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ON COMPOSED POISSON DISTRIBUTIONS, IV

(Remarks on the theory of differential processes)

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Introduction

In the present paper we shall use the following notations. Let ξ_t be a stochastic process. For the difference $\xi_{t_2} - \xi_{t_1}$ let us introduce the notation

$$\xi_J = \xi_{t_2} - \xi_{t_1},$$

where J means the finite time interval (t_1, t_2). Let further $W_\lambda(J)$ denote the following probability

$$W_\lambda(J) = Pr(\xi_J = \lambda)$$

and let $F(x, J)$ denote the probability distribution of the variable ξ_J. For the characteristic function of the distribution $F(x, J)$ we shall write $f(u, J)$:

$$f(u, J) = \int_{-\infty}^{\infty} e^{iux} \, \mathrm{d}F(x, J).$$

The process on which our discussions are based will be considered in a finite, closed time interval I in which the process satisfies the following conditions. (By J we shall henceforth denote a subinterval of I.)

A) If $J_1, J_2, \dots, J_n$ denote a subdivision of the interval I so that $I = J_1 + J_2 + \cdots + J_n$, the corresponding variables $\xi_{J_1}, \xi_{J_2}, \dots, \xi_{J_n}$ are independent.

B) The variables ξ_J can only assume the values of a countable set of real numbers $\lambda_0 = 0, \lambda_1, \lambda_2, \dots$ which set is independent of the special selection of J.

It follows form condition *A)* that this set must be closed under addition, since for every λ_k and λ_l it must contain a number λ_m such that $\lambda_k + \lambda_l = \lambda_m$, that is to say, it must form a semigroup with respect to addition.

C) $1 - W_0(J) = 1 - Pr\ (\xi_J = 0)$ is a continuous interval function, that is,

$$1 - W_0(J) \to 0,$$

if J contracts to a fixed point.

It follows from condition *C)* that $1 - W_0(J)$ is also uniformly continuous in I_1.[1] It follows likewise form condition *C)* that $1 - f(u, J)$ is also uniformly continuous, notably, in a manner independent of u, for

$$|1 - f(u, J)| \leq 2(1 - W_0(J)),$$

and, further, that the process ξ_t is weakly continuous, since

$$Pr(|\xi_J| > \varepsilon) \leq 1 - Pr\ (\xi_J = 0) \to 0.$$

In consequence of condition *A)* and *C)*, ξ_I possesses an infinitely divisible distribution,[2] and therefore $\log f(u, I)$ can be represented in the canonical form:

$$\begin{aligned} &\log f(u, I) = i\gamma(I)u - \frac{\sigma^2(I)}{2}u^2 \\ (1)\qquad &+ \int_{-\infty}^{0} \left(e^{iux} - 1 - \frac{iux}{1+x^2}\right) \mathrm{d}M(x, I) + \int_{0}^{\infty} \left(e^{iux} - 1 - \frac{iux}{1+x^2}\right) \mathrm{d}N(x, I), \end{aligned}$$

where $\gamma(I)$ and $\sigma(I)$ are constants, $M(x, I)$ and $N(x, I)$ are non-decreasing functions in the intervals $(-\infty, 0)$ and $(0, \infty)$, respectively; $M(-\infty) = N(\infty) = 0$ and

$$\int_{-1}^{0} x^2\, \mathrm{d}M(x, I) + \int_{0}^{1} x^2\, \mathrm{d}N(x, I) < \infty.$$

In our case the general form (1) reduces to a simpler expression in which the concrete meaning of the functions $M(x, I)$ and $N(x, I)$ can be indicated more specifically.

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[1] Let $\varphi(J)$ be an interval function continuous in the finite closed interval I; in this case $\varphi(J)$ is also uniformly continuous in I. For, in the contrary case it would be possible to find an ε_0 for which no δ exists, that is, if $\delta_n \to 0$, by selecting a suitable sequence J_n, we would obtain that $|\varphi(J_n)| > \varepsilon_0$ contrary to $|J_n| \leq \delta_n \to 0$. Let d denote a point of condensation of the centres of the intervals J_n, then J_n will have a subsequence J_{k_n} whose centres converge to d. As the sequence of intervals J_{k_n} contracts to a single point, to d, it follows that $\varphi(J_{k_n}) \to 0$, which is a contradiction.

[2] see [1], pp. 161–163.

§ 1. The meaning of $M(x, I)$ and $N(x, I)$

THEOREM. *If conditions A), B), C) are fulfilled, then the general form* (1) *can be written in the following manner:*

$$\log f(u, I) = \sum_{k=1}^{\infty} C_{\lambda_k}(I)(e^{i\lambda_k u)} - 1), \tag{2}$$

where

$$C_{\lambda_k}(I) = \int_I W_{\lambda_k}(J), \qquad \lambda_k \neq 0 \tag{3}$$

and

$$\sum_{k=1}^{\infty} C_{\lambda_k}(I) = \int_I (1 - W_0(J)) < \infty. \tag{4}$$

In (3) and (4) the integrals in the interval I are taken in the sense of BURKILL.

PROOF. We first show that $\sigma(I) = 0$. In fact, the function $f(u, I)$ in formula (1) is the product of two characteristic functions:

$$f(u, I) = f_1(u, I) f_2(u, I),$$

where

$$f_1(u, I) = e^{-\frac{\sigma^2(I)}{2} u^2}$$

is the characteristic function of the normal distribution

$$F_1(x, I) = \frac{1}{\sqrt{2\pi}\sigma(I)} \int_{-\infty}^{x} e^{-\frac{y^2}{2\sigma^2(I)}} \,\mathrm{d}y.$$

As

$$|F_1(x + h, I) - F_1(x, I)| \leq \frac{|h|}{\sqrt{2\pi}\sigma(I)} \quad \text{and} \quad F(x, I) = \int_{-\infty}^{\infty} F_1(x - y, I) \,\mathrm{d}F_2(y, I),$$

it follows that

$$|f(x + h, I) - F(x, I)| \leq \frac{|h|}{\sqrt{2\pi}\sigma(I)}.$$

This is, however, a contradiction, because we have supposed that $F(x, I)$ is a step function. (For the proof, see [3], pp. 94–95.)

$f(u, I)$ is an almost periodic function of the variable u,

$$f(u, I) = \sum_{k=0}^{\infty} W_{\lambda_k}(I) e^{i\lambda_k u}.$$

Together with $f(u,I)$, $\log f(u,I)$ is also an almost periodic function. For the proof we have only to show that $|f(u,I)| \geq \delta > 0$, because $\log z$ is continuous in the region $0 < \delta \leq |z| \leq 1$, and it is well known that any continuous function of an almost periodic function is itself almost periodic. In fact, if the decomposition $I = J_1 + J_2 + \cdots + J_n$ is carried out, where $|J_k|$ (the length of J_k) is sufficiently small to permit that in accordance with condition *C)* $|f(u,J_k)| \geq \eta > 0$, it will follow that

$$|f(u,I)| = \prod_{k=1}^{n} |f(u,J_k)| \geq \eta^n = \delta > 0.$$

Let us now multiply both sides of (1) by $\dfrac{1}{2T}$ and integrate in the interval $(-T,T)$; then by FUBINI's theorem, first integrating with respect to u, we obtain

$$-\frac{1}{2T}\int_{-T}^{T} \log f(u,I)\,\mathrm{d}u = \int_{-\infty}^{0}\left(1-\frac{\sin Tx}{Tx}\right)\mathrm{d}M(x,I) + \int_{0}^{\infty}\left(1-\frac{\sin Tx}{Tx}\right)\mathrm{d}N(x,I).$$

If $T \to \infty$, then, as $\log f(u,I)$, is almost periodic, we shall obtain on the left side a finite limit and therefore, taking into account also that

$$1-\frac{\sin Tx}{Tx} \geq 0,$$

we may apply FATOU's theorem to obtain

$$\int_{\infty}^{0}\mathrm{d}M(x,I) + \int_{0}^{\infty}\mathrm{d}N(x,I) < \infty. \tag{5}$$

Accordingly, $M(x,I)$ and $N(x,I)$ are of bounded variation and have a finite limit at the point $x = 0$. In view of (5), formula (1) may be brought to the following from:

$$\log f(u,I) = i\gamma'(I)u + \int_{-\infty}^{0}(e^{iux}-1)\,\mathrm{d}M(x,I) + \int_{0}^{\infty}(e^{iux}-1)\,\mathrm{d}N(x,I), \tag{6}$$

where

$$\gamma'(I) = \gamma(I) - \int_{-\infty}^{0}\frac{x}{1+x^2}\,\mathrm{d}M(x,I) - \int_{0}^{\infty}\frac{x}{1+x^2}\,\mathrm{d}N(x,I).$$

Here we have $\gamma'(I) = 0$, since the left side of (6) and the two last members of its right side are bounded functions of u.

$M(x,I)$ and $N(x,I)$ are step functions; this fact follows at once if we take into account that on the left side of (6) there stands an almost periodic function. For if we decompose the functions

$$M(x,I) = M_1(x,I) + M_2(x,I), \qquad N(x,I) = N_1(x,I) + N_2(x,I),$$

so that on the right sides the first member is a step function, while the second member is a continuous, non-decreasing function, then from (6), by a suitable rearrangement, we obtain

$$\varphi(u, I) = \int_{-\infty}^{0} e^{iux}\,\mathrm{d}M(x, I) + \int_{0}^{\infty} e^{iux}\,\mathrm{d}N(x, I),$$

if the almost periodic part is denoted by $\varphi(u, I)$. The right hand side can, however, be almost periodic only if $M(x, I) \equiv$ const. and $N(x, I) \equiv$ const. Indeed, this is readily seen if we form the expression[3] $M(\varphi(u, I)e^{-i\lambda u})$ and prove that this vanishes for all λ's, owing to the continuity of $M_2(x, I)$ and $N_2(x, I)$.

We shall now show that $\log f(u, I)$ possesses the same exponents as $f(u, I)$, and further, that the other assertions of the theorem are also valid. For the proof we need a lemma.

Lemma. If a stochastic process satisfies conditions *A), B)*, and *C)*, then

$$\log f(u, I) = \int_I (f(u, J) - 1), \tag{7}$$

and the sequence in the definition of the Burkill integral will uniformly converge with respect to u to $\log f(u, I)$, or, more precisely,

$$\left| \log f(u, I) - \sum_{k=1}^{n} (f(u, J_k) - 1) \right| \leq K \max(1 - W_0(J_k)) \to 0 \tag{8}$$
$$\text{if } \max |J_k| \to 0.$$

Proof. We know that $|f(u, J)|^2$, together with $f(u, J)$, is an almost periodic characteristic function, and that, owing to $|f(u, J)|^2 \geq \delta^2 > 0$, the function $\log |f(u, J)|^2$ is likewise almost periodic. Let us denote in the distribution defined by $|f(u, J)|^2$ the probability of the value 0 by $W_0^*(J)$, then

$$W_0^*(J) = \sum_{k=0}^{\infty} W_{\lambda_k}^2(J) = M(|f(u, J)|^2). \tag{9}$$

From (9) we have

$$W_0^*(J) - 1 \leq W_0^2(J) + 1 - W_0(J) - 1 = W_0(J)(W_0(J) - 1)$$

and therefore, if $|J|$ is so small that $W_0(J) > \dfrac{1}{2}$, then

$$1 - W_0(J) < 2(1 - W_0^*(J)). \tag{10}$$

[3] $M(\varphi(u, I)e^{-i\lambda u}) = \lim\limits_{T\to\infty} \dfrac{1}{2T} \displaystyle\int_{-T}^{T} \varphi(u, I)e^{-i\lambda u}\,\mathrm{d}u.$

We shall now show that $1 - W_0(J)$ is of bounded variation.

$$
\begin{aligned}
\sum_{k=1}^{n}(1 - W_0(J_k)) &\leq 2\sum_{k=1}^{n}(1 - W_0^*(J_k)) = 2M\left(\sum_{k=1}^{n}(1 - |f(u, J_k)|^2)\right) \\
&\leq -2M\left(\sum_{k=1}^{n}\log|f(u, J_k)|^2\right) = -2M(\log|f(u, I)|^2) = K,
\end{aligned} \tag{11}
$$

where use has been made of the inequality $1 - x \leq -\log x$ $(0 < x \leq 1)$. Thus (8) can be established in the following manner:

$$
\begin{aligned}
&\left|\log f(u, I) - \sum_{k=1}^{n}(f(u, J_k) - 1)\right| \leq \sum_{k=1}^{n}|\log f(u, J_k) - (f(u, J_k) - 1)| \\
&\quad \leq \sum_{k=1}^{n}\sum_{l=2}^{\infty}\frac{1}{l}|f(u, J_k) - 1|^l \leq \frac{1}{2}\sum_{k=1}^{n}\frac{|f(u, J_k) - 1|}{1 - |f(u, J_k) - 1|} \\
&\quad \leq \sum_{k=1}^{n}|f(u, J_k) - 1|^2 \leq \sum_{k=1}^{n}(1 - W_0(J_k))^2 \leq K\max(1 - W_0(J_k)) \to 0.
\end{aligned}
$$

As the sequence figuring in the expression (8) converges uniformly to $\log f(u, I)$, it follows that

$$
M\left(\sum_{k=1}^{n}(f(u, J_k) - 1)e^{-i\lambda u}\right) \to M(\log f(u, I)e^{-i\lambda u}),
$$

that is, if the Fourier coefficients of $\log f(u, I)$ are denoted by $C_{\lambda_k}(I)$, then

$$
\begin{cases}
C_{\lambda_k}(l) = \int_I W_{\lambda_k}(J), & \lambda_k \neq 0 \\
C_0(l) \ = \int_I (W_0(J) - 1), &
\end{cases} \tag{12}
$$

and, in view of (6), we have

$$
-C_0(l) = \sum_{k=1}^{\infty}C_{\lambda_k}(l) < \infty. \tag{12a}
$$

Consequently, the theorem is proved. □

It may happen that in the formula (2) $C_{\lambda_k}(l)$ belonging to a certain λ_k is equal to 0. A sufficient condition for this is that in every point of the interval we have

$$
\frac{W_{\lambda_k}(J)}{|J|} \to 0
$$

during J contracts to the point in question. For, in this case, we have

$$\int_I W_{\lambda_k}(J) = 0. \tag{13}$$

The proof is very simple. As the interval function $\dfrac{W_{\lambda_k}(J)}{|J|}$ is continuous in the closed interval I, it is therefore also uniformly continuous. In other words, if $\varepsilon > 0$ is arbitrary and $\max |J_l| < \delta$, it follows that $\dfrac{W_{\lambda_k}(J_l)}{|J_l|} < \varepsilon$ and therefore $\sum_{l=1}^{n} W_{\lambda_k}(J_l) < \varepsilon|l|$, whence (13) follows.

§ 2. A direct proof

For the theorem we have just proved, we can also give a direct proof,[4] if we make use of a theorem known in the theory of almost periodic functions.

Thus in this case we do not make use of formula (1), but we require the lemma. For by the lemma, as is seen from (12), the Fourier coefficients of the function $\log f(u, I) - C_0(I)$ are $C_{\lambda_k}(l) \geq 0$ $(\lambda_k \neq 0)$. It is known that if all Fourier coefficients of an almost periodic function are non-negative, then their sum is convergent:[5]

$$\sum_{k=1}^{\infty} C_{\lambda_k}(I) < \infty. \tag{14}$$

It follows from (14) that the Fourier series of $f(u, I) - C_0(I)$ converges uniformly, and therefore

$$\log f(u, I) - C_0(I) = \sum_{k=1}^{\infty} C_{\lambda_k}(I) e^{i\lambda_k u}. \tag{15}$$

If 0 is substituted for u, we obtain

$$-C_0(I) = \sum_{k=1}^{\infty} C_{\lambda_k}(l), \tag{16}$$

therefore

$$\log f(u, I) = \sum_{k=1}^{\infty} C_{\lambda_k}(I)(e^{i\lambda_k u} - 1)$$

[4] A direct proof of formula (2) is discussed in the first section of [3] for the case when the totality of the numbers λ_k is identical with the set of non-negative integers.

[5] See [1], p. 62.

which exactly proves (2), whilst (12), (14) and (16) are proving (3) and (4).

The proof of (14) can simply be carried out in case the totality of the numbers λ_k is identical with the totality of integers. In fact, in this case $\log f(u,I)$ is a periodic function with period 2π. As $f(u,I) = \overline{f(-u,I)}$, the real part of $\log f(u,I)$ is an even function, whilst its imaginary part is an odd function. Consequently,

$$\frac{1}{2\pi}\int_0^{2\pi} (\log|f(u,I)| - C_0(I)) \cos ku \,\mathrm{d}u = \begin{cases} \dfrac{C_k(I) + C_{-k}(I)}{2} & \text{if} \quad k \neq 0, \\ 0 & \text{if} \quad k = 0. \end{cases}$$

The integrand is a continuous even function and its Fourier coefficients are non-negative, whence[6]

$$\sum_{k\neq 0} (C_k(I) + C_{-k}(I)) < \infty.$$

§ 3. The explicit form of $W_k(I)$

Let us consider the case in which the values λ_k are integers. Then (2) can be written in the following manner:

$$\log f(u,I) = \sum_{k\neq 0} C_k(I)(e^{i\lambda_k u} - 1). \tag{17}$$

Let us now express the probabilities $W_k(I)$ with the aid of the interval functions $C_k(I)$. It follows from (17) that the variable ξ_I which will now be denoted by $\xi(I)$, can be represented as follows:

$$\xi(I) = \sum_{k=-\infty}^{\infty} k\xi_k(I), \quad \text{or} \quad \xi(I) = \sum_{k=1}^{\infty} k(\xi_k(I) - \xi_{-k}(I)), \tag{18}$$

where the variables $\xi_k(I)$ are independent and have a Poisson distribution with a mean value $C_k(I)$. Let us now consider the second expression of (18). Here the differences of variables of Poisson distribution are standing. In general, if ξ and η are two independent variables of Poisson distribution with mean values λ and μ, the difference

$$\zeta = \xi - \eta$$

[6] A special case of Paley's theorem which can easily be proved is the following: if the Fourier coefficients of an even and continuous function are non-negative, then their sum will converge. As a matter of fact,

$$\frac{1}{2}S_n(0) \leq \frac{n+1}{2n+1}S_n(0) \leq \frac{1}{2n+1}\sum_{k=n}^{2n} S_k(0) \leq \sigma_{2n}(0)$$

where σ_n denotes the n-th Fejér mean.

can assume arbitrary integral values and

$$Pr(\zeta = n) = \lambda^n e^{-(\lambda+\mu)} \sum_{k=0}^{\infty} \frac{(\lambda\mu)^k}{k!(k+n)!} \qquad (n = 0, 1, 2, \ldots)$$

and

$$Pr(\zeta = -n) = \mu^n e^{-(\lambda+\mu)} \sum_{k=0}^{\infty} \frac{(\lambda\mu)^k}{k!(k+n)!} \qquad (n = 0, 1, 2, \ldots).$$

These probabilities can be expressed with the aid of the Bessel function of n-th order

$$J_n(x) = \left(\frac{x}{2}\right)^n \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{x}{2}\right)^{2k}}{k!(k+n)!}$$

in the following manner:

$$\begin{aligned} Pr(\zeta = n) &= \left(\frac{1}{i}\sqrt{\frac{\lambda}{\mu}}\right)^n e^{-(\lambda+\mu)} J_n(2i\sqrt{\lambda\mu}) \\ & \qquad\qquad\qquad\qquad (n = 0, 1, 2, \ldots) \\ Pr(\zeta = -n) &= \left(\frac{1}{i}\sqrt{\frac{\mu}{\lambda}}\right)^n e^{-(\lambda+\mu)} J_n(2i\sqrt{\lambda\mu}), \end{aligned}$$

or, with a uniform method of writing:

$$Pr(\zeta = n) = \left(\frac{1}{i}\right)^{|n|} \left(\frac{\lambda}{\mu}\right)^{\frac{n}{2}} e^{-(\lambda+\mu)} J_{|n|}(2i\sqrt{\lambda\mu}). \tag{19}$$

By making use of (19) and on the basis of (18) we obtain

$$\begin{aligned} W_k(I) &= e^{-\lambda(I)} \sum_{\sum nr_n = k} \prod \left(\frac{1}{i}\right)^{|r_n|} \left(\frac{C_n(I)}{C_{-n}(I)}\right)^{\frac{r_n}{2}} J_{|r_n|}(2i\sqrt{C_n(I)C_{-n}(I)}), \\ & k = 0, \pm 1, \pm 2, \ldots \qquad (n = 1, 2, \ldots;\ r_n \text{ is an integer}), \end{aligned} \tag{20}$$

where the product is extended over all finite systems of values r_n for which $\sum nr_n = k$, whilst the summation relates to all such systems, and

$$\lambda(I) = \sum_{n \neq 0} C_n(I).$$

Finally, I express my sincere thanks to Alfred Rényi for his valuable remarks.

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[3] A. Rényi, On composed Poisson distributions, II, *Acta Math. Acad. Sci. Hung.*, **2** (1951), pp. 83–98.