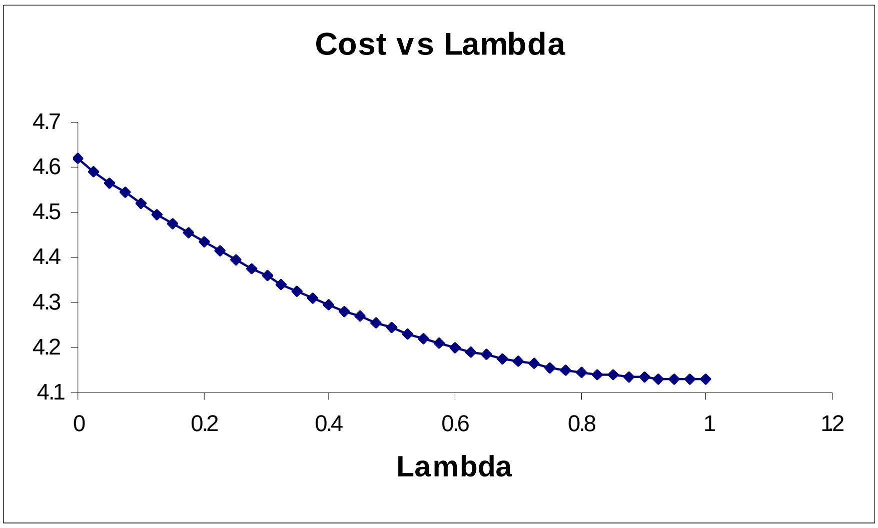

Figure 2

Figure 1 shows the additional safety stock levels in scenarios 2 and 6 for various values of the parameter λ while **Figure 2** shows the corresponding optimal objective function values.

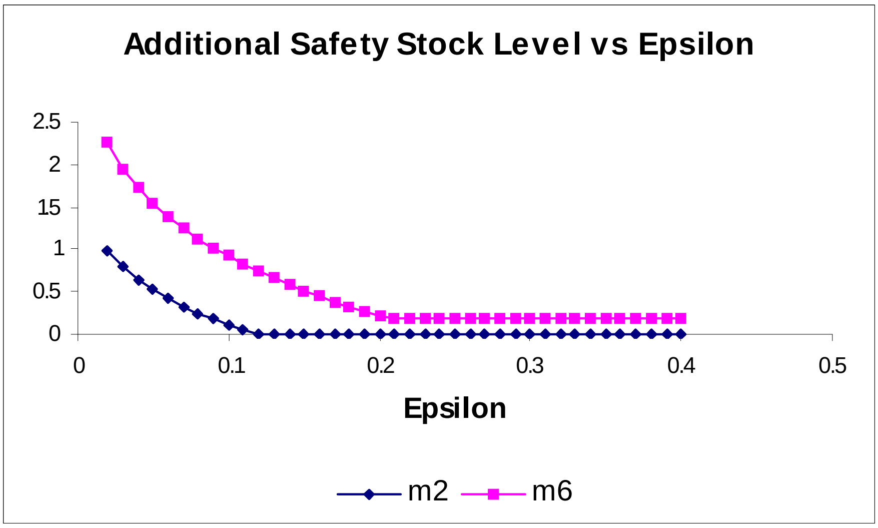

Figure 3

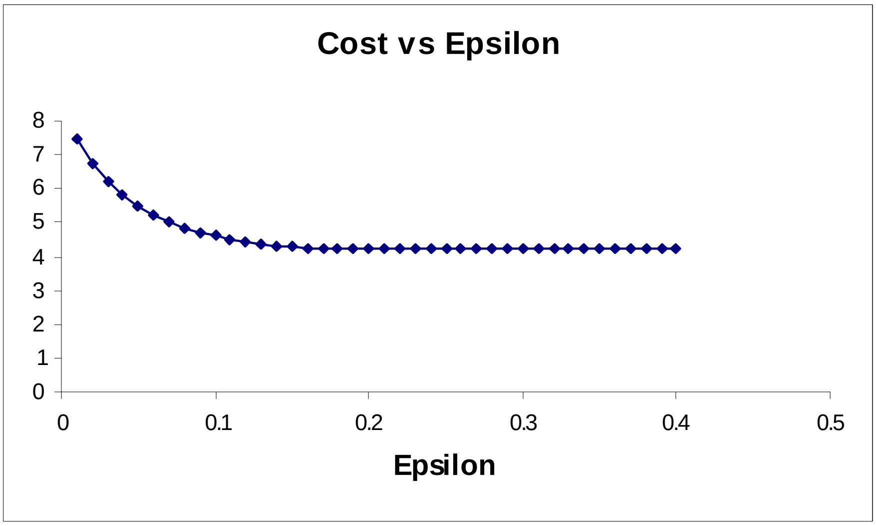

Figure 4

Similarly, **Figure 3** and **Figure 4** depict the changes in additional safety stock levels (for scenarios 2 and 6) and in the cost when modifying ϵ, where $1-\epsilon$ is the prescribed probability of consumption without disruption.

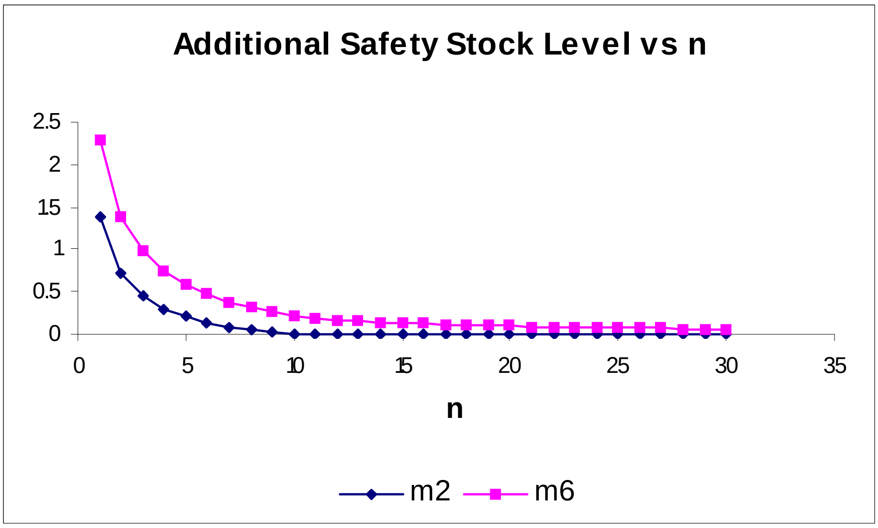

Figure 5

Figure 5 illustrates the effect of changing the number of deliveries on the additional safety stock levels, again for scenarios 2 and 6. Whereas **Figure 6** illustrates the effect of changing the number of deliveries on the optimal objective function value.

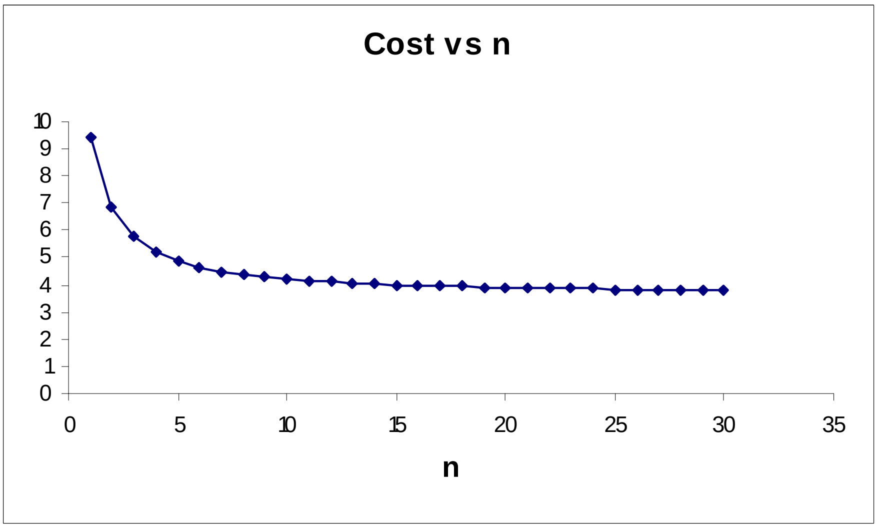

Figure 6

Finally, **Figure 7** shows the changes in additional safety stock levels for the above scenarios when adjusting the shortage cost factor q^-.

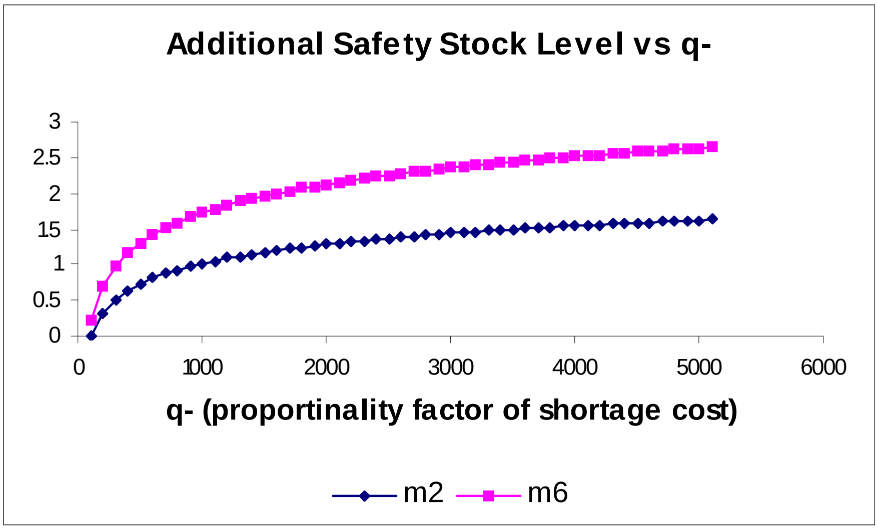

Figure 7

6.2 Two-item problem

As an example we consider a product whose production involves two items. We assume that two delivery processes are stochastically independent (in practice it is assured if, e.g., there are two different suppliers). Otherwise we have the same assumptions and cost functions as in the single-item case.

6.2.1 Parameters for the base case

The number of deliveries in the time interval is $n^{(1)} = 10$ for item 1 and $n^{(2)} = 10$ for item 2. The parameters $\lambda^{(1)}$ and $\lambda^{(2)}$ are equal to 0.6.

$m^{(1)} = 2$ and $m^{(2)} = 1.9$ are the safety stock amounts of items 1 and 2 at the beginning of the delivery period, respectively.

The unit size of item 1 is $a^{(1)} = 1$ and the unit size of item 2 is $a^{(2)} = 1.2$.

Let $s^{(1)}(x) = x$ and $s^{(2)}(x) = 1.2x$ be the cost functions of storage capacities corresponding to item 1 and item 2, respectively.

The demand that equals the expected total consumption is a discrete random variable with possible values $\mathbf{c}_{01} = (3.2, 2.4)$, $\mathbf{c}_{02} = (3.4, 2.8)$, $\mathbf{c}_{03} = (3.6, 3.2)$, $\mathbf{c}_{04} = (3.8, 3.6)$, $\mathbf{c}_{05} = (4, 4)$, $\mathbf{c}_{06} = (4.2, 4.4)\}$, where $\mathbf{c}_{0u} = (c_{0u}^{(1)}, c_{0u}^{(2)})$. In all scenarios the standard deviation of the consumption rate is assumed to be $\sigma_{c_u}^{(1)} = 0.375 c_{0u}^{(1)}$ for item 1 and $\sigma_{c_u}^{(2)} = 0.4 c_{0u}^{(2)}$ for item 2.

The probabilities of the various scenarios are $p_1 = 0.04982, p_2 = 0.16095, p_3 = 0.28923, p_4 = 0.28923, p_5 = 0.16095, p_6 = 0.04982$.

The proportionality factors for the inventory holding and the shortage costs are $q^{+(1)} = 1.2$, $q^{+(2)} = 1$, $q^{-(1)} = 120$ and $q^{-(2)} = 100$.

6.2.2 Numerical Results

The results for the base case are given in **Table 2**.

		u=1	u=2	u=3	u=4	u=5	u=6
Given initial stock	Item1	2	2	2	2	2	2
	Item2	1.9	1.9	1.9	1.9	1.9	1.9
Optimal additional stock amount	Item1	0	0	0.0993	0.2225	0.3455	0.5640
	Item2	0	0	0.0381	0.2738	0.5091	0.6522
Total initial safety stock	Item1	2	2	2.0993	2.2225	2.3455	2.564
	Item2	1.9	1.9	1.9381	2.1738	2.4091	2.5522
Total expected consumption	Item1	3.2	3.4	3.6	3.8	4	4.2
	Item2	2.4	2.8	3.2	3.6	4	4.4
Proportion of initial safety stock to total exp. consumption	Item1	62.5%	58.82%	58.31%	58.49%	58.64%	61.05%
	Item2	79.17%	67.86%	60.57%	60.38%	60.23%	58.00%
Probability of ensuring continuous production of	Item1	0.9167	0.8980	0.8951	0.8961	0.8970	0.9097
	Item2	0.9615	0.9271	0.8937	0.8927	0.8919	0.8794
Probability of ensuring continuous production		0.8815	0.8325	0.80	0.80	0.80	0.80
		The optimal objective function value is 8.674					

Table 2

In our first example we multiply the expectation and standard deviation of the consumption rate of item 1 in each scenario by a factor represented on the horizontal axis and observe the changes in additional safety stock levels for both items 1 and 2. The results for scenario 6 can be seen in **Figure 8**.

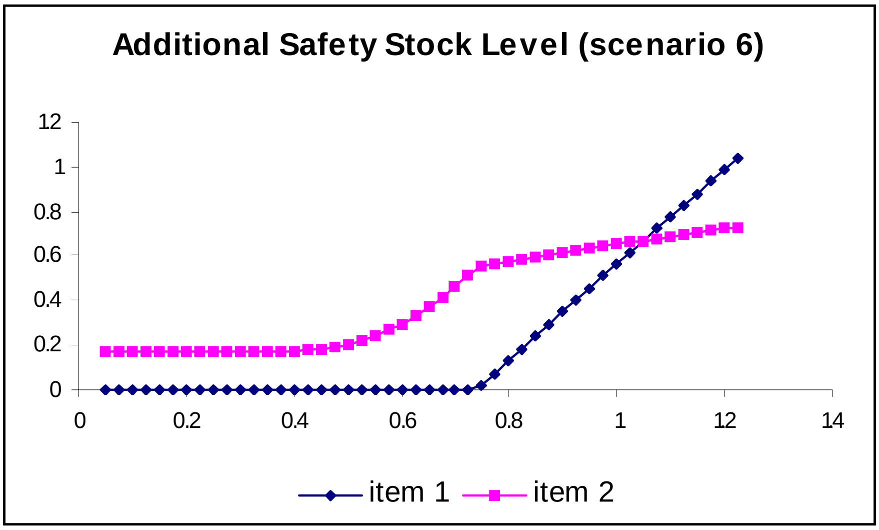

Figure 8

Figure 9 and **Figure 10** show the effects of changing the inventory holding cost of item 1 (represented on the horizontal axis) on the additional safety stock levels of items 1 and 2 for scenarios 4 and 6, respectively.

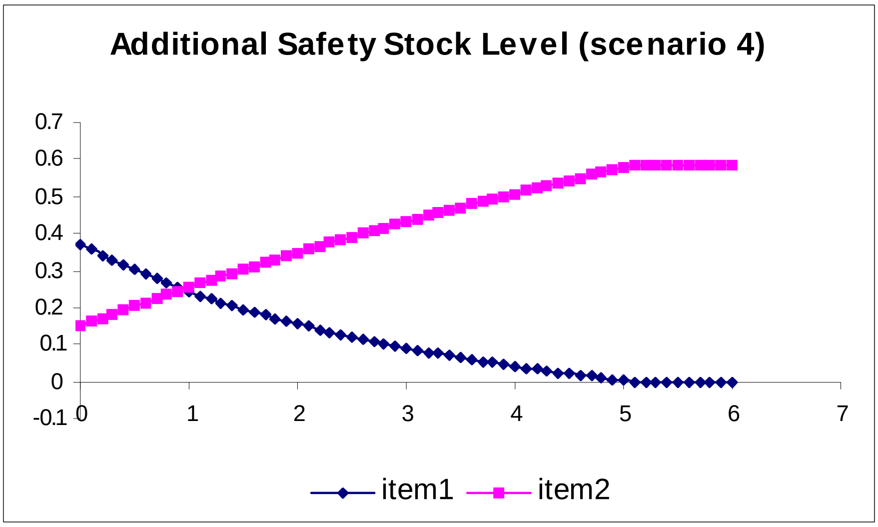

Figure 9

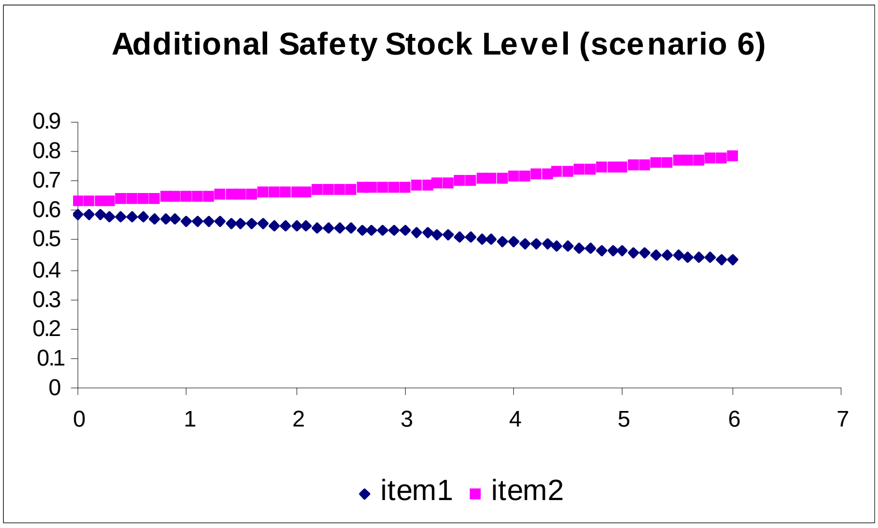

Figure 10

Finally we vary the initial safety stock level of item 1; the changes in additional safety stock levels of items 1 and 2 are shown in **Figure 11** for scenario 6.

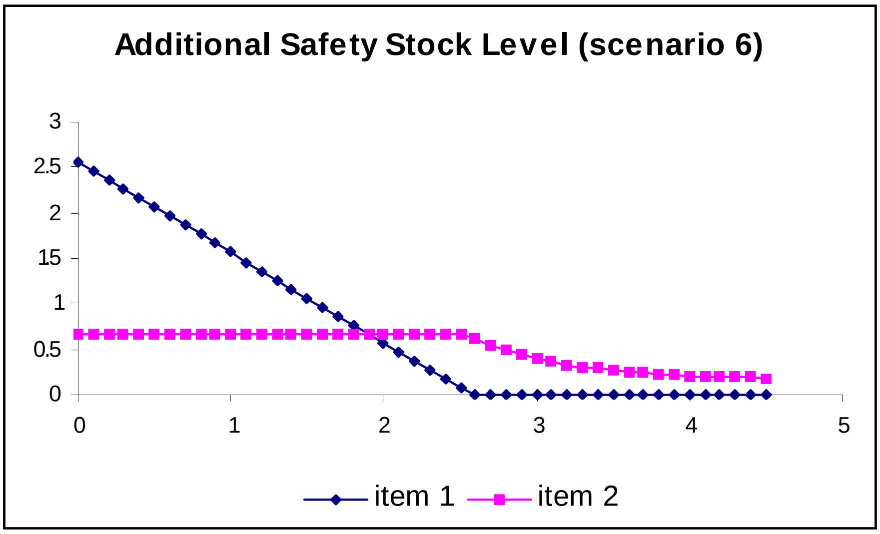

Figure 11

7 Conclusion

In the Hungarian inventory control model both the deliveries of the ordered amounts and/or consumption take place in intervals, according to some random processes, rather than at discrete time epochs. Inventory control models of this type have been introduced by Prékopa (1965) and Ziermann (1964). It seems to us that in many practical applications one encounters similar situations (see Morris et al., 1987 and Segal, 1997) and thus we think that the Hungarian inventory models deserve more attention than that has been paid to these models in the literature so far. The original Hungarian inventory control model is about to calculate an initial safety stock for one period with a prescribed service (reliability) level. No costs are taken into consideration. Since then, however, a few dynamic variants of it have appeared which, in addition, included cost parameters as well. In a recent paper by Prékopa (2004) a class of dynamic type Hungarian inventory models has been introduced where high service level is ensured by probabilistic constraint and various costs are taken into account. The problems in these models are, however, difficult to solve because we need the calculation of the multivariate Dirichlet type c.d.f values along with nonlinear programming algorithm.

In this paper we keep some of the main characteristics of the new models in Prékopa (2004) but introduce simplifying assumptions: we assume that the consumption process is linear with random, normally distributed slope and the expectation of total consumption minus the delivery process is approximated by a Brownian bridge. The Brownian bridge is then transformed into a Brownian motion process, of which the probability distribution of the maximum functional is available in closed form (see Bachelier, 1900 and Takács, 1967). In our setting this formula provides us with the c.d.f. of the supremum of consumption minus delivery process. First we formulated a static type probabilistically constrained multi-item inventory control model. Then we obtained two-stage type stochastic programming model which is supplemented by probabilistic constraints on the solvability of the second stage problem.

The numerical solutions for our problems are convex nonlinear programming problems like in Prékopa (2004). However, under our simplifying assumptions the 'Hungarian inventory control model' becomes computationally tractable. We have used a MATLAB implementation of a nonlinear optimization algorithm based on the Sequential Quadratic Programming (SQP) method and we are able to efficiently solve problems of moderate-size. For example, it takes 15 minutes (wallclock time) to solve a problem with 30 items and 10 scenarios on a PC with Pentium 4 2.40 GHz. We have also applied sensitivity analysis by changing the parameters in the numerical examples and demonstrated that the results produced by the model are consistent with the expected behavior of an inventory control system.

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References

[1] Auslender, A. and M. Haddou, 1995, An interior proximal method for convex linearly constrained problems and its extension to variational inequalities, Mathematical Programming 71, 77–100.

[2] Bachelier, L., 1900, Thorie de la spculation, Thses prsentes a la facult des sciences de Paris (Gauthier-Villars, Paris).

[3] Baxter, G. and M. D. Donsker, 1957, On the distribution of the supremum functional for processes with stationary independent increments, Transactions of the American Mathematical Society 85, 73–87.

[4] Boggs, P. T., A. J. Kearsley, and J. W. Tolle, 1994, A practical algorithm for general large scale nonlinear optimization problems, SIAM Journal on Optimization 3, Vol. 9, 755–778.

[5] Coleman, T.F. and Y. Li, 1996, An interior trust region approach for nonlinear minimization subject to bounds, SIAM Journal on Optimization 6, 418–445.

[6] Conn, A. R., N. I. M. Gould and Ph. L. Toint, 1994, Large-scale nonlinear constrained optimization: a current survey, Technical Report TR/PA/94/03, CERFACS (Toulouse, France).

[7] Dentcheva, D., A. Prékopa and A. Ruszczy'nski, 1999, Bounds for probabilistic integer programming problems, RUTCOR-Rutgers Center for Operations Research RRR 31-99.

[8] Donsker, M. D., 1952, Justification and extension of Doob's heuristic approach to the Kolmogorov–Smirnov theorems, Annals of Mathematical Statistics 23, 277–281.

[9] Doob, J. L., 1949, Heuristic approach to the Kolmogorov–Smirnov theorems, Annals of Mathematical Statistics 20, 393–403.

[10] Doob, J. L., 1953, Stochastic processes, Wiley.

[11] Gill, P. E., W. Murray, and M. H. Wright., 1981, Practical optimization, Academic Press (London).

[12] Gould, N. I. M. and Ph. L. Toint, 2000, SQP methods for large-scale nonlinear programming, in: M. J. D. Powell and S. Scholtes, eds., System modelling and optimization, methods, theory and applications, 149–178, Kluwer Academic Publishers (Dordrecht, The Netherlands).

[13] Han, S. P., 1977, A globally convergent method for nonlinear programming, Journal of Optimization Theory and Applications 22, 297.

[14] Kelle, P., 1984, On the safety stock problem for random delivery processes, European Journal of Operational Research 17, 191–200.

[15] Morris, P. A., J. Sandling, R. B. Fancher, M. A. Kohn, H. Chao and S. W. Chapel, 1987, An utility fuel inventory control model, Operations Research 35, 169–183.

[16] Powell, M. J. D., 1978a, The convergence of variable metric methods for nonlinearly constrained optimization calculations, Journal Nonlinear Programming 3, in: O. L. Mangasarian, R. R. Meyer, and S. M. Robinson, eds., Academic Press, 27–63.

[17] Powell, M. J. D., 1978b, A fast algorithm for nonlineary constrained optimization calculations, Numerical Analysis, in: G. A. Watson, ed., Lecture notes in mathematics, Springer Verlag, Vol. 630.

[18] Powell, M. J. D., 1983, Variable metric methods for constrained optimization, Mathematical Programming, in: A. Bachem, M. Grotschel and B. Korte, eds., The state of the art, Springer Verlag, 288–311.

[19] Prékopa, A., 1965, Reliability equation for an inventory problem and its asymptotic solutions, in: A. Prékopa, ed., Application of the Mathematics to Economics, Publication House of the Hungarian Academy of Science, 317–327.

[20] Prékopa, A., 1973a, Generalizations of the theorems of Smirnov with application to a reliability type inventory problem, Mathematische Operationsforschung und Statistik 4, 283–297.

[21] Prékopa, A., 1973b, On logarithmic concave measures and functions, Acta Scientiarum Mathematicarum (Szeged) 34, 335–343.

[22] Prékopa, A., P. Kelle, 1978, Reliability type inventory models based on stochastic programming, Mathematical Programming Study 9, 43–58.

[23] Prékopa, A., 1980, Reliability type inventory models, A survey, in: A. Chikán, ed., Studies in production and engineering ecomonics (Elsevier, Amsterdam) 477–490.

[24] Prékopa, A., 1990, Dual method for a one-stage stochastic programming problem with random RHS obeying a discrete probability distribution, Zeithschrifth für Operations Research 34, 441–461.

[25] Prékopa, A., 1995, Stochastic programming, Kluwer Academic Publishers (Dordrecht).

[26] Prékopa, A., 2003, Probabilistic programming, in: A. Ruszczy'nski and A. Shapiro, eds., Handbooks in operations research and management science, Vol. 10 (Elsevier, New York) 267–351.

[27] Prékopa, A., 2004, On the Hungarian inventory control model, RUTCOR-Rutgers Center for Operations Research RRR 17-04, To appear in European Journal of Operational Research, available online 14 March, 2005.

[28] Segal, M., 1997, PAPYRUS, a forecasting optimization and planning system for the paper industry, Presentation at the ORSA New Jersey Chapter meeting on November 6 at RUTCOR (Rutgers Center for Operations Research, Rutgers University, New Brunswick, N.J).

[29] Shanno, D. F., 1970, Conditioning of quasi-newton methods for function minimization, Mathematics of Computing 24, 647–656.

[30] Takács, L., 1967, Combinatorial methods in the theory of stochastic processes, Wiley (New York).

[31] Ziermann, M., 1964, Application of Smirnov's theorems for an inventory control problem, Publications of the Mathematical Institute of the Hungarian Academy of Science Series B 8, 509–518 (in Hungarian).