Acta Math. Acad. Sci. Hung. **8** (1957), pp. 107–126.

ON ADDITIVE AND MULTIPLICATIVE TOTALS

András Prékopa (Budapest)

(Presented by A. Rényi)

Mathematical Institute of the Hungarian Academy of Sciences

Received: October 18, 1956

Introduction

In the present paper terminology "total" is used for a generalization of the Burkill integral and multiplicative integral, respectively. The functions, the totals of which are considered, take their values from a Banach algebra $\mathcal{B}$ with a unity. This means a Banach space $\mathcal{B}$ in which for every pair $f \in \mathcal{B}$, $g \in \mathcal{B}$ a product $fg \in \mathcal{B}$ is defined such that $\|fg\| \leq \|f\| \, \|g\|$ and if $h \in \mathcal{B}$, then $f(g+h) = fg + fh$, $(g+h)f = gf + hf$, finally there is an $e \in \mathcal{B}$ with the properties $ef = fe = f$, $\|e\| = 1$. It is proved that under some conditions the additive (and multiplicative) total of a multiplicative (and additive, resp.) set function exists. The theorems of this type are useful in solving some functional equations (see e.g. [3] and § 6) and studying the properties of multiplicative set functions by tracing the problems to those formulated in terms of additive set functions.

The multiplicative integral (on the real axis for matrix-valued functions) has been introduced by V. Volterra [12], [13], [14] and considered by several authors: L. Schlesinger [8], [9], [10], [11], G. Rasch [7], R.L Dobrušin [3]; G. Birkhoff [2] has given a generalization of this integral by using more general notions instead of matrices. The additive integral (on the real axis for matrix-valued functions) has been considered by M. Fréchet [4] and R. L. Dobrušin [3]. The theorems proved in the present paper are analogous to those of Dobrušin [3] and will be used in the theory of stochastic set functions.[1] In § 6 we anticipate an example from this as in this special case (Example 1) the relevant statements can be proved at once by the aid of the present paper.

§ 1. Notions and notations

Throughout the paper the basic space will be denoted by H which is supposed to be metric and compact. We suppose that we are given a class of sets $\mathcal{K}$ consisting of some subsets of H and satisfying the following conditions:

[1] A. Prékopa, On stochastic set functions. III, *Acta Math. Acad. Sci. Hung.*, **8** (1957), 375–400.

a) $\mathcal{K}$ is a semi-ring, i.e. if $A_1 \in \mathcal{K}$, $A_2 \in \mathcal{K}$, then $A_1A_2 \in \mathcal{K}$ and if $A_1 \subseteq A_2$, then there exists a finite number of sets $C_1, C_2, \ldots, C_n$ such that $C_i \in \mathcal{K}$ $(i = 1, 2, \ldots, n)$, $C_iC_k = 0$ if $i \neq k$ and $A_2 - A_1 = \sum_{i=1}^{n} C_i$.[2] We suppose furthermore that this can be done always so that $n \leq R$, where R is a positive integer independent of the sets A_1, A_2.

b) If $h \in H$, then $\{h\} \in \mathcal{K}$.

c) If $A \in \mathcal{K}$, then for every positive integer r and every $\varepsilon > 0$ there is a decomposition $A_1, A_2, \ldots, A_r$ of the set A into pairwise disjoint sets of $\mathcal{K}$ such that $\max\limits_{1 \leq k \leq r} d(A_k) \leq \varepsilon$.[3]

A finite sequence of sets $A_1, A_2, \ldots, A_r$, for which $A_i \in \mathcal{K}$ $(i = 1, 2, \ldots, r)$, $A_iA_k = 0$ if $i \neq k$ and $A = \sum_{k=1}^{r} A_k \in \mathcal{K}$, will be called a decomposition of the set A (or briefly decomposition) and will be denoted by $\mathfrak{z} = \{A_1, A_2, \ldots, A_r\}$. If $\mathfrak{z}_1 = \{A_i^{(1)}\}$, $\mathfrak{z}_2 = \{A_i^{(2)}\}$ are two decompositions and every $A_i^{(2)}$ can decomposed by means of some $A_i^{(1)}$, then we write $\mathfrak{z}_2 \sqsubset \mathfrak{z}_1$. We shall use the following definitions:

Definition 1 Let us suppose that to every decomposition $\mathfrak{z} = \{A_1, A_2, \ldots, A_r\}$ there corresponds a permutation $\mathcal{P}(\mathfrak{z}) = (A_{i_1}, A_{i_2}, \ldots, A_{i_r})$ of the sets of $\mathfrak{z}$ such that if $\mathfrak{z}_1 \sqsubset \mathfrak{z}_2$ and $\mathcal{P}(\mathfrak{z}_1)$ is given by $(A_{i_1}^{(1)}, A_{i_2}^{(1)}, \ldots, A_{i_r}^{(1)})$, then in $\mathcal{P}(\mathfrak{z}_2)$ first come those sets of $\mathfrak{z}_2$ which decompose $A_{i_1}^{(1)}$, then those which decompose $A_{i_2}^{(1)}$ etc.

A correspondence between the decompositions and permutations described above will be called a permutation function.

Definition 2 Let $f(A)$ $(A \in \mathcal{K})$ be a set function with values in the Banach algebra $\mathcal{B}$. If for every pair A_1, A_2 of disjoint sets of $\mathcal{K}$, for which $A = A_1 + A_2 \in \mathcal{K}$, the relation

$$(*) \qquad f(A) = f(A_1) + f(A_2)$$

holds, then the set function $f(A)$ will be called additive. $(*)$ implies that $f(0) = 0$.

Definition 3 Let $g(A)$ $(A \in \mathcal{K})$ be a set function with values in the Banach algebra $\mathcal{B}$. If there is a permutation function $\mathcal{P}$ such that for every system $A_1, A_2, \ldots, A_r$ of disjoint sets of $\mathcal{K}$, for which $A = \sum_{k=1}^{r} A_k \in \mathcal{K}$, the relation

$$g(A) = \prod_{k=1}^{r} g(A_{i_k})$$

holds where $\mathfrak{z} = \{A_1, A_2, \ldots, A_r\}$ and $\mathcal{P}(\mathfrak{z}) = (A_{i_1}, A_{i_2}, \ldots, A_{i_r})$, then the set function $g(A)$ will be called multiplicative. We suppose in this case that $f(0) = e$.

[2]This notion of semi-rings, which is more general than that of P. Halmos (cf. *Measure theory*, Chapter 1, § 4), is due to Á. Császár.

[3]If $B \subseteq H$, then $d(B)$ denotes the diameter of the set B, i.e. $d(B) = \sup \rho(h_1, h_2)$, where $h_1 \in B$, $h_2 \in B$ and $\rho(h_1, h_2)$ is the distance between h_1 and h_2.

If $\mathcal{B}$ is commutative, then we do not require the existence of a permutation function $\mathcal{P}$. All the statements in this paper are to be taken in the sense that if $\mathcal{B}$ is commutative, then we omit the requirements regarding to the permutation function.

Definition 4 A set function $\alpha(A)$ $(A \in \mathcal{K})$ with values in $\mathcal{B}$ is called of bounded variation if there is a number K such that for every system $A_1, A_2, \ldots, A_r$ of disjoint sets of $\mathcal{K}$ we have

$$(**) \qquad \sum_{k=1}^{r} \|\alpha(A_k)\| \leq K.$$

If $A_i \subseteq A$ $(i = 1, 2, \ldots, r)$, then the smallest K for which relation $(**)$ holds will be denoted by $\mathrm{Var}_\alpha(A)$.

Definition 5 A set function $\alpha(A)$ $(A \in \mathcal{K})$ with values in $\mathcal{B}$ is said to be v-continuous if for every sequence $B_1, B_2, \ldots$ of sets of $\mathcal{K}$, for which $\lim\limits_{k\to\infty} B_k = 0$ and $\lim\limits_{k\to\infty} d(B_k) = 0$, the relation

$$\lim_{k\to\infty} \mathrm{Var}_\alpha(B_k) = 0$$

holds.

Definition 6 Let $\alpha(A)$ $(A \in \mathcal{K})$ be a set function with values in $\mathcal{B}$. Suppose that there exists a $\beta(B)$ such that for every $\varepsilon > 0$ a number $\delta > 0$ can be found with the property

$$\left\| \sum_{k=1}^{r} \alpha(A_k) - \beta(B) \right\| \leq \varepsilon,$$

provided that $\max\limits_{1\leq k\leq r} d(A_k) \leq \delta$ where $A_1, A_2, \ldots, A_r$ is a decomposition of the set $B \in \mathcal{K}$. In this case we say that the additive total of $\alpha(A)$ exists in B and we denote it by

$$\beta(B) = \mathsf{S}_B\, \alpha(dA).$$

Definition 7 Let $\alpha(A)$ $(A \in \mathcal{K})$ be a set function with values in $\mathcal{B}$. Suppose that there exists a permutation function $\mathcal{P}$ and a $\gamma(B) \in \mathcal{B}$ such that for every $\varepsilon > 0$ a number $\delta > 0$ can be found with the property

$$\left\| \prod_{k=1}^{r} \alpha(A_{i_k}) - \gamma(B) \right\| \leq \varepsilon,$$

provided that $\max\limits_{1\leq k\leq r} d(A_k) \leq \delta$ where $(A_{i_1}, A_{i_2}, \ldots, A_{i_r})$ is the permutation corresponding to the decomposition $\mathfrak{z} = \{A_1, A_2, \ldots, A_r\}$ of the set $B \in \mathcal{K}$. In this case we say that the multiplicative total of $\alpha(A)$ exists in B relative to the permutation function $\mathcal{P}$. This total will be denoted by

$$\gamma(B) = {}_{\mathcal{P}}\sqcap_B \alpha(dA).$$

It is easy to see that both totals are uniquely determined and

$$\mathsf{S}_{A_1+A_2}\alpha(dA) = \mathsf{S}_{A_1}\alpha(dA) + \mathsf{S}_{A_2}\alpha(dA),$$

$${}_{\mathcal{P}}\prod_{A_1+A_2+\ldots+A_r}\alpha(dA) = {}_{\mathcal{P}}\prod_{A_{i_1}}\alpha(dA)\cdot {}_{\mathcal{P}}\prod_{A_{i_2}}\alpha(dA)\cdots {}_{\mathcal{P}}\prod_{A_{i_r}}\alpha(dA),$$

where $A_i \in \mathcal{K}$ $(i = 1, 2, \ldots, r)$, $A_iA_k = 0$ if $i \neq k$ and $(A_{i_1}, A_{i_2}, \ldots, A_{i_r})$ is the permutation given by $\mathcal{P}$, provided that the totals on both sides exist. If $\alpha_1(A)$ and $\alpha_2(A)$ are two set functions defined on $\mathcal{K}$ and both are additively totalizable in B, then the same holds for $\alpha(A) = c_1\alpha_1(A) + c_2\alpha_2(A)$ and

$$\mathsf{S}_B\,\alpha(dA) = c_1\mathsf{S}_B\,\alpha_1(dA) + c_2\mathsf{S}_B\,\alpha_2(dA),$$

where c_1, c_2 are constants. An analogous relation holds also for the multiplicative total if $\mathcal{B}$ is commutative. In this case if the multiplicative totals of $\alpha_1(A)$ and $\alpha_2(A)$ exist in $B \in \mathcal{K}$, then that of $\alpha(A) = \alpha_1(A)\alpha_2(A)$ in B exists too and

$$\prod_B \alpha(dA) = \prod_B \alpha_1(dA) \prod_B \alpha_2(dA).$$

§ 2. Preliminary lemmas

In this § we prove some simple inequalities for Banach algebras and lemmas for additive and multiplicative set functions.

Lemma 1 *If $f_i \in \mathcal{B}$, $g_i \in \mathcal{B}$ $(i = 1, 2, \ldots, n)$ and*

$$\left\|\prod_{i=1}^{j} f_i\right\| \leq K, \qquad \left\|\prod_{i=j}^{n} g_i\right\| \leq K \qquad (j = 1, 2, \ldots, n),$$

where K is a constant, then

$$\left\|\prod_{i=1}^{n} f_i - \prod_{i=1}^{n} g_i\right\| \leq K^2 \sum_{i=1}^{n} \|f_i - g_i\|. \tag{1}$$

Proof. Since

$$\prod_{i=1}^{n} f_i - \prod_{i=1}^{n} g_i = \sum_{i=1}^{n} f_1 \ldots f_{i-1}(f_i - g_i)g_{i+1}\ldots g_n,$$

it follows that

$$\left\|\prod_{i=1}^{n} f_i - \prod_{i=1}^{n} g_i\right\| \leq \sum_{i=1}^{n} \|f_1 \ldots f_{i-1}\|\,\|f_i - g_i\|\,\|g_{i+1}\ldots g_n\| \leq K^2 \sum_{i=1}^{n} \|f_i - g_i\|.$$

Lemma 2 *Let $f_1, f_2, \dots, f_n$ be such elements of $\mathcal{B}$ that*

$$\sum_{i=1}^{n} \|f_i\| \le c < 1.$$

In this case if $r < n$, then

$$\left\| \prod_{i=1}^{n}(e+f_i) - \left(e + \sum_{i=1}^{n} f_i + \sum_{1\le i_1<i_2\le n} f_{i_1}f_{i_2}\dots f_{i_r} + \cdots \right.\right.$$

$$\left.\left. + \sum_{1\le i_1<i_2\dots<i_r\le n} f_{i_1}f_{i_2}\dots f_{i_r} \right) \right\| \le \frac{1}{1-c}\left(\sum_{i=1}^{n}\|f_i\|\right)^{r+1}. \tag{2}$$

Proof. Let us start from the identity

$$\prod_{i=1}^{n}(e+f_i) = e + \sum_{i=1}^{n} f_i + \sum_{1\le i_1<i_2\le n} f_{i_1}f_{i_2} + \cdots + f_1f_2\dots f_n.$$

It follows from this that

$$\left\| \prod_{i=1}^{n}(e+f_i) - \left(e + \sum_{i=1}^{n} f_i + \sum_{1\le i_1<i_2\le n} f_{i_1}f_{i_2} + \cdots \right.\right.$$

$$\left.\left. + \sum_{1\le i_1<i_2\dots<i_r\le n} f_{i_1}f_{i_2}\cdots f_{i_r} \right) \right\|$$

$$\le \sum_{1\le i_1<i_2<\dots<i_{r+1}\le n} \|f_{i_1}\|\,\|f_{i_2}\|\cdots\|f_{i_{r+1}}\| + \cdots + \|f_1\|\,\|f_2\|\cdots\|f_n\|$$

$$\le \left(\sum_{i=1}^{n}\|f_i\|\right)^{r+1} + \cdots + \left(\sum_{i=1}^{n}\|f_i\|\right)^{n}$$

$$\le \frac{1}{1-\sum_{i=1}^{n}\|f_i\|}\left(\sum_{i=1}^{n}\|f_i\|\right)^{r+1}$$

$$\le \frac{1}{1-c}\left(\sum_{i=1}^{n}\|f_i\|\right)^{r+1}$$

what was to be proved. □

For $r = 0$ and $r = 1$ we obtain as special cases of Lemma 2 the inequalities

$$\left\| \prod_{i=1}^{n}(e+f_i) - e \right\| \le \frac{1}{1-c}\sum_{i=1}^{n}\|f_i\|, \tag{3}$$

$$\left\| \prod_{i=1}^{n}(e+f_i) - \left(e + \sum_{i=1}^{n} f_i\right) \right\| \le \frac{1}{1-c}\left(\sum_{i=1}^{n}\|f_i\|\right)^{2}. \tag{4}$$

LEMMA 3 *If $\alpha(A)$ $(A \in \mathcal{K})$ is a set function of bounded variation with values in the Banach algebra $\mathcal{B}$, then there exists a countable set $H_1 \subseteq H$ such that $\alpha(h) = 0$ if $h \in H - H_1$ (we use the notation $\alpha(h)$ for $\alpha(\{h\})$).*

PROOF. Since $\alpha(A)$ is of bounded variation, the set of those h's for which $\|\alpha(h)\| \geq \frac{1}{n}$, is finite. If n runs over the positive integers, then we obtain all the points h for which $\|\alpha(h)\| > 0$. Thus Lemma 3 is proved. □

LEMMA 4 *Let $\alpha(A)$, $(A \in \mathcal{K})$ be a multiplicative set function with values in the Banach algebra $\mathcal{B}$ for which $K = \mathrm{Var}_{\alpha-e}(H) < \infty$. If $B_1 \in \mathcal{K}$, $B_2 \in \mathcal{K}$, $B_1 \subseteq B_2$ and $B_2 - B_1 = \sum_{k=1}^{r} C_k$ where $C_k \in \mathcal{K}$ $(k = 1, 2, \ldots, r)$, then*

$$\mathrm{Var}_{\alpha-e}(B_2 - B_1) \leq (K+1)^{2r} \sum_{k=1}^{r} \mathrm{Var}_{\alpha-e}(C_i).$$

PROOF. Let $A_1, A_2, \ldots, A_n$ be a system of disjoint sets of $\mathcal{K}$,

$$\sum_{k=1}^{n} A_k \subseteq B_2 - B_1.$$

In this case

$$A_k = \sum_{i=1}^{r} A_k C_i \qquad (k = 1, 2, \ldots, n).$$

If $\mathfrak{z}_k = \{A_k C_1, A_k C_2, \ldots, A_k C_r\}$ and $\mathcal{P}(\mathfrak{z}_k) = (A_k C_{i_1}, A_k C_{i_2}, \ldots, A_k C_{i_r})$, then

$$\alpha(A_k) = \prod_{l=1}^{r} \alpha(A_k C_{i_l}).$$

Since $\|\alpha(B)\| \leq \|\alpha(B) - e\| + 1 \leq K + 1$ $(B \in \mathcal{K})$, it follows that

$$\left.\begin{aligned} \left\| \prod_{l=r_1}^{r} \alpha(A_k C_{i_l}) \right\| &\leq (K+1)^r, \\ \left\| \prod_{l=1}^{r_1} \alpha(A_k C_{i_l}) \right\| &\leq (K+1)^r, \end{aligned}\right\} \qquad (1 \leq r_1 \leq r).$$

Hence, applying the inequality (1), we get

$$\sum_{k=1}^{n} \|e - \alpha(A_k)\| \leq \sum_{k=1}^{n} (K+1)^{2r} \sum_{l=1}^{r} \|e - \alpha(A_k C_l)\| \leq (K+1)^{2r} \sum_{l=1}^{r} \mathrm{Var}_{\alpha-e}(C_l).$$

Thus Lemma 4 is proved. □

If α is additive, then we have the stronger relation

$$\mathrm{Var}_{\alpha}(B_2 - B_1) \leq \sum_{k=1}^{r} \mathrm{Var}_{\alpha}(C_i).$$

Lemma 5 *Let us suppose that the set function $\alpha(A)$ $(A \in \mathcal{K})$ with values in the Banach algebra $\mathcal{B}$ is multiplicative (and additive, resp.), $K = \mathrm{Var}_{\alpha-\alpha(0)}(H) < \infty$ and $\alpha(A) - \alpha(0)$ is v-continuous. Then for every $\varepsilon > 0$ there can be found a $\delta > 0$ such that if $U \in \mathcal{K}$ is a set with $d(U) \leq \delta$ and $\|\alpha(h) - \alpha(0)\| \leq \varepsilon$ for $h \in U$, then*

$$\mathrm{Var}_{\alpha-\alpha(0)}(U) \leq 2(K+1)^{2R}\varepsilon. \tag{5}$$

Proof. Contrary to the assertion let us suppose that there exists an $\varepsilon_0 > 0$ and a sequence of sets U_k for which

$$\lim_{k\to\infty} d(U_k) = 0, \qquad \|\alpha(h) - \alpha(0)\| \leq \varepsilon_0$$

if $h \in \sum_{k=1}^{\infty} U_k$ and

$$\mathrm{Var}_{\alpha-\alpha(0)}(U_k) > 2(K+1)^{2R}\varepsilon_0. \tag{6}$$

Since H is a compact metric space, we may suppose without restricting the generality that all the sequences h_k, where $h_k \in U_k$ $(k = 1, 2, \ldots)$, are convergent. Let h' denote their common limit element. We can distinguish two cases.

In the first case h' is contained at most in a finite number of sets U_n. This implies that $\lim\limits_{n\to\infty} U_n = 0$ and thus, since $\alpha - \alpha(0)$ is v-continuous,

$$\lim_{n\to\infty} \mathrm{Var}_{\alpha-\alpha(0)}(U_n) = 0$$

which contradicts (6).

In the second case, if h' is contained in infinitely many U_n, then we choose a number N such that

$$\mathrm{Var}_{\alpha-\alpha(0)}(U_N - h') < \varepsilon_0.$$

The possibility of this assured by Lemma 4 and the v-continuity of $\alpha - \alpha(0)$. According to (6) there exists in U_N a system of disjoint sets $A_1, A_2, \ldots, A_r$ that satisfies the inequality

$$\sum_{k=1}^{r} \|\alpha(A_k) - \alpha(0)\| > 2(K+1)^{2R}\varepsilon_0.$$

The element h' is contained at most in one of the sets A_i. If $h' \notin \sum_{i=1}^{r} A_i$, then

$$2\varepsilon_0(K+1)^{2R} < \sum_{k=1}^{r} \|\alpha(A_k) - \alpha(0)\| \leq \mathrm{Var}_{\alpha-\alpha(0)}(U_N - h') < \varepsilon_0$$

which is a contradiction. On the other hand, if $h' \in A_m$, then there exist disjoint sets $C_1, C_2, \ldots, C_n$ $(n \leq R)$ of the class of sets $\mathcal{K}$ such that

$$A_m = \{h'\} + \sum_{k=1}^{n} C_k.$$

Now, let α be multiplicative. Applying Lemma 4 for $B_2 = A_m$, $B_1 = 0$, we get

$$\|e - \alpha(A_m)\| \leq (K+1)^{2R} \left(\sum_{k=1}^{n} \mathrm{Var}_{\alpha-e}(C_k) + \|e - \alpha(h')\| \right).$$

Using this relation we conclude

$$\begin{aligned} 2(K+1)^{2R}\varepsilon_0 &< \sum_{k=1}^{r} \|e - \alpha(A_k)\| \\ &= \sum_{k \neq m} \|e - \alpha(A_k)\| + \|e - \alpha(A_m)\| \\ &\leq (K+1)^{2R} \Bigg(\sum_{k \neq m} \|e - \alpha(A_k)\| + \sum_{k=1}^{n} \mathrm{Var}_{\alpha-e}(C_k) \\ &\quad + \|e - \alpha(h')\| \Bigg) \\ &\leq (K+1)^{2R} (\mathrm{Var}_{\alpha-e}(U_N - h') + \|e - \alpha(h')\|) \\ &\leq 2\varepsilon_0 (K+1)^{2R}. \end{aligned}$$

If α is additive, then

$$\begin{aligned} 2(K+1)^{2R}\varepsilon_0 &< \sum_{k=1}^{r} \|\alpha(A_k)\| \\ &= \sum_{k \neq m} \|\alpha(A_k)\| + \|\alpha(A_m)\| \\ &\leq \sum_{k \neq m} \|\alpha(A_k)\| + \sum_{k=1}^{n} \|\alpha(C_k)\| + \|\alpha(h')\| \\ &\leq \mathrm{Var}_{\alpha}(U_N - h') + \|\alpha(h')\| \\ &\leq 2\varepsilon_0 \leq 2\varepsilon_0 (K+1)^{2R}. \end{aligned}$$

In both cases we arrived at contradictions, hence our lemma is proved. □

Lemma 6 *If $\alpha(B)$ ($\alpha(B) \in \mathcal{B}$, $B \in \mathcal{K}$) is a set function of bounded variation, then for every system $B_1, B_2, \ldots, B_r$ of disjoint sets of $\mathcal{K}$*

$$\left\| \prod_{k=1}^{r} (e + \alpha(B_k)) \right\| \leq K_1,$$

where K_1 is a constant.

Proof. Let us select from the above product those $\alpha(B_k)$'s for which $\|\alpha(B_k)\| \geq \frac{1}{2}$. The number of these elements is at most $N_1 = [2\,\mathrm{Var}_{\alpha}(H)]$. After this we form a maximal number of groups of the remaining $\alpha(B_i)$'s such that in every group the sum of the norms fall in the interval $\left[\frac{1}{4}, \frac{1}{2}\right]$. The number of these groups is at most $N_2 = [4 \operatorname{Var}_\alpha(H)]$. The sum of the norms of the remaining elements does not exceed $\frac{1}{4}$. Hence,

$$\left\| \prod_{k=1}^{r} (e + \alpha(B_k)) \right\| \leq 2^{N_2+1} (\operatorname{Var}_\alpha(H) + 1)^{N_1} = K_1. \qquad \square$$

LEMMA 7 *Let us suppose that the set function $\alpha(A)$ ($A \in \mathcal{K}$) with values in the Banach algebra $\mathcal{B}$ is multiplicative (and additive, resp.), $K = \operatorname{Var}_{\alpha-\alpha(0)}(H) < \infty$ and $\alpha(A) - \alpha(0)$ is v-continuous. If $B_1, B_2 \ldots$ is a sequence of sets of $\mathcal{K}$, $h \in B_k$ ($k = 1, 2, \ldots$), $\lim_{k\to\infty} d(B_k) = 0$, then*

$$\lim_{k\to\infty} \|\alpha(B_k) - \alpha(h)\| = 0.$$

PROOF. Let $B_k = \{h\} + \sum_{i=1}^{R} C_i^{(k)}$, $C_i^{(k)} \in \mathcal{K}$ ($i = 1, 2, \ldots, R;\ k = 1, 2, \ldots$) (we may always choose R such sets since if we had $r < R$, then we should complete this system by $R - r$ void sets) and let α be multiplicative. Applying Lemma 6 for $\alpha - e$ instead of α, moreover, using the inequality (1) we obtain

$$\|\alpha(B_k) - \alpha(h)\| \leq K_1^2 \sum_{i=1}^{R} \|\alpha(C_i^{(k)}) - e\| \leq K_1^2 \sum_{i=1}^{R} \operatorname{Var}_{\alpha-e}(C_i^{(k)}).$$

Since $\lim_{k\to\infty} C_i^{(k)} = 0$ ($i = 1, 2, \ldots, R$) and $\alpha(A) - e$ is v-continuous,

$$\lim_{k\to\infty} \operatorname{Var}_{\alpha-e}(C_i^{(k)}) = 0 \qquad (i = 1, 2, \ldots, R)$$

which proves the assertion.

Let us now consider the case of an additive α. Since $\alpha(0) = 0$, we get

$$\|\alpha(B_k) - \alpha(h)\| = \left\| \sum_{i=1}^{R} \alpha(C_i^{(k)}) \right\| \leq \sum_{i=1}^{R} \operatorname{Var}_\alpha(C_i^{(k)}).$$

The right-hand side tends to 0 when $k \to \infty$, hence our statement is completely proved. $\square$

§ 3. The additive total

In this § our purpose is to prove the following

THEOREM 1 *Let $f(A) \cdot (f(A) \in \mathcal{B})$ be a multiplicative set function defined on the elements of the class of sets $\mathcal{K}$. Suppose that $K = \operatorname{Var}_{f-e}(H)$ is finite and the set function $f(A) - e$ is v-continuous. In this case the additive total*

$$g(B) = \mathsf{S}_B(f(dA) - e)$$

exists for every $B \in \mathcal{K}$ and

$$\operatorname{Var}_g(B) \leq \operatorname{Var}_{f-e}(B). \tag{7}$$

PROOF. For the proof of the Theorem we need two lemmas.

LEMMA 8 *For every $h \in H$ and every $\varepsilon > 0$ there can be found a $\delta > 0$ such that if $A_1, A_2, \ldots, A_r$ are pairwise disjoint sets of the class of sets $\mathcal{K}$ with the property that $A = \sum_{k=1}^r A_k \in \mathcal{K}$, $h \in A$ and $d(A) \leq \delta$, then*

$$\left\| f(A) - e - \sum_{k=1}^{r} (f(A_k) - e) \right\| \leq \varepsilon. \tag{8}$$

PROOF OF LEMMA 8 Let δ be such a number that satisfies the conditions in Lemma 5 for $\dfrac{\varepsilon}{4R(K+1)^{4R}}$ instead of ε. The number δ can be chosen so small that the sphere with the centre h and the radius δ does not contain an $h' \in H$ $(h' \neq h)$ with $\|f(h') - e\| > \dfrac{\varepsilon}{4R(K+1)^{4R}}$. We choose furthermore δ so small that $\|f(B) - f(h)\| \leq \dfrac{\varepsilon}{4}$ if $h \in B \in \mathcal{K}$, $d(B) \leq \delta$. By Lemma 7 this is always possible. Let $h \in A_l$. Obviously

$$\begin{aligned} \left\| f(A) - e - \sum_{k=1}^{r} (f(A_k) - e) \right\| &\leq \|f(A) - f(A_l)\| + \operatorname{Var}_{f-e}(A - h) \\ &\leq \|f(A) - f(h)\| + \|f(A_l) - f(h)\| + \operatorname{Var}_{f-e}(A - h). \end{aligned}$$

Let $C_1, C_2, \ldots, C_n$ $(n \leq R)$ be a system of disjoint sets of $\mathcal{K}$ for which $A - \{h\} = \sum_{k=1}^n C_k$. According to Lemmas 4 and 5

$$\operatorname{Var}_{f-e}(A-h) \leq (K+1)^{2R} \sum_{k=1}^{n} \operatorname{Var}_{f-e}(C_k) \leq (K+1)^{2R} \frac{n\varepsilon}{2R(K+1)^{2R}} \leq \frac{\varepsilon}{2}.$$

On the other hand, we have chosen δ in such a way that

$$\|f(A) - f(h)\| \leq \frac{\varepsilon}{4}, \qquad \|f(A_l) - f(h)\| \leq \frac{\varepsilon}{4},$$

hence our lemma is proved. □

LEMMA 9 *Let $B \in \mathcal{K}$. To every $\varepsilon > 0$ there can be found a number $\delta > 0$ such that if $A_1, A_2, \ldots, A_r$ $(A_k \in \mathcal{K};\ k = 1, 2, \ldots, r)$ is a system of disjoint subsets of B, $\max\limits_{1 \leq k \leq r} d(A_k) \leq \delta$ and $A_k^{(1)}, A_k^{(2)}, \ldots, A_k^{(r_k)}$ is a decomposition of the set A_k into pairwise disjoint sets of $\mathcal{K}$, then*

$$\sum_{k=1}^{r} \left\| f(A_k) - e - \sum_{i=1}^{r_k} (f(A_k^{(i)}) - e) \right\| \leq \varepsilon. \tag{9}$$

PROOF OF LEMMA 9. We may suppose that the sets $A_k^{(i)}$ are so numbered that if $\mathfrak{z}_k = \{A_k^{(1)}, A_k^{(2)}, \ldots, A_k^{(r_k)}\}$, then $\mathcal{P}(\mathfrak{z}_k) = (A_k^{(1)}, A_k^{(2)}, \ldots, A_k^{(r_k)})$. Since the variation of $f(A) - e$ is equal to K, the number of the points $h \in H$ for which $\|f(h) - e\| > \dfrac{\varepsilon}{8K(K+1)^{2R}}$ is at most $\dfrac{8K^2}{\varepsilon}(K+1)^{2R}$. If such a point exists in B, then we renumber the sets A_k so that those sets, which contain these points, be $A_1, A_2, \ldots, A_l$ $\left(l \le r,\ l \le \dfrac{8K^2(K+1)^{2R}}{\varepsilon}\right)$. By Lemma 8 δ can be chosen so small that

$$\left\| f(A_k) - e - \sum_{i=1}^{r_k} (f(A_k^{(i)}) - e) \right\| \le \frac{\varepsilon}{2} \frac{\varepsilon}{8K^2(K+1)^{2R}} \le \frac{\varepsilon}{2l} \qquad (k = 1, 2, \ldots, l). \tag{10}$$

If there are sets which do not contain such points and these are $A_{l+1}, \ldots, A_r$, then we choose δ so small that besides (10) the following inequality holds:

$$\mathrm{Var}_{f-e}(A_k) \le \frac{\varepsilon}{4K} \qquad (k = l+1, l+2, \ldots, r).$$

By Lemma 5 this is possible since the sets $A_{l+1}, A_{l+2}, \ldots, A_r$ do not contain points h with $\|f(h) - e\| > \dfrac{\varepsilon}{8K(K+1)^{2R}}$. Applying the inequality (4) for $f_i = f(A_k^{(i)}) - e$ ($i = 1, 2, \ldots, r_k$) we get (we may suppose that $0 < \varepsilon \le 2K$)

$$\begin{aligned} \left\| f(A_k) - e - \sum_{i=1}^{r_k} (f(A_k^{(i)}) - e) \right\| &= \left\| \prod_{i=1}^{r_k} f(A_k^{(i)}) - e - \sum_{i=1}^{r_k} (f(A_k^{(i)}) - e) \right\| \\ &\le 2 \left(\sum_{i=1}^{r_k} \|f(A_k^{(i)}) - e\| \right)^2 \\ &\le 2(\mathrm{Var}_{f-e}(A_k))^2 \le 2 \frac{\varepsilon}{4K} \mathrm{Var}_{f-e}(A_k). \end{aligned} \tag{11}$$

Our statement follows from the inequalities (10) and (11).

After these preparations we can complete the proof of Theorem 1. We point out that if $B \in \mathcal{K}$ is a fixed set and $\mathfrak{z}_n = \{A_{nk}\}$ is a sequence of decompositions of B with $\lim_{n\to\infty} \max_k d(A_{nk}) = 0$, then the sequence

$$\sum_k (f(A_{nk}) - e)$$

satisfies the Cauchy's convergence criterion. For this purpose we consider two decompositions $A_1, A_2, \ldots, A_r$ and $A'_1, A'_2, \ldots, A'_{r'}$, of the set B into pairwise disjoint sets of $\mathcal{K}$ with $\max_{1\le k\le r} d(A_k) \le \delta$, $\max_{1\le k\le r'} d(A'_k) \le \delta$ where δ is the same number as in Lemma 9. If δ is so small that (9) is satisfied with $\dfrac{\varepsilon}{2}$ instead of ε, then, considering the superposition of the two decompositions, a well-known argument shows that

$$\left\| \sum_{k=1}^{r} (f(A_k) - e) - \sum_{k=1}^{r'} (f(A'_k) - e) \right\| \le \varepsilon$$

and thus $\mathsf{S}_B(f(dA) - e)$ exists. The relation (7) can be proved in an obvious way. □

§ 4. The multiplicative total

In this section we prove the following

THEOREM 2 *Let $g(B)$ $(g(B) \in \mathcal{B})$ be an additive set function defined on the elements of the class of sets $\mathcal{K}$. Suppose that $\mathrm{Var}_g(H) < \infty$ and the set function $g(B)$ $(B \in \mathcal{K})$ is v-continuous. In this case for any permutation function $\mathcal{P}$ and for every $A \in \mathcal{K}$ the total*

$$f(A) = {}_{\mathcal{P}}\sqcap_A (g(dB) + e)$$

exists and

$$\mathrm{Var}_{f-e}(A) \leq L \,\mathrm{Var}_g(A), \tag{12}$$

where L is a constant independent of the set A.

PROOF. First we prove two lemmas.

LEMMA 10 *For every $h \in H$ and $\varepsilon > 0$ there can be found a $\delta > 0$ such that if $B_1, B_2, \ldots, B_r$ are disjoint sets of $\mathcal{K}$ with $B = \sum_{k=1}^{r} B_k \in \mathcal{K}$, $h \in B$, $d(B) \leq \delta$, then*

$$\left\| g(B) + e - \prod_{k=1}^{r} (g(B_k) + e) \right\| \leq \varepsilon. \tag{13}$$

PROOF OF LEMMA 10. Let us suppose that $h \in B_l$. Using Lemma 6 and the inequality (1) it follows that

$$\begin{aligned} &\left\| g(B) + e - \prod_{k=1}^{r} (g(B_k) + e) \right\| \\ &\quad = \left\| e(g(B) + e)e - \prod_{k=1}^{l-1} (g(B_k) + e)(g(B_l) + e) \prod_{k=l+1}^{r} (g(B_k) + e) \right\| \\ &\quad \leq K_1^2 \left(\left\| \prod_{k=1}^{l-1} (g(B_k) + e) - e \right\| + \left\| \prod_{k=l+1}^{r} (g(B_k) + e) - e \right\| + \| g(B) - g(B_l) \| \right). \end{aligned}$$

Let $\varepsilon \leq K_1^2$. By Lemmas 5 and 7 (taking into account the remark made after Lemma 4) δ can be chosen so that

$$\mathrm{Var}_g(C - h) \leq \frac{\varepsilon}{4K_1^2}, \tag{14}$$

moreover

$$\| g(C) - g(h) \| \leq \frac{\varepsilon}{4K_1^2}, \tag{15}$$

where $C \in \mathcal{K}$, $h \in C$, $d(C) < \delta$; then from (3) and (14) it follows

$$\left\| \prod_{k=1}^{l-1} (g(B_k) + e) - e \right\| + \left\| \prod_{k=l+1}^{r} (g(B_k) + e) - e \right\|$$
$$\leq 2 \sum_{k=1}^{l-1} \|g(B_k)\| + 2 \sum_{k=l+1}^{r} \|g(B_k)\| \leq 2 \operatorname{Var}_g(B - h) \leq \frac{\varepsilon}{2K_1^2};$$

relation (15) implies that

$$\|g(B) - g(B_l)\| \leq \frac{\varepsilon}{2K_1^2},$$

hence our assertion holds. □

Lemma 11 *Let $A \in \mathcal{K}$. To every $\varepsilon > 0$ there can be found a number $\delta > 0$ such that if $B_1, B_2, \ldots, B_r$ ($B_k \in \mathcal{K}$; $k = 1, 2, \ldots, r$) is a system of disjoint subsets of A and $B_k^{(1)}, B_k^{(2)}, \ldots, B_k^{(r_k)}$ is a decomposition of the set B_k into pairwise disjoint sets of $\mathcal{K}$, $d(B_k) \leq \delta$, then*

$$\sum_{k=1}^{r} \left\| e + g(B_k) - \prod_{i=1}^{r_k} (e + g(B_k^{(i)})) \right\| \leq \varepsilon. \tag{16}$$

Proof of Lemma 11. If for a k we have $\operatorname{Var}_g(B_k) \leq \frac{1}{2}$, then by the inequality (4) it follows that

$$\left\| e + g(B_k) - \prod_{i=1}^{r_k} (e + g(B_k^{(i)})) \right\| \leq 2 \left(\sum_{i=1}^{r_k} \|g(B_k^{(i)})\| \right)^2 \leq 2(\operatorname{Var}_g(B_k))^2.$$

The remaining part of the proof can be accomplished by the aid of Lemma 10 in a similar way as we have proved Lemma 9 by the aid of Lemma 8.

In order to complete the proof of Theorem 2 let us consider the decompositions $\{B_1, B_2, \ldots, B_r\}$, $\{B'_1, B'_2, \ldots, B'_{r'}\}$ of the set A into pairwise disjoint sets of $\mathcal{K}$ satisfying $\max_{1 \leq k \leq r} d(B_k) \leq \delta$, $\max_{1 \leq k \leq r'} d(B'_k) \leq \delta$ where δ is a number fixed in Lemma 11. Let $\{B''_1, B''_2, \ldots, B''_{r''}\}$ denote the superposition of these two decompositions. If δ is so small that the inequality (16) holds for $\frac{\varepsilon}{2K_1^2}$ instead of ε, then

$$\left\| \prod_{k=1}^{r} (g(B_{i_k}) + e) - \prod_{k=1}^{r'} (g(B'_{j_k}) + e) \right\| \leq \left\| \prod_{k=1}^{r} (g(B_{i_k}) + e) - \prod_{k=1}^{r''} (g(B''_{n_k}) + e) \right\|$$
$$+ \left\| \prod_{k=1}^{r'} (g(B'_{j_k}) + e) - \prod_{k=1}^{r''} (g(B''_{n_k}) + e) \right\|$$

$$\leq K_1^2 \sum_{k=1}^{r} \left\| g(B_{i_k}) + e - \prod_{B''_{n_s} \subseteq B_{i_k}} (g(B''_{n_s}) + e) \right\|$$

$$+ K_1^2 \sum_{k=1}^{r'} \left\| g(B'_{j_k}) + e - \prod_{B''_{n_s} \subseteq B'_{j_k}} (g(B''_{n_s}) + e) \right\|$$

$$\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

where the sequences i_k, j_k, n_k are fixed by the permutation function $\mathcal{P}$ and in the products $\prod_{B''_{n_s} \subseteq B_{i_k}}$, $\prod_{B''_{n_s} \subseteq B'_{j_k}}$ the factors are arranged according to the same order as they were in the first row. Hence the existence of

$$f(A) = {}_{\mathcal{P}}\sqcap_A (g(dB) + e)$$

follows in an obvious way.

The proof of (12) can be accomplished as follows. By making use of Lemma 1 we get

$$\left\| \prod_{k=1}^{r_s} (g(B_s^{(i_k)}) + e) - e \right\| \leq K_1^2 \sum_{k=1}^{r_s} \| g(B_s^{(i_k)}) \| \leq K_1^2 \operatorname{Var}_g(B_s).$$

Hence it follows

$$\| f(B_s) - e \| \leq K_1^2 \operatorname{Var}_g(B_s)$$

and

$$\sum_{s=1}^{r} \| f(B_s) - e \| \leq K_1^2 \sum_{s=1}^{r} \operatorname{Var}_g(B_s) \leq K_1^2 \operatorname{Var}_g(A)$$

whence

$$\operatorname{Var}_{f-e}(A) \leq K_1^2 \operatorname{Var}_g(A)$$

what was to be proved. □

§ 5. Connection between the additive and multiplicative totals

According to Theorems 1 and 2 the indefinite additive (and multiplicative) total of a multiplicative (and additive, resp.) and v-continuous set function is also v-continuous. Hence it may be a starting function of further totalization. In this § we prove that the latter total (with respect to some permutation function) coincides with the original one. First we prove

THEOREM 3 *Let* $g(B)$ $(g(B)\in\mathcal{B})$ *be an additive set function defined on the elements of the class of sets* $\mathcal{K}$ *and satisfying the conditions of Theorem 2. If* $A\in\mathcal{K}$ *and*

$$f(A) = {}_{\mathcal{P}}\textstyle\bigsqcap_A (g(dB)+e), \tag{17}$$

then the additive total of $f(A)-e$ *exists in every* $B\in\mathcal{K}$ *and*

$$g(B) = \mathsf{S}_B (f(dA)-e). \tag{18}$$

PROOF. Let $B_1, B_2, \ldots, B_r$ be a system of disjoint sets of $\mathcal{K}$ with the property that $A=\sum_{k=1}^{r} B_k\in\mathcal{K}$, $\max\limits_{1\le k\le r} d(B_k)\le\delta$. Relation (16) implies that in case of a small δ

$$\sum_{k=1}^{r}\|e+g(B_k)-f(B_k)\|\le\frac{\varepsilon}{2}. \tag{19}$$

If we introduce the notation

$$g'(B) = \mathsf{S}_B(f(dA)-e),$$

then by (9), choosing δ small enough, we obtain

$$\sum_{k=1}^{r}\|g'(B_k)-f(B_k)-e)\|\le\frac{\varepsilon}{2}. \tag{20}$$

Relations (19) and (20) imply

$$\sum_{k=1}^{r}\|g(B_k)-g'(B_k)\|\le\varepsilon.$$

It follows that

$$\|g(A)-g'(A)\| = \left\|\sum_{k=1}^{r} g(B_k) - \sum_{k=1}^{r} g'(B_k)\right\| \le \sum_{k=1}^{r}\|g(B_k)-g'(B_k)\|\le\varepsilon$$

and thus $g(A)=g'(A)$. □

A similar theorem can be proved if we start from a multiplicative set function. In advance we remind of Definition 3 that to every multiplicative set function a permutation function is attached. Our statement is expressed in

THEOREM 4 *Let* $f(A)$ $(f(A)\in\mathcal{B})$ *be a multiplicative set function defined on the elements of the class of sets* $\mathcal{K}$ *and satisfying the conditions of Theorem* 1*. If* $B\in\mathcal{K}$ *and*

$$g(B) = \mathsf{S}_B(f(dA)-e), \tag{21}$$

moreover $\mathcal{P}$ *is a permutation function attached to* f*, then* ${}_{\mathcal{P}}\bigsqcap_A(g(dB)+e)$ *exists for every* $A\in\mathcal{K}$ *and*

$$f(A) = {}_{\mathcal{P}}\textstyle\bigsqcap_A (g(dB)+e). \tag{22}$$

PROOF. Let us introduce the notation

$$f'(A) = {}_{\mathcal{P}}\bigsqcap_A (g(dB) + e).$$

If $A_1, A_2, \ldots, A_r$ is a system of disjoint sets of $\mathcal{K}$ with the property that $\max\limits_{1 \le k \le r} d(A_k) \le \delta$ and $B = \sum_{k=1}^r A_k$, then by formula (9) (for a small δ)

$$\sum_{k=1}^{r} \|f(A_k) - e - g(A_k)\| \le \frac{\varepsilon}{2}. \tag{23}$$

According to (16) we get furthermore that if δ is small enough, then

$$\sum_{k=1}^{r} \|e + g(A_k) - f'(A_k)\| \le \frac{\varepsilon}{2}. \tag{24}$$

Hence, by (23) and (24),

$$\sum_{k=1}^{r} \|f(A_k) - f'(A_k)\| \le \varepsilon. \tag{25}$$

Let $\mathcal{P}(\mathfrak{z}) = (A_{i_1}, A_{i_2}, \ldots, A_{i_r})$ be the permutation corresponding to $\mathfrak{z} = \{A_1, A_2, \ldots, A_r\}$. By Lemma 6 there exists a K_1 such that

$$\begin{aligned} \left\| \prod_{k=1}^{j} f(A_{i_k}) \right\| &= \left\| \prod_{k=1}^{j} [f(A_{i_k}) - e) + e] \right\| \le K_1, \\ \left\| \prod_{k=j}^{r} f'(A_{i_k}) \right\| &= \left\| \prod_{k=j}^{r} [f'(A_{i_k}) - e) + e] \right\| \le K_1. \end{aligned} \tag{26}$$

Taking into account (25) and (26) and applying Lemma 1 we obtain

$$\|f(A) - f'(A)\| = \left\| \prod_{k=1}^{r} f(A_{i_k}) - \prod_{k=1}^{r} f'(A_{i_k}) \right\| \le K_1^2 \varepsilon,$$

hence $f(A) = f'(A)$. □

The last two theorems show that the indefinite total of an additive (and multiplicative, resp.) set function entirely determines the original set function.

§ 6. Examples

1. The weighted random point distribution

Let us consider a random selection of a finite number of points of H where each selected point is weighted with a (positive or negative) integer. We suppose that the sum of the weights in a set $A \in \mathcal{K}$ is a random variable which we denote by $\xi(A)$. We suppose furthermore that if $A_1, A_2, \dots, A_r$ are disjoint sets of $\mathcal{K}$, then the random variables $\xi(A_1), \xi(A_2), \dots, \xi(A_r)$ are independent. Let us introduce the notation

$$P_k(A) = \mathbb{P}(\xi(A) = k). \tag{27}$$

The sequences $P(A) = (\dots, P_{-2}(A), P_{-1}(A), P_0(A), P_1(A), P_2(A), \dots)$ are elements of the Banach algebra $\mathcal{B}$ of the sequences the corresponding series of which are absolutely convergent with the norm of the sum of the absolute values. If the product of two elements $(\dots, a_{-1}, a_0, a_1, \dots)$, $(\dots, b_{-1}, b_0, b_1, \dots)$ of $\mathcal{B}$ is the convolution

$$\left(\sum_{k=-\infty}^{\infty} a_{n-k} b_k; \quad n = 0, \pm 1, \pm 2, \dots \right), \tag{28}$$

then $\mathcal{B}$ is commutative and has as unity element that one for which the member corresponding to the index 0 is equal to 1 and the others are 0. In this case $P(A)$ is a multiplicative set function. We may establish a more stronger statement, namely

$$P(A) = P(A_1)P(A_2)\dots = \lim_{n\to\infty} \prod_{k=1}^{n} P(A_k) \tag{29}$$

if $A_i \in \mathcal{K}$ $(i = 1, 2, \dots)$, $A_i A_k = 0$ for $i \neq k$ and $A = \sum_{i=1}^{\infty} A_i \in \mathcal{K}$.

We shall show that $P(A) - e$ is of bounded variation and v-continuous. Since

$$\|P(A) - e\| = \sum_{k\neq 0} P_k(A) + 1 - P_0(A) = 2(1 - P_0(A)),$$

we have to prove the assertion for the real-valued set function $1 - P_0(A)$. Let us first extend the definition of $\xi(A)$ to $\mathcal{R}(\mathcal{K})$ which is the smallest ring[4] containing $\mathcal{K}$. Let $A_1, A_2, \dots$ be a sequence of disjoint sets of $\mathcal{R}(\mathcal{K})$. Since we have selected only a finite number of points from H, the sequence of independent random variables

$$\sum_{k=1}^{\infty} \xi(A_k)$$

converges with probability 1. Hence, by the three series theorem of Kolmogorov or simply by the Borel–Cantelli lemma,

$$\sum_{k=1}^{\infty} (1 - P_0(A_k)) < \infty.$$

On the other hand, $1 - P_0(A)$ is completely subadditive in the following sense: if $A_1, A_2, \dots$ is a sequence of disjoint sets of $\mathcal{K}$ for which $A = \sum_{k=1}^{\infty} A_k \in \mathcal{R}(\mathcal{K})$, then

$$1 - P_0(A) \leq \sum_{k=1}^{\infty} (1 - P_0(A_k)).$$

[4]A class of sets $\mathcal{R}$ is called a ring if $A + B \in \mathcal{R}$, $A - B \in \mathcal{R}$, provided that $A \in \mathcal{R}$, $B \in \mathcal{R}$. This extension is obviously possible and relation (29) will be satisfied also in $\mathcal{R}(\mathcal{K})$.

In fact, the event $\xi(A) \neq 0$ implies that at least one of $\xi(A_1) \neq 0$, $\xi(A_2) \neq 0$, ... occurs. Hence, by Lemma 4 of [5], $1 - P_0(A)$ is of bounded variation and by Lemma 1 of [5] $\mathrm{Var}_{1-P_0}(A)$ $(A \in \mathcal{R}(\mathcal{K}))$ is a bounded measure. This last property implies that $1 - P_0(A)$ is v-continuous. Thus the additive total

$$Q(B) = \mathsf{S}_B(P(dA) - e) \tag{30}$$

exists for every $B \in \mathcal{K}$. This implies obviously the existence of the totals

$$\begin{aligned} Q_0(B) &= \mathsf{S}_B(P_0(dA) - 1), \\ Q_k(B) &= \mathsf{S}_B P_k(dA) \qquad (k \neq 0). \end{aligned} \tag{31}$$

Since the convergence in (30) holds in the norm, it follows that

$$Q_0(B) = -\sum_{k \neq 0} Q_k(B). \tag{32}$$

Moreover, the relation

$$Q_k(B) \leq -Q_0(B) \leq \mathrm{Var}_{1-P_0}(B) \qquad (B \in \mathcal{K})$$

implies that with $\mathrm{Var}_{1-P_0}(A)$ the additive set functions $-Q_0(B)$, $Q_k(B)$ $(k = \pm 1, \pm 2, \ldots)$ are also bounded measures. Finally, if $h \in H$, then

$$\mathsf{S}_{\{h\}}(1 - P_0(dA)) = 1 - P_0(\{h\}) \leq 1,$$

hence

$$-Q_0(\{h\}) \leq 1. \tag{33}$$

It is interesting to write all the solutions of (29) in a closed form, provided that the sequence $P(A)$ is a probability distribution. This can be done as follows. We start from a sequence of bounded measures $-Q_0(B)$, $Q_k(B)$ $(k = \pm 1, \pm 2, \ldots)$ satisfying (32) and (33). Then every solution of (29) can be represented as

$$P(A) = \sqcap_A(e + Q(dB)) \qquad (A \in \mathcal{K}), \tag{34}$$

where $Q(B) = (\ldots, Q_{-1}(B), Q_0(B), Q_1(B), \ldots)$. In fact, if $P(A)$ is a solution of (29), then the $Q(B)$ defined by (30) has the mentioned properties, hence by Theorem 4 (34) holds. Conversely, if $Q(B)$ has the mentioned properties, then by Theorem 2 (34) exists and the properties of $Q(B)$ imply that $P(A)$ is a probability distribution for every $A \in \mathcal{K}$ and (29) holds. By Theorems 3 and 4 the correspondence between the set functions $P(A)$ and $Q(B)$ is one-to-one.

It is not difficult to see that $Q_k(B)$ is the expected number of points weighted with k in the set B. (This can be deduced immediately from the results of [6] too.) If there are no $h \in H$ such that $1 - P_0(\{h\}) > 0$, then $P(A)$ is a compound Poisson distribution, i.e. it has the characteristic function

$$\exp \sum_{k \neq 0} Q_k(A)(e^{iku} - 1). \tag{35}$$

If all the random points are weighted by 1, then from (35) we obtain a Poisson distribution

$$\exp Q_1(A)(e^{iu} - 1). \tag{36}$$

The proof of (35) can be accomplished with the aid of the relation (34). Similar statements are proved in [6].

2. The linear integer-valued Markov process

Let ξ_t be a Markov process on the linear interval $a \le t \le b$. We suppose that ξ_t can take on only the values $1, 2, \ldots, N$. Let $\mathcal{K}$ be the semi-ring of all the subintervals of $[a, b]$ (we permit here closed, open, semi-closed, degenerated intervals equally). If the right-hand and left-hand limits of the transition probability matrices $P(t_1, t_2)$ exists when t_1 or t_2 tend to a limit, then we can correspond to every $I \in \mathcal{K}$ a matrix $P(I)$. $P(I)$ is an element of the (non-commutative) Banach algebra of N-rowed quadratic matrices, where the norm is the maximal value among the absolute column-sums. Obviously $\mathcal{B}$ has a unite element. The set function $P(I)$ is then multiplicative relative to the natural permutation of linear intervals. Hence, if $P(I)$ is of bounded variation and v-continuous, then we can write the solution of the equation

$$P(I_1 + I_2) = P(I_1)P(I_2) \qquad (I_1 \in \mathcal{K},\ I_2 \in \mathcal{K},\ I_1 + I_2 \in \mathcal{K}) \tag{37}$$

in a closed form. We will not enter into the details since in the paper of Dobrušin [3] this is profoundly investigated under somewhat general assumptions. An analogous treatment can be given for the Markov processes having a countable number of possible states. To this question the author will return later.

Finally, I express my thank to Á. Császár for his valuable remarks.

References

[1] Arley, N. (1948). *On the Theory of Stochastic Processes and Their Applications to the Theory of Cosmic Radiation*, New York.

[2] Birkhoff, G. (1937). On Product Integration, *Journal of Math. and Phys.*, **16**, 104–132.

[3] Dobrušin, R. L. (1953). Obobscsenie uravnenij Kolmogorova dlja markovszkih processzov sz konecsnüm csiszlom vozmozsnüh szosztojanij, *Mat. Szbornik*, **33**, 566–596.

[4] Fréchet, M. (1938). *Recherches théoriques modernes sur le calcul des probabilités. II*, Paris, p. 210.

[5] Prékopa, A. (1956). Extension of Multiplicative Set Functions with Values in a Banach Algebra, *Acta Math. Acad. Sci. Hung.*, **7**, 201–213.

[6] Prékopa, A. (1957). On Poisson and Composed Poisson Stochastic Set Functions, *Studia Math.*, **16** 142–157.

[7] Rasch, G. (1943). Zur Theorie und Anwendung des Produkintegrals, *Journal f. reine u. angew. Math.*, **171**, 65–119.

[8] Schlesinger, L. (1908). *Vorlesungen über lineare Differentialgleichungen*, Berlin.

[9] Schlesinger, L. (1922). *Einführung in die Theorie der gewöhnlichen Differentialgleuchungen auf funktionentheoretischer Grundlage*, Berlin.

[10] Schlesinger, L. (1931). Neue Grundlagen für einen Infinitesimalkalkül der Matrizen, *Math. Zeitschrift*, **33**, 33–61.

[11] Schlesinger, L. (1932). Weitere Beiträge zum Infinitesimalkalkül der Matrizen, *Math. Zeitschrift*, **35**, 485–501.

[12] Volterra, V. (1887). Sui fondamenti della teoria delle equazioni differenziali lineari, *Mem. Soc. Ital. delle Scienze* (3), **6**, p. 107.

[13] Volterra, V. (1888). Sulla teoria delle equazioni differenziali lineari, *Rendiconti del Circolo Mat. di Palermo*, **3**, 69–75.

[14] Volterra, V. (1902). Sui fondamenti della teoria delle equazioni differenziali lineari, *Mem. Soc. Ital. delle Scienze* (3), **12**, 3–68.