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# ON THE CONVERGENCE OF SERIES OF INDEPENDENT RANDOM VARIABLES

To the memory of Professor Tibor Szele

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## Introduction

Let $\xi_1, \xi_2, \ldots$ denote a sequence of independent random variables. In the present paper two theorems concerning the convergence (regardless of the order of summation) of the series

$$\sum_{k=1}^{\infty} \xi_k \tag{1}$$

are proved.

Throughout the paper I denote by $\mathcal{J}$ the set of all finite subsets and by $\mathcal{S}$ the set of all subsets of the set of natural numbers. Let

$$\xi(A) = \sum_{k \in A} \xi_k \tag{2}$$

provided the series on the right converges with probability 1 regardless of the order of summation. Let $F(x, A)$ denote the distribution function and $f(t, A)$ the characteristic function of the random variable $\xi(A)$.

Let $0 < \lambda < 1$, and let $\xi$ denote a random variable. We say that $Q(\lambda)$ is a $\lambda$-quantile of $\xi$ if $P(\xi \leq Q(\lambda)) \geq \lambda$ and $P(\xi \geq Q(\lambda)) \geq 1 - \lambda$. An arbitrary $\lambda$-quantile of the random variable $\xi(A)$ will be denoted by $Q(\lambda, A)$.

In the literature many necessary and sufficient conditions are given concerning the convergence of the series (1). These conditions are expressed generally in terms of mean values, dispersions and characteristic functions. In what follows such conditions are given in terms of the compactness of certain sets of distribution functions and of the quantiles.

## § 1. Preliminary lemmas

In this § three lemmas are proved. The first two contain some assertions relative to the set of random variables $\{\xi_z,\ z \in Z\}$, where $Z$ is an arbitrary given set. Let $F(x,z)$ denote the distribution function and $f(t,z)$ the characteristic function of the random variable $\xi_z$.

P. Lévi has introduced the notion of distance between two distribution functions. The distance $L(F_1,F_2)$ between $F_1(x)$ and $F_2(x)$ is defined as the lower bound of the values $h$, for which

$$F_1(x-h)-h \leq F_2(x) \leq F_1(x+h)+h. \tag{3}$$

It is known that the axioms of the metric space are fulfilled relative to the distance $L$:

a) $L(F_1,F_2)=0$ if and only if $F_1(x)\equiv F_2(x)$;

b) $L(F_1,F_2)=L(F_2,F_1)$;

c) $L(F_1,F_3)\leq L(F_1,F_2)+L(F_2,F_3)$.

Let us consider the set $\mathcal{F}$ of all one dimensional distribution functions. From what has been said it follows that $\mathcal{F}$ is a metric space relative to the distance $L$. According to (3), $\mathcal{F}$ is bounded and by Theorem 2 of [2], p. 42. it is also complete. In Theorem 1 of [2], p. 38. it is proved, that the relation $L(F_n,F)\to 0$ holds if and only if $F_n(x)\to F(x)$ at every continuity point of $F(x)$. From this fact it is easy to see that the space $\mathcal{F}$ is not compact. For instance sequences of distribution functions can be given, which converge in every point to 0. The first two lemmas give answers to the question: under what conditions is a subset $\mathcal{F}'=\{F(x,z),\ z\in Z\}$ of the space $\mathcal{F}$ compact?

To fixed values of $\lambda$, and $z$ there corresponds generally not only one $Q(\lambda,z)$, therefore the set $\{Q(\lambda,z),\ z\in Z\}$ is generally not uniquely determined.

Lemma 1 *If to every $\lambda$, for which $0<\lambda<1$, the quantiles $Q(\lambda,z)$ can be chosen in such a way that $|Q(\lambda,z)|<K(\lambda)$ for $z\in Z$, where $K(\lambda)$ is independent of $z$, then the set $\mathcal{F}'$ is compact.*

*Conversely, if the set $\mathcal{F}'$ is compact, then to every $\lambda$ for which $0<\lambda<1$, there corresponds a $K(\lambda)$ such that $|Q(\lambda,z)|<K(\lambda)$ for every $\lambda$-quantile $Q(\lambda,z)$ of the variable $\xi_z$.*

Proof of the first part. Let $\varepsilon>0$, $0<\lambda<\varepsilon$ and $x<-K(\lambda)$. From our hypothesis it follows that

$$F(x,z)\leq F(-K(\lambda),z)=P(\xi_z<-K(\lambda))\leq P(\xi_z<Q(\lambda,z))\leq\lambda<\varepsilon.$$

On the other hand if $\lambda>1-\varepsilon$ and $x>K(\lambda)$, then we obtain

$$F(x,z)\geq F(K(\lambda),z)=P(\xi_z<K(\lambda))\geq P(\xi_z\leq Q(\lambda,z))\geq\lambda>1-\varepsilon.$$

It follows that

$$F(x,z)\to 0 \text{ if } x\to-\infty,\quad F(x,z)\to 1 \text{ if } x\to+\infty$$

uniformly in $z$. According to HELLY's theorem we can choose a sequence $F(x, z_k)$ which converges to a non-decreasing, left continuous function $F(x)$ at every point of continuity of the latter. By the preceding relation $F(x)$ must be a distribution function.

PROOF OF THE SECOND PART. Obviously the relations

$$F(x,z) \to 0 \text{ if } x \to -\infty, \quad F(x,z) \to 1 \text{ if } x \to +\infty$$

hold uniformly in $z$. If $0 < \varepsilon < 1$, then there is a positive number $K(\varepsilon)$ such that

$$\begin{aligned} F(x,z) < \varepsilon \quad &\text{if} \quad x \le -\frac{K(\varepsilon)}{2}, \\ 1 - F(x,z) < \varepsilon \quad &\text{if} \quad x \ge \frac{K(\varepsilon)}{2}. \end{aligned}$$

In this case

$$-K(\varepsilon) < Q(\varepsilon, z) < K(\varepsilon) \text{ for } z \in Z,$$

as was to be proved. □

LEMMA 2 *The set $\mathcal{F}'$ is compact if and only if the characteristic functions $f(t,z)$, $z \in Z$ are equicontinuous at $t = 0$, i.e. if to every $\varepsilon > 0$ there corresponds a $\delta$ such that*

$$|1 - f(t,z)| < \varepsilon \quad \text{if} \quad |t| < \delta. \tag{4}$$

PROOF OF THE FIRST PART. Suppose that for a fixed pair $\varepsilon$, $\delta$ of positive numbers the relation (4) holds. In view of

$$1 - \frac{\sin x}{x} > \frac{1}{10} \quad \text{if} \quad |x| \ge 1,$$

we obtain

$$\begin{aligned} \varepsilon &> \frac{1}{2\delta} \int_{-\delta}^{\delta} |1 - f(t,z)| \, \mathrm{d}t \ge \frac{1}{2\delta} \int_{-\infty}^{\infty} \int_{-\delta}^{\delta} (1 - e^{itx}) \, \mathrm{d}t \, \mathrm{d}F(x,z) \\ &= \int_{-\infty}^{\infty} \left(1 - \frac{\sin \delta x}{\delta x}\right) \mathrm{d}F(x,z) \ge \int_{|x| > \frac{1}{\delta}} \left(1 - \frac{\sin \delta x}{\delta x}\right) \mathrm{d}F(x,z) \ge \frac{1}{10} P\left(|\xi_z| > \frac{1}{\delta}\right), \end{aligned}$$

whence

$$F(-x,z) + 1 - F(x,z) < 10\varepsilon \text{ if } x > \frac{1}{\delta}.$$

To every $\varepsilon$ there corresponds such a $\delta$, therefore if $F(x, z_k)$ is a sequence of $\mathcal{F}'$, $\lim_{k \to \infty} F(x, z_k) = F(x)$ at every point of continuity of the latter, where $F(x)$ is a left continuous function, then $F(x)$ must be a distribution function.

PROOF OF THE SECOND PART. If $\varepsilon > 0$, then there is a number $K$ such that

$$P(|\xi_z| > K) < \frac{\varepsilon}{4}.$$

Let $\delta = \dfrac{\varepsilon}{2K}$. Then for $|t| < \delta$ we have

$$\begin{aligned} |1 - f(t,z)| &= \left| \int_{-\infty}^{\infty} (1 - e^{itx})\,\mathrm{d}F(x,z) \right| \\ &\le \int_{-\infty}^{\infty} |1 - e^{itx}|\,\mathrm{d}F(x,z) \\ &\le |t| \int_{|x|\le K} |x|\,\mathrm{d}F(x,z) + 2P(|\xi_z| > K) < |t|K + \frac{\varepsilon}{2} < \varepsilon. \end{aligned}$$ □

LEMMA 3 *Let $\eta_1, \eta_2, \ldots$ denote a sequence of random variables for which*

$$P\left(\lim_{n\to\infty} \eta_n = \infty\right) = 1.$$

*Then if $0 < \lambda < 1$ and $Q(\lambda, n)$ is a $\lambda$-quantile of $\eta_n$, we have*

$$\lim_{n\to\infty} Q(\lambda, n) = \infty.$$

PROOF. Let $\Omega_{mn}$ denote the event that

$$\eta_n > m.$$

Since

$$\lim_{n\to\infty} P(\Omega_{mn}) = 1,$$

there exists a number $N$ such that if $n > N$,

$$P(\Omega_{mn}) > 1 - \lambda.$$

But in this case

$$Q(\lambda, n) > m.$$

As $m$ can be chosen arbitrarily, this implies the assertion of Lemma 3. □

COROLLARY *If the random variables $\xi_1, \xi_2, \ldots$ are non negative and there exists a number $\lambda$ $(0 < \lambda < 1)$ for which*

$$Q(\lambda, A) \le K \quad \textit{if} \quad A \in \mathcal{J},$$

*where $K$ is a constant, then the series* (1) *converges with probability* 1.

PROOF. By the zero-one law (see [3], p. 60.) either

$$P\left(\sum_{k=1}^{\infty} \xi_k < \infty\right) = 0,$$

or

$$P\left(\sum_{k=1}^{\infty} \xi_k < \infty\right) = 1.$$

But now the first case is impossible because it would imply that for every $\lambda$

$$\lim_{n\to\infty} Q(\lambda, \{1, 2, \ldots, n\}) = \infty.$$

## § 2. Necessary and sufficient conditions of the convergence of the series (1)

In this § two theorems are proved.

THEOREM 1 *If the set* $\{F(x,A),\ A \in \mathcal{J}\}$ *is compact, the series* (1) *converges with probability* 1 *regardless of the order of summation.*

*Conversely, if the series* (1) *converges with probability* 1 *regardless of the order of summation, then the set* $\{F(x,A),\ A \in \mathcal{S}\}$ *is compact.*

PROOF OF THE FIRST PART. Let us consider the sequence

$$F(x,\{1\}), F(x,\{1,2\}), \dots, F(x,\{1,2,\dots,n\}), \dots .$$

Since the set $\{F(x,A),\ A \in \mathcal{J}\}$ is compact, there is an increasing subsequence $n_k$ of the natural numbers with the property that the sequence

$$F(x,\{1,2,\dots,n_k\})$$

converges to a distribution function $F(x)$ at every point of continuity of the latter. It follows that if

$$f(t) = \int_{-\infty}^{\infty} e^{itx}\,\mathrm{d}F(x),$$

then

$$\lim_{k\to\infty} \prod_{l=1}^{n_k} f(t,\{l\}) = f(t).$$

Since the sequence

$$\prod_{l=1}^{n} |f(t,\{l\})|$$

is non-increasing, we obtain that

$$\prod_{l=1}^{\infty} |f(t,\{l\})| = |f(t)|.$$

Since $f(t)$ is a characteristic function, there is a number $T > 0$ such that $|f(t)| > 0$ if $|t| \le T$. According to Theorems 2.7 and 2.6 of [1], Chapter III, there is a sequence $c_1, c_2, \dots$ of real numbers such that the series

$$\sum_{k=1}^{\infty} (\xi_k - c_k)$$

converges with probability 1 regardless of the order of summation. Therefore it is sufficient to prove that

$$\sum_{k=1}^{\infty} |c_k| < \infty.$$

Let us suppose that this is not true. In this case there exists a rearrangement $c_{i_1}, c_{i_2}, \ldots$ of the sequence $c_1, c_2, \ldots$, for which either $\sum_{k=1}^{\infty} c_{i_k} = +\infty$ or $\sum_{k=1}^{\infty} c_{i_k} = -\infty$. It suffices to consider the first case as the second can be reduced to the first by considering the variables $\zeta_k = -\xi_k$. It follows that

$$\sum_{k=1}^{n} \xi_{i_k} = \sum_{k=1}^{n} (\xi_{i_k} - c_{i_k}) + \sum_{k=1}^{n} c_{i_k} \to \infty \text{ if } n \to \infty. \tag{5}$$

Let $A_n = \{i_1, i_2, \ldots, i_n\}$. By (5) and Lemma 3 we obtain that for every fixed $\lambda$ $(0 < \lambda < 1)$

$$Q(\lambda, A_n) \to \infty \text{ if } n \to \infty.$$

But this contradicts our supposition that the set $\{F(x, A),\ A \in \mathcal{J}\}$ is compact because by Lemma 1 to every $\lambda$ $(0 < \lambda < 1)$ there corresponds a number $K(\lambda)$ for which

$$|Q(\lambda, A)| \leq K(\lambda), \quad A \in \mathcal{J}.$$

Proof of the second part. If the series (1) converges with probability 1 regardless of the order of summation then the series

$$\sum_{k=1}^{\infty} |1 - f(t, \{k\})|$$

converges uniformly in every finite $t$-interval (See [1], Chapter III, Theorem 2.7). Using the inequality

$$|1 - z_1 z_2 \ldots z_r| \leq |1 - z_1| + |1 - z_2| + \cdots + |1 - z_r|$$

valid for every sequence $z_1, z_2, \ldots, z_r$ of complex numbers with the property $|z_1| \leq 1$, $|z_2| \leq 1, \ldots, |z_r| \leq 1$, we obtain that

$$\left|1 - \prod_{l=1}^{r} f(t, \{j_l\})\right| \leq \sum_{l=1}^{r} |1 - f(t, j_l)|.$$

Let us put $A = \{j_1, j_2, \ldots\}$. If $r \to \infty$, we obtain the following inequality:

$$|1 - f(t, A)| \leq \sum_{k=1}^{\infty} |1 - f(t, \{j_l\})| \leq \sum_{k=1}^{\infty} |1 - f(t, \{k\})|. \tag{6}$$

The series on the right of (6) is uniformly convergent, hence the continuity of the terms implies the continuity of the sum. Since the relation (6) holds for every $A \in \mathcal{S}$, the characteristic functions $f(t, A)$, $A \in \mathcal{S}$, are equicontinuous at $t = 0$, and thus by Lemma 2 the set $\{F(x, A),\ A \in \mathcal{S}\}$ is compact, as it was to be proved. □

Corollary *If for every $\lambda$ $(0 < \lambda < 1)$ the quantiles $Q(\lambda, A)$ $(A \in \mathcal{J})$ can be chosen in such a way that they are bounded, then the series* (1) *converges with probability* 1 *regardless of the order of summation.*

*Conversely, if the series* (1) *converges with probability* 1 *regardless of the order of summation, then to every $\lambda$ $(0 < \lambda < 1)$ there is a $K(\lambda)$ such that*

$$|Q(\lambda, A)| \leq K(\lambda), \quad A \in \mathcal{S}.$$

PROOF. The assertion is an immediate consequence for Lemma 1 and Theorem 1.

In the above corollary to Theorem 1 the condition implying the convergence of the series (1) can be essentially reduced. This fact is contained in the following theorem.

THEOREM 2 *Let us suppose that there can be found two numbers* $\lambda_1$, $\lambda_2$ *for which* $0 < \lambda_1 < \lambda_2 < 1$, *such that the quantiles* $Q(\lambda_1, A)$ *and* $Q(\lambda_2, A)$ *can be chosen in such a manner, that the sets* $Q_1 = \{Q(\lambda_1, A),\ A \in \mathcal{J}\}$ *and* $Q_2 = \{Q(\lambda_2, A),\ A \in \mathcal{J}\}$ *are bounded. Then the series* (1) *converges with probability* 1 *regardless of the order of summation.*

PROOF. Let $K$ be such a number for which by choosing appropriately the values $Q(\lambda_1, A)$ and $Q(\lambda_2, A)$, we have

$$|Q(\lambda_1, A)| \le K, \quad Q(\lambda_1, A) \in Q_1$$

and

$$|Q(\lambda_2, A)| \le K, \quad Q(\lambda_2, A) \in Q_2$$

for $A \in \mathcal{J}$. It follows that

$$\sup_{A\in\mathcal{J}} P(|\xi(A)| > K) \le \sup_{A\in\mathcal{J}} P(\xi(A) < -K) + \sup_{A\in\mathcal{J}} P(\xi(A) > K) \le \lambda_1 + 1 - \lambda_2 < 1.$$

It follows simply that

$$\inf_{A\in\mathcal{J}} P(|\xi(A)| \le K) = \rho > 0.$$

Thus

$$\overline{\lim_{n\to\infty}} P\left(\left|\sum_{k=1}^{n} \xi_k\right| \le K\right) \ge \rho > 0.$$

The last inequality, together with Theorems 2.9 and 2.6 of [1], Chapter III, implies the existence of constants $c_1, c_2, \ldots$ which have the property that the series

$$\sum_{k=1}^{\infty} (\xi_k - c_k)$$

converges with probability 1 regardless of the order of summation. Applying the same reasoning as in the proof of Theorem 1 it can be shown that

$$\sum_{k=1}^{\infty} |c_k| < \infty,$$

because in the contrary case all the sets $\{Q(\lambda, A),\ A \in \mathcal{J}\}$, $0 < \lambda < 1$ were unbounded. Taking into account the equality

$$\sum_{k=1}^{n} \xi_{i_k} = \sum_{k=1}^{n} (\xi_{i_k} - c_{i_k}) + \sum_{k=1}^{n} c_{i_k},$$

we obtain that the series (1) converges with probability 1 regardless of the order of summation. Hence Theorem 2 is proved. □

Corollary *Let us suppose that the random variables $\xi_k$ are symmetrically distributed. Then if there exists $\lambda \neq \frac{1}{2}$ such that with a convenient choice of the quantiles $Q(\lambda, A)$, the set $\{Q, (\lambda, A),\ A \in \mathcal{J}\}$ is bounded, the series* (1) *converges with probability* 1 *regardless of the order of summation.*

Proof. The assertion follows from the fact that in this case 0 is a $\frac{1}{2}$-quantile of $\xi(A)$ for $A \in \mathcal{J}$, hence the conditions of Theorem 2 are fulfilled. □

Remark It is easy to see that the boundedness of only one set of the type $\{Q, (\lambda, A),$ $A \in \mathcal{J}\}$ does not imply the convergence of the series (1). For instance if the random variables $\xi_k$ are equally and symmetrically distributed, moreover $P(\xi_k = 0) < 1$, then the series (1) diverges for every ordering of the terms, but 0 is a $\frac{1}{2}$-quantile of $\xi(A)$ for $A \in \mathcal{J}$.

Finally I mention a problem. Let us suppose that the series (1) converges with probability 1 regardless of the order of summation. Let $A_1, A_2, \ldots$ be a sequence of sets of $\mathcal{S}$ such that if $B_n = A_n + A_{n+1} + \cdots$, then $\prod_{n=1}^{\infty} B_n = 0$. By the inequality (6) it follows that

$$\xi(A_k) \Rightarrow 0, \quad \text{if} \quad k \to \infty.$$

Does the ordinary limit of the sequence $\xi(A_k)$ exist with probability 1? If the series (1) converges absolutely, then this is true, but the series (1) may be convergent by every ordering of the terms and not absolutely convergent. Such series is the following:

$$\sum_{n=1}^{\infty} \frac{\text{sgn } \sin 2^n \pi x}{n}, \qquad 0 \leq x \leq 1.$$

## References


[1] Doob, J. L. (1953). *Stochastic Processes*, New York, London.

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[3] Kolmogoroff, A. N. (1933). *Grundbegriffe der Wahrscheinlichkeitsrechnung*, Berlin.