RUTCOR
RESEARCH
REPORT

# On Convex Probabilistic Programming with Discrete Distributions

Darinka Dentcheva[a] András Prékopa[b]
Andrzej Ruszczyński[c]

RRR 49-2000, September, 2000

RUTCOR
Rutgers Center for
Operations Research
Rutgers University
640 Bartholomew Road
Piscataway, New Jersey
08854-8003
Telephone: 732-445-3804
Telefax: 732-445-5472
Email: rrr@rutcor.rutgers.edu
http://rutcor.rutgers.edu/~rrr

[a]RUTCOR, Rutgers University, 640 Bartholomew Road, Piscataway, NJ 08854-8003, USA
[b]RUTCOR, Rutgers University, 640 Bartholomew Road, Piscataway, NJ 08854-8003, USA
[c]RUTCOR, Rutgers University, 640 Bartholomew Road, Piscataway, NJ 08854-8003, USA

Rutcor Research Report
RRR 49-2000, September, 2000

# On Convex Probabilistic Programming with Discrete Distributions

Darinka Dentcheva   András Prékopa   Andrzej Ruszczyński

**Abstract.** We consider convex stochastic programming problems with probabilistic constraints involving integer-valued random variables. The concept of a $p$-efficient point of a probability distribution is used to derive various equivalent problem formulations. Next we introduce the concept of $r$-concave discrete probability distributions and analyse its relevance for problems under consideration. These notions are used to derive lower and upper bounds for the optimal value of probabilistically constrained convex stochastic programming problems with discrete random variables.

# 1 Introduction

Let $f : \mathbb{R}^n \to \mathbb{R}$ and $g : \mathbb{R}^n \to \mathbb{R}^m$ be concave functions, and let $\mathcal{D}$ be a closed convex set. If in the convex program

$$\begin{aligned} \max \quad & f(x) \\ \text{subject to} \quad & g(x) \geq \xi, \\ & x \in \mathcal{D}, \end{aligned}$$

the vector $\xi$ is random, we require that $g(x) \geq \xi$ shall hold at least with some prescribed probability $p \in (0, 1)$, rather than *for all* possible realizations of the right hand side. This leads to the *convex programming problem with probabilistic constraints*:

$$\begin{aligned} \max \quad & f(x) \\ \text{subject to} \quad & \mathbb{P}\{g(x) \geq \xi\} \geq p, \\ & x \in \mathcal{D}, \end{aligned} \tag{1}$$

where the symbol $\mathbb{P}$ denotes probability. Programming under probabilistic constraints was initiated by Charnes, Cooper and Symonds in [4]. They formulated probabilistic constraints individually for each stochastic constraint. Joint probabilistic constraints for independent random variables were used first by Miller and Wagner in [10]. The general case was introduced and first studied by the second author of the present paper in [12, 15].

Much is known about problem (1) in the case where $\xi$ has a continuous probability distribution (see [17] and the references therein). However, the convex case with a discrete distribution has not been addressed yet.

Although we concentrate on integer random variables, all our results easily extend to other discrete distributions with non-uniform grids, under the condition that a uniform lower bound on the distance of grid points in each coordinate can be found.We use $\mathbb{Z}$ and $\mathbb{Z}_+$ to denote the set of integers and nonnegative integers, respectively. The inequality '$\geq$' for vectors is always understood coordinate-wise.

# 2 $p$-Efficient Points

Let us define the set

$$\mathcal{Z}_p = \{y \in \mathbb{R}^s : \mathbb{P}(\xi \leq y) \geq p\}. \tag{2}$$

Clearly, problem (1) can be compactly rewritten as

$$\begin{aligned} \max \quad & f(x) \\ \text{subject to} \quad & g(x) \in \mathcal{Z}_p, \\ & x \in \mathcal{D}. \end{aligned} \tag{3}$$

The structure of $\mathcal{Z}_p$ needs to be analysed in more detail. Let $F$ denote the probability distribution function of $\xi$, and $F_i$ the marginal probability distribution function ofthe $i$th component $\xi_i$. By assumption, the set $\mathcal{Z}$ of all possible values of the random vector $\xi$ is included in $\mathbb{Z}^s$. We shall use the concept of a $p$-efficient point, introduced in [16].

**Definition 2.1** *Let $p \in [0,1]$. A point $v \in \mathbb{R}^s$ is called a* $p$-efficient *point of the probability distribution function $F$, if $F(v) \geq p$ and there is no $y \leq v$, $y \neq v$ such that $F(y) \geq p$.*

Obviously, for a scalar random variable $\xi$ and for every $p \in (0,1)$ there is exactly one $p$-efficient point: the smallest $v$ such that $F(v) \geq p$. Since $F(v) \leq F_i(v_i)$ for every $v \in \mathbb{R}^s$ and $i = 1, \dots, s$, we have the following result.

**Lemma 2.2** *Let $p \in (0,1)$ and let $l_i$ be the $p$-efficient point of the one-dimensional marginal distribution $F_i$, $i = 1, \dots, s$. Then every $v \in \mathbb{R}^s$ such that$F(v) \geq p$ must satisfy the inequality $v \geq l = (l_1, \dots, l_s)$.*

Rounding down to the nearest integer does not change the value of the distribution function, so $p$-efficient points of a random vector with all integer components (shortly, *integer random vector*) must be integer. We can thus use Lemma 2.2 to get the following fact.

**Theorem 2.3** *For each $p \in (0,1)$ the set of $p$-efficient points of an integer random vector is nonempty and finite.*

**Proof.** The result follows from Dickson's Lemma [1, Cor. 4.48] and Lemma 2.2. □

Let $p \in (0,1)$ and let $v^j$, $j \in J$, be *all* $p$-efficient points of $\xi$. By Theorem 2.3, $J$ is a finite set. Let us define the cones

$$K_j = v^j + \mathbb{R}^s_+, \quad j \in J.$$

**Remark 2.4** $\mathcal{Z}_p = \bigcup_{j \in J} K_j$.

**Proof.** If $y \in \mathcal{Z}_p$ then either $y$ is $p$-efficient or there exists an integer $v \leq y$, $v \neq y$, $v \in \mathcal{Z}_p$. By Lemma 2.2, one must have $l \leq v$. Since there are only finitely many integer points $l \leq v \leq y$ one of them, $v_j$, must be $p$-efficient, and so $y \in K_j$. □ Thus, we obtain (for $0 < p < 1$) the following *disjunctive* formulation of (3):

$$\begin{array}{rl} \max & f(x) \\ \text{subject to} & g(x) \in \bigcup_{j \in J} K_j, \\ & x \in \mathcal{D}. \end{array} \tag{4}$$

Its main advantage is an insight into the nature of the non-convexity of the feasible set.

A straightforward way to solve (1) is to find all $p$-efficient points $v^j$, $j \in J$, and to process all convex programming problems

$$\begin{array}{rl} \max & f(x) \\ \text{subject to} & g(x) \geq v^j, \\ & x \in \mathcal{D}. \end{array} \tag{5}$$

Specialized bounding–pruning techniques can be used to avoid solving all of them.

For multi-dimensional random vectors $\xi$ the number of $p$-efficient points can be very large and their straightforward enumeration – very difficult. It would be desirable, therefore, to avoid the complete enumeration and to search for promising $p$-efficient points only. We shall return to this issue in section 5.

# 3 $r$-Concave Discrete Distribution Functions

Since the set $\mathcal{Z}_p$ need not be convex, it is essential to analyse its properties and to find equivalent formulations with more convenient structures. To this end we shall recall and adapt the notion of $r$-concavity of a distribution function. It uses the *generalized mean function* $m_r : \mathbb{R}_+ \times \mathbb{R}_+ \times [0,1] \to \mathbb{R}$ defined as follows:

$$m_r(a,b,\lambda) = 0 \quad \text{for} \quad ab = 0,$$

and if $a > 0$, $b > 0$, $0 \leq \lambda \leq 1$, then

$$m_r(a,b,\lambda) = \begin{cases} a^\lambda b^{1-\lambda} & \text{if } r = 0, \\ \max\{a,b\} & \text{if } r = \infty, \\ \min\{a,b\} & \text{if } r = -\infty, \\ (\lambda a^r + (1-\lambda)b^r)^{1/r} & \text{otherwise.} \end{cases}$$

**Definition 3.1** *A distribution function $F : \mathbb{R}^s \to [0,1]$ is called $r$-concave, where $r \in [-\infty, \infty]$, if*

$$F(\lambda x + (1-\lambda)y) \geq m_r(F(x), F(y), \lambda)$$

*for all $x, y \in \mathbb{R}^s$ and all $\lambda \in [0,1]$,*

If $r = -\infty$ we call $F$ *quasi-concave*, for $r = 0$ it is known as *log-concave*, and for $r = 1$ the function $F$ is concave in the usual sense.

The concept of a log-concave probability measure (the case $r = 0$) was introduced and studied in [13, 14]. The notion of $r$-concavity and corresponding results were given in [2, 3]. For detailed description and proofs, see [17].

By monotonicity, $r$-concavity of a distribution function is equivalent to the inequality

$$F(z) \geq m_r(F(x), F(y), \lambda)$$

for all $z \geq \lambda x + (1-\lambda)y$. Clearly, distribution functions of integer random variables are not continuous, and cannot be $r$-concave in the sense of the above definition. Therefore, we relax Definition 3.1 in the following way.

**Definition 3.2** *A distribution function $F$ is called* $r$-concaveon the set $\mathcal{A} \subset \mathbb{R}^s$ *with $r \in [-\infty, \infty]$, if*

$$F(z) \geq m_r(F(x), F(y), \lambda)$$

*for all $z, x, y \in \mathcal{A}$ and $\lambda \in (0,1)$ such that $z \geq \lambda x + (1-\lambda)y$.*

To illustrate the relation between the two definitions let us consider the case of integer random vectors which are roundups of continuously distributed random vectors.

**Remark 3.3** *If the distribution function of a random vector $\eta$ is $r$-concave on $\mathbb{R}^s$ then the distribution function of $\xi = \lceil \eta \rceil$ is $r$-concave on $\mathbb{Z}^s$.*

The last property follows from the observation that at integer points both distribution functions coincide. For the relations between the $r$-concavity of the distribution function of $\eta$ and the $r$-concavity of its density the Reader is referred to [2, 3, 19]. The concept of $r$-concavity on a set can be used to find an equivalent representation of the set $\mathcal{Z}_p$ given by (2).

**Theorem 3.4** *Let $\mathcal{Z}$ be the set of all possible values of an integer random vector $\xi$. If the distribution function $F$ of $\xi$ is $r$-concave on $\mathcal{Z} + \mathbb{Z}^s_+$, for some $r \in [-\infty, \infty]$, then for every $p \in (0,1)$ one has*

$$\mathcal{Z}_p = \{y \in \mathbb{R}^s : \; y \geq z \geq \sum_{j\in J} \lambda_j v^j, \; \sum_{j \in J} \lambda_j = 1, \; \lambda_j \geq 0, \; z \in \mathbb{Z}^s\},$$

*where $v^j$, $j \in J$, are the $p$-efficient points of $F$.*

**Proof.** By the monotonicity of $F$ we have $F(y) \geq F(z)$ if $y \geq z$. It is,therefore, sufficient to show that $\mathbb{P}(\xi \leq z) \geq p$ for all $z \in \mathbb{Z}^s$ such that $z \geq \sum_{j\in J} \lambda_j v^j$ with $\lambda_j \geq 0, \sum_{j \in J} \lambda_j = 1$. We consider five cases with respect to $r$.

*Case 1:* $r = \infty$. It follows from the definition of $r$-concavity that $F(z) \geq \max\{F(v^j), j \in J : \lambda_j \neq 0\} \geq p$.

*Case 2:* $r = -\infty$. Since $F(v^j) \geq p$ for each index $j \in J$ such that $\lambda_j \neq 0$, the assertion follows as in Case 1.

*Case 3:* $r = 0$. By the definition of $r$-concavity,

$$F(z) \geq \prod_{j\in J} [F(v^j)]^{\lambda_j} \geq \prod_{j \in J} p^{\lambda_j} = p.$$

*Case 4:* $r \in (-\infty, 0)$. By the definition of $r$-concavity,

$$[F(z)]^r \leq \sum_{j \in J} \lambda_j [F(v^j)]^r \leq \sum_{j\in J} \lambda_j p^r = p^r.$$

Since $r < 0$, we obtain $F(z) \geq p$.

*Case 5:*$r \in (0, \infty)$. By the definition of $r$-concavity,

$$[F(z)]^r \geq \sum_{j \in J} \lambda_j [F(v^j)]^r \geq \sum_{j\in J} \lambda_j p^r = p^r.$$

□ Under the conditions of Theorem 3.4, problem (4) can be formulated in the following equivalent way:

$$\begin{aligned} \max \quad & f(x) && (6)\\ \text{subject to} \quad & x \in \mathcal{D}, && (7)\\ & g(x) \geq z, && (8)\\ & z \in \mathbb{Z}^s, && (9)\\ & z \geq \sum_{j\in J} \lambda_j v^j, && (10)\\ & \sum_{j\in J} \lambda_j = 1, && (11)\\ & \lambda_j \geq 0,\ j \in J. && (12) \end{aligned}$$

So, the probabilistic constraint has been replaced by linear equations and inequalites, together with the integrality requirement (9).This condition cannot be dropped, in general. If $\xi$ takes values on a non-uniform grid, condition (9) should be replaced by the requirement that $z$ is a grid point. The difficulty comes from the implicitly given $p$-efficient points $v_j$, $j \in J$. Our objective will be to avoid their enumeration and to develop an approach that generates them only when needed. An obvious question arises: which distributions are $r$-concave in our sense? We devote the remaining part of this section to some useful observations on this topic.

Directly from the definition and Hölder's inequality we obtain the following property.

**Remark 3.5** *If a distribution function $F$ is $r$-concave on the set $\mathcal{A} \subset \mathbb{R}^s$ with some $r \in [-\infty, \infty]$, then it is $\rho$-concave on $\mathcal{A}$ for all $\rho \in [-\infty, r]$.*

For binary random vectors we have the strongest possible property.

**Proposition 3.6** *Every distribution function of an $s$-dimensional binary random vector is $r$-concave on $\mathbb{Z}^s_+$ for all $r \in [-\infty, \infty]$.*

**Proof.** Let $x, y \in \mathbb{Z}^s_+$, $\lambda \in (0,1)$ and let $z \geq \lambda x + (1-\lambda)y$. By projecting $x$ and $y$ on $\{0,1\}^s$ we get some $x'$ and $y'$ such that $F(x') = F(x)$, $F(y') = F(y)$ and $z \geq \lambda x' + (1-\lambda)y'$. Since $z$ is integer and $x'$ and $y'$ binary, then $z \geq x'$ and $z \geq y'$. Thus $F(z) \geq \max(F(x'), F(y')) = \max(F(x), F(y))$. Consequently, $F$ is $\infty$-concave and the result follows from Remark 3.5. □
For scalar integer random variables our definition of $r$-concavity is related to log-concavity of sequences. A sequence $p_k$, $k = \dots, -1, 0, 1, \dots$, is called *log-concave*, if $p_k^2 \geq p_{k-1}p_{k+1}$ for all $k$. By [6] (see also [17, Thm. 4.7.2]) and Remark 3.5, we have the following property.

**Proposition 3.7** *Suppose that for a scalar integer random variable $\xi$ the probabilities $p_k = \mathbb{P}\{\xi = k\}$, $k = \dots, -1, 0, 1, \dots$, form a log-concave sequence. Then the distribution function of $\xi$ is $r$-concave on $\mathbb{Z}$ for every $r \in [-\infty, 0]$.*

Many well-known one-dimensional discrete distributions satisfy the conditions of Proposition 3.7: the Poisson distribution, the geometrical distribution, the binomial distribution [17, p. 109].

We end this section with sufficient conditions for the $r$-concavity of the joint distribution function in the case of integer-valued independent subvectors. Our assertion, presented in the next proposition is the discrete version of an observation from [11]. The same proof, using Hölder's inequality, works in our case as well.

**Proposition 3.8** *Assume that $\xi = (\xi^1, \dots, \xi^L)$, where the $s_l$-dimensional subvectors $\xi_l$, $i = l, \dots, L$, are independent ($\sum_{l=1}^L s_l = s$). Furthermore, let the marginal distribution functions $F_l : \mathbb{R}^{s_l} \to [0,1]$ be $r_l$-concave on sets $\mathcal{A}_l \subset \mathbb{Z}^{s_l}$.*

*(i) If $r_l > 0$, $l = 1, \dots, L$, then $F$ is $r$-concave on $\mathcal{A} = \mathcal{A}_1 \times \cdots \times \mathcal{A}_L$ with $r = \left(\sum_{l=1}^L r_l^{-1}\right)^{-1}$;*

*(ii) If $r_l = 0$, $l = 1, \dots, L$, then $F$ is log-concave on $\mathcal{A} = \mathcal{A}_1 \times \cdots \times \mathcal{A}_L$.*

# 4 Lagrangian Relaxation

Let us split variables in problem (3):

$$\begin{aligned} \max \quad & f(x) \\ & g(x) \ge z, \\ & x \in \mathcal{D}, \\ & z \in \mathcal{Z}_p. \end{aligned} \tag{13}$$

Associating Lagrange multipliers $u \in \mathbb{R}^s_+$ with constraints (13) we obtain the Lagrangian function:

$$L(x, z, u) = f(x) + u^T(g(x) - z).$$

The dual functional has the form

$$\Psi(u) = \sup_{(x,z) \in \mathcal{D} \times \mathcal{Z}_p} L(x, z, u) = h(u) - d(u),$$

where

$$h(u) = \sup\{f(x) + u^T g(x) \mid x \in \mathcal{D}\}, \tag{14}$$

$$d(u) = \inf\{u^T z \mid z \in \mathcal{Z}_p\}. \tag{15}$$

For any $u \in \mathbb{R}^s_+$ the value of $\Psi(u)$ is a upper bound on the optimal value $F^*$ of the original problem. This is true irrespectively whether the distribution function of $\xi$ is or is not $r$-concave.

The best Lagrangian upper bound will be given by

$$D^* = \inf_{u \geq 0} \Psi(u). \tag{16}$$

By Remark 2.4, for $u \geq 0$ the minimization in (15) may be restricted to finitely many $p$-efficient points $v^j$, $j \in J$. For $u \not\geq 0$ one has $d(u) = -\infty$. Therefore, $d(\cdot)$ is concave and polyhedral. We also have

$$d(u) = \inf\{u^T z \mid z \in \operatorname{co} \mathcal{Z}_p\}. \tag{17}$$

Let us consider the *convex hull problem*:

$$\max \quad f(x) \tag{18}$$
$$g(x) \geq z, \tag{19}$$
$$x \in \mathcal{D}, \tag{20}$$
$$z \in \operatorname{co} \mathcal{Z}_p. \tag{21}$$

We shall make the following assumption.

**Constraint Qualification Condition.** *There exist points $x^0 \in \operatorname{ri} \mathcal{D}$ and $z^0 \in \operatorname{co} \mathcal{Z}_p$ such that $g(x^0) > z^0$.*

If the Constraint Qualification Condition is satisfied, from the duality theory in convex programming [9, Sec.1.1.2] we know that there exists $\hat{u} \geq 0$ at which the maximum in (16) is attained, and $D^* = \Psi(\hat{u})$ is the optimal value of the convex hull problem (18)–(21).

We now study in detail the structure of the dual functional $\Psi$. Properties of $d(\cdot)$ can be analysed in a more explicit way. Define

$$V(u) = \{v \in \mathbb{R}^m : u^T v = d(u) \text{ and } v \text{ is a } p\text{-efficient point}\}, \tag{22}$$
$$C(u) = \{d \in \mathbb{R}^s_+ : d_i = 0 \text{ if } u_i > 0,\ i = 1, \ldots, s\}. \tag{23}$$

**Lemma 4.1** *For every $u \geq 0$ the solution set of (15) is nonempty and has the following form: $\hat{Z}(u) = V(u) + C(u)$.*

**Proof.** The result follows from Remark 2.4. Let us at first consider the case $u > 0$. Suppose that a solution $z$ to (15) is not a $p$-efficient point. Then there is a $p$-efficient $v \in \mathcal{Z}_p$ such that $v \leq z$, so $u^T v < u^T z$, a contradiction. Thus, for all $u \geq 0$ all solutions to (15) are $p$-efficient. In the general case $u \geq 0$, if a solution $z$ is not $p$-efficient, we must have $u^T v = u^T z$ for all $p$-efficient $v \leq z$. This is equivalent to $z \in \{v\} + C(u)$, as required. □ The last result allows us to calculate the subdifferential of $d$ in a closed form.

**Lemma 4.2** *For every $u \geq 0$ one has $\partial d(u) = V(u) + C(u)$.*

**Proof.** From (15) it follows that $d(u) = -\delta^*_{\mathcal{Z}_p}(-u)$, where $\delta^*_{\mathcal{Z}_p}(\cdot)$ is the support function of $\mathcal{Z}_p$ and, consequently, of $\operatorname{co}\mathcal{Z}_p$. This fact follows from the structure of $\mathcal{Z}_p$ (Remark 2.4) by virtue of Corolarry 16.5.1 in [20]. By [20, Thm 23.5], $g \in \partial\delta^*_{\mathcal{Z}_p}(-u)$ if and only if $\delta^*_{\mathcal{Z}_p}(-u) + \delta_{\operatorname{co}\mathcal{Z}_p}(g) = -g^T u$, where $\delta_{\operatorname{co}\mathcal{Z}_p}(\cdot)$ is the indicator function of $\operatorname{co}\mathcal{Z}_p$. It follows that $g \in \operatorname{co}\mathcal{Z}_p$ and $\delta^*_{\mathcal{Z}_p}(-u) = -g^T u$. Thus, $g$ is a convex combination of solutions to (15) and the result follows from Lemma 4.1. □

Let us turn now to the function $h(\cdot)$. Define the set of maximizers in (14),

$$X(u) = \{x \in \mathcal{D} : f(x) + u^T g(x) = h(u)\}.$$

**Lemma 4.3** *Assume that the set $\mathcal{D}$ is compact. Then the function $h$ is convex on $\mathbb{R}^m$ and for every $u \in \mathbb{R}^m$,*

$$\partial h(u) = \operatorname{co}\{g(x) : x \in X(u)\}.$$

Therefore the following necessary and sufficient optimality conditions for problem (16) can be formulated.

**Theorem 4.4** *Assume that the Constraint Qualification Condition is satisfied and the set $\mathcal{D}$ is compact. A vector $u \geq 0$ is an optimal solution of (16) if and only if there exists a point $x \in X(u)$, points $v^1, \dots, v^{m+1} \in V(u)$ and scalars $\beta_1 \dots, \beta_{m+1} \geq 0$ with $\sum_{j=1}^{m+1} \beta_j = 1$, such that*

$$g(x) - \sum_{j=1}^{m+1} \beta_j v^j \in C(u). \tag{24}$$

*where $C(u)$ is given by (23).*

**Proof.** Since $-C(u)$ is the normal cone to the positive orthant at $u \geq 0$, the necessary and sufficient condition for (16) has the form

$$\partial\Psi(u) \cap C(u) \neq \emptyset$$

(cf. [9, Sec.1.1.2]). Using Lemma 4.2 and Lemma 4.3, we conclude that there exist

$$\begin{aligned}
&p\text{-efficient points } v^j \in V(u), \quad j = 1, \dots, m+1,\\
&\beta^j \geq 0, \quad j = 1, \dots, m+1, \quad \sum_{j=1}^{m+1} \beta_j = 1,\\
&x^j \in X(u), \quad j = 1, \dots, m+1,\\
&\alpha^j \geq 0, \quad j = 1, \dots, m+1, \quad \sum_{j=1}^{m+1} \alpha_j = 1,
\end{aligned} \tag{25}$$

such that

$$\sum_{j=1}^{m+1} \alpha_j g(x^j) - \sum_{j=1}^{m+1} \beta_j v^j \in C(u). \tag{26}$$

Let us define

$$x = \sum_{j=1}^{m+1} \alpha_j x^j.$$

We have

$$f(x) + \sum_{i=1}^{m} u_i g_i(x) = f(x^j) + \sum_{i=1}^{m} u_i g_i(x^j), \quad j = 1, \dots, m+1. \tag{27}$$

Indeed, the inequality $\geq$ above follows from the concavity of $f$ and $g_j$, and the inequality $\leq$ is implied by (25).

By the concavity, $g_i(x) \geq \sum_{j=1}^{m+1} \alpha_j g_i(x^j)$. Suppose that $u_i > 0$. Then we must have $g_i(x) = \sum_{j=1}^{m+1} \alpha_j g_i(x^j)$, since the strict inequality contradicts (27). It follows that

$$g(x) - \sum_{j=1}^{m+1} \alpha_j g(x^j) \in C(u).$$

Therefore relation (26) can be simplified as (24), as required. □

Since the set of $p$-efficient points is not known, we need a numerical method for solving the convex hull problem (18)–(21) or its dual (16).

# 5 The cone generation method

The idea of a numerical method for calculating Lagrangian bounds is embedded in the convex hull formulation(18)–(21). We shall develop for it a new specialized method, which separates the generation of $p$-efficient points and the solution of the approximation of the original problem using these points. It is related to column generation methods, which have been known since the classical work [7] as extremely useful tools of large scale linear and integer programming.

### The Method

**Step 0:** Select a $p$-efficient point $v^0$. Set $J_0 = \{0\}$, $k = 0$.

**Step 1:** Solve the *master problem*

$$\max \quad f(x) \tag{28}$$
$$g(x) \geq \sum_{j \in J_k} \lambda_j v^j, \tag{29}$$
$$\sum_{j \in J_k} \lambda_j = 1, \tag{30}$$
$$x \in \mathcal{D}, \ \lambda \geq 0. \tag{31}$$

Let $u^k$ be the vector of simplex multipliers associated with the constraint (29).

**Step 2:** Calculate $\overline{d}(u^k) = \min_{j \in J_k} (u^k)^T v^j$.

**Step 3:** Find a $p$-efficient solution $v^{k+1}$ of the subproblem: $\min_{z \in \mathcal{Z}_p} (u^k)^T z$ and calculate $d(u^k) = (v^{k+1})^T u^k$.

**Step 4:** If $d(u^k) = \overline{d}(u^k)$ then stop; otherwise set $J_{k+1} = J_k \cup \{k+1\}$, increase $k$ by one and go to Step 1.

A few comments are in order. The first $p$-efficient point $v^0$ can be found by solving the subproblem at Step 3 for an arbitrary $u \geq 0$. All master problems will be solvable, if the first one is solvable, i.e., if the set $\{x \in \mathcal{D} : g(x) \geq v^0\}$ is nonempty. If not, adding a penalty term $M\mathbb{1}^T t$ to the objective, and replacing (29) by

$$g(x) + t \geq \sum_{j \in J_k} \lambda_j v^j,$$

with $t \geq 0$ and a very large $M$, is the usual remedy ($\mathbb{1}^T = [1\ 1\ \ldots\ 1]$). The calculation of the upper bound at Step 2 is easy, because one can simply select $j_k \in J_k$ with $\lambda_{j_k} > 0$ and set $\overline{d}(u^k) = (u^k)^T v^{j_k}$. At Step 3 one may search for $p$-efficient solutions only, due to Lemma 4.1. The algorithm is finite. Indeed, the set $J_k$ cannot grow indefinitely, because there are finitely many $p$-efficient points (Theorem 2.3). If the stopping test of Step 4 is satisfied, optimality conditions of Theorem 4.4 are satisfied. Moreover $\hat{J}_k = \{j \in J_k : \langle v^j, u^k \rangle = d(u^k)\} \subseteq \hat{J}(u)$.

# 6 Primal feasible solution and upper bounds

Let us consider the optimal solution $x^{\text{low}}$ of the convex hull problem (18)–(21) and the corresponding multipliers $\lambda_j$. Define $J^{\text{low}} = \{j \in J : \lambda_j > 0\}$.

If $J^{\text{low}}$ contains only one element, the point $x^{\text{low}}$ is feasible and therefore optimal for the disjunctive formulation (4). If, however, there are more positive $\lambda$'s, we need to generate a feasible point. A natural possibility is to consider the *restricted disjunctive* formulation:

$$\begin{aligned} \max \quad & f(x) \\ \text{subject to} \quad & g(x) \in \textstyle\bigcup_{j \in J^{\text{low}}} K_j, \\ & x \in \mathcal{D}. \end{aligned} \tag{32}$$

It can be solved by simple enumeration of all cases for $j \in J^{\text{low}}$:

$$\begin{array}{rl} \max & f(x) \\ \text{subject to} & g(x) \geq v^j, \\ & x \in \mathcal{D}. \end{array} \tag{33}$$

An alternative strategy would be to solve the corresponding upper bounding problem (33) every time a new $p$-efficient point is generated. If $U_j$ denotes the optimal value of (33), the upper bound at iteration $k$ is

$$\bar{U}^k = \min_{0 \leq j \leq k} U_j. \tag{34}$$

If the distribution function of $\xi$ is $r$-concave on the set of possible values of $\xi$, Theorem 3.4 provides an alternative formulation of the upper bound problem (32):

$$\begin{array}{rl} \max & f(x) \\ \text{subject to} & x \in \mathcal{D}, \\ & g(x) \geq z, \\ & z \in \mathbb{Z}^s, \\ & z \geq \displaystyle\sum_{j \in J_k} \lambda_j v^j, \\ & \displaystyle\sum_{j \in J_k} \lambda_j = 1, \\ & \lambda_j \geq 0,\ j \in J_k. \end{array} \tag{35}$$

Problem (35) is more accurate than the bound (34), because the set of integer $z$ dominated by convex combinations of $p$-efficient points in $J_k$ is not smaller than $J_k$. In fact, we need to solve this problem only at the end, with $J_k$ replaced by $J^{\text{low}}$.

# References

[1] T. Becker and V. Weispfenning, *Gröbner Bases*, Springer-Verlag, New York, 1993.

[2] C. Borell, Convex Set Functions in d-Space, *Periodica Mathematica Hungarica* 6 (1975) 111–136.

[3] H.J. Brascamp and E.H. Lieb, On Extensions of the Brunn–Minkowski and Prékopa–Leindler Theorems, Including Inequalities for Log-Concave Functions and with an Application to the Diffusion Equation, *Journal of Functional Analysis* 22 (1976) 366–389.

[4] A. Charnes, W.W.Cooper and G.H. Symonds, Cost Horizonsand Certainty Equivalents: An Approach to Stochastic Programming of Heating Oil, *Management Science* 4 (1958) 235–263.

[5] D. Dentcheva, A. Prékopa and A. Ruszczyński, Concavity and $p$-efficient points of discrete distributions in probabilistic programming, Rutcor Research Report 9-2000, Rutgers University Center for Operations Research.

[6] M. Fekete and G. Polya, Über ein Problem von Laguerre, *Rediconti dei Circolo Matematico di Palermo*, 23 (1912) 89–120.

[7] L.R Ford and D.R. Fulkerson, A Suggested Computation for Maximal Multicommodity Network Flows, *Management Science* 5 (1958) 97–101.

[8] J.-B. Hiriart-Urruty and C. Lemaréchal, *Convex Analysis and Minimization Algorithms I and II*, Springer-Verlag, Berlin 1993.

[9] A.D. Ioffe and V.M. Tikhomirov, *Theory of Extremal Problems*, Nauka, Moscow 1974.

[10] L.B. Miller and H. Wagner, Chance-Constrained Programming with Joint Constraints, *Operations Research* **13** (1965) 930–945.

[11] V.I. Norkin and N.V. Roenko, $\alpha$-Concave Functions and Measures and Their Applications, *Kibernet. Sistem. Anal.* 189 (1991) 77–88 (in Russian); translation in: *Cybernet. Systems Anal.* 27 (1991) 860–869.

[12] A. Prékopa, On Probabilistic Constrained Programming, *Proceedings of the Princeton Symposium on Mathematical Programming.* Princeton University Press, 1970, pp. 113–138.

[13] A. Prékopa, Logarithmic Concave Functions with Applications to Stochastic Programming, *Acta Sci. Math. (Szeged)* 32 (1971) 301–316.

[14] A. Prékopa, On Logarithmic Concave Measures and Functions, *Acta Sci. Math. (Szeged)* 34 (1973) 339–343.

[15] A. Prékopa, Contributions to the Theory of Stochastic Programming, *Mathematical Programming* 4 (1973) 202–221.

[16] A. Prékopa, Dual Method for a One-Stage Stochastic Programming Problem with Random RHS Obeying a Discrete Probability Distribution, *Zeitschrift für Operations Research* 34 (1990) 441–461.

[17] A. Prékopa, *Stochastic Programming*, Kluwer, Dordrecht, Boston, 1995.

[18] A. Prékopa, B. Vizvári, and T. Badics, Programming Under Probabilistic Constraint with Discrete Random Variable, in: *New Trends in Mathematical Programming*, L. Grandinetti et al. (Ed.), Kluwer, Dordrecht, Boston, 1998, pp. 235–255.

[19] Y. Rinott, On Convexity of Measures, *Annals of Probability* 4 (1976) 1020–1026.

[20] R. T. Rockafellar, *Convex Analysis*, Princeton University Press, Princeton, 1970.