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A CLASS OF STOCHASTIC PROGRAMMING DECISION PROBLEMS

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Abstract

A class of probabilistic constrained programming problems are considered where the probabilistic constraint is of the form $\mathsf{P}\{g_i(\mathbf{x},\boldsymbol{\xi}) \geq 0,\ i = 1,\dots,r\} \geq p$ and the functions g_i, $i = 1,\dots,r$ are concave. It is shown that the $\mathbf{x}$-function on the left hand side is logarithmic concave provided $\boldsymbol{\xi}$ has a logarithmic concave density. Special cases are mentioned and algorithmic solution of problems containing such constraint is discussed.

1 Remarks on logarithmic concave measures

The notion of logarithmic concave probability measure was introduced in [3]. Let P be a probability measure defined on the measurable subsets of $\mathbb{R}^n$. It is called *logarithmic concave* if for every pair of convex sets A, $B \subset \mathbb{R}^n$ and for every $0 < \lambda < 1$ the following inequality holds

$$\mathsf{P}\{\lambda A + (1-\lambda)B\} \geq (\mathsf{P}\{A\})^{\lambda}(\mathsf{P}\{B\})^{1-\lambda}. \tag{1.1}$$

It is proved in [3] that if P is a continuous probability measure the density of which is of the following form

$$f(\mathbf{x}) = e^{-Q(\mathbf{x})}, \qquad \mathbf{x} \in \mathbb{R}^n, \tag{1.2}$$

where $Q(\mathbf{x})$ is a convex function in the entire n-dimensional space, then P is a logarithmic concave measure. The value $+\infty$ is also allowed for the function Q.

It follows from (1.1) that *if D is an arbitrary convex subset of $\mathbb{R}^n$ then the function*

$$\mathsf{P}\{D + \mathbf{x}\} = \int_{D+\mathbf{x}} f(\mathbf{z})\,\mathrm{d}\mathbf{z}, \qquad \mathbf{x} \in \mathbb{R}^n \tag{1.3}$$

is logarithmic concave in the entire space (see [3]).

A non-negative function $h(\mathbf{x})$ defined in a convex set $C \subset \mathbb{R}^n$ is called *logarithmic concave* if for every $\mathbf{x}_1, \mathbf{x}_2 \in C$ and $0 < \lambda < 1$ we have $h\ (\lambda\mathbf{x}_1 + (1-\lambda)\mathbf{x}_2) \geq [h(\mathbf{x}_1)]^\lambda [h(\mathbf{x}_2)]^{1-\lambda}$. If in particular $D = \{\mathbf{z} \mid \mathbf{z} \leq \mathbf{0}\}$ then

$$\mathsf{P}\{D + \mathbf{x}\} = \int_{\mathbf{z} \leq \mathbf{x}} f(\mathbf{z})\, \mathrm{d}\mathbf{z} = F(\mathbf{x}), \qquad \mathbf{x} \in \mathbb{R}^n, \tag{1.4}$$

where $F(\mathbf{x})$ is the probability distribution function belonging to the probability density $f(\mathbf{x})$ and we see that $F(\mathbf{x})$ is logarithmic concave in the entire space. Examples for logarithmic concave multivariate probability distributions are the normal, the Wishart, the Dirichlet and the beta distributions (see [3]).

It is easy to see that the statement concerning the function (1.3) can be converted. *If* P *is a continuous, logarithmic concave probability measure in* $\mathbb{R}^n$ *and its probability density* $f(\mathbf{x})$ *is continuous in an open convex set* C*, then* $f(\mathbf{x})$ *is logarithmic concave in* C*.*

In fact if $\mathbf{x}_1, \mathbf{x}_2 \in C$ and D is a spherical neighbourhood of the point $\mathbf{0}$, then

$$\lambda(D + \mathbf{x}_1) + (1-\lambda)(D + \mathbf{x}_2) = D + [\lambda\mathbf{x}_1 + (1-\lambda)\mathbf{x}_2] \tag{1.5}$$

and $D + \mathbf{x}_1$, $D + \mathbf{x}_2$, $D + [\lambda\mathbf{x}_1 + (1-\lambda)\mathbf{x}_2]$ are spherical neighbourhoods of the points $\mathbf{x}_1$, $\mathbf{x}_2$, $\lambda\mathbf{x}_1 + (1-\lambda)\mathbf{x}_2$. In view of (1.5) and the logarithmic concavity of the P measure, we have

$$\mathsf{P}\{D + [\lambda\mathbf{x}_1 + (1-\lambda)\mathbf{x}_2]\} \geq (\mathsf{P}\{D + \mathbf{x}_1\})^\lambda (\mathbf{P}\{D + \mathbf{x}_2\})^{1-\lambda}. \tag{1.6}$$

Dividing on both sides by the volume of D and let the radius of D tend to zero we obtain the required inequality

$$f(\lambda\mathbf{x}_1 + (1-\lambda)\mathbf{x}_2) \geq (f(\mathbf{x}_1))^\lambda (f(\mathbf{x}_2))^{1-\lambda}. \tag{1.7}$$

2 The class of stochastic programming models

Let $g_i(\mathbf{x}, \mathbf{y})$, $i = 1, \ldots, r$ be concave functions in $\mathbb{R}^{n+q}$ where $\mathbf{x}$ is an n-component $\mathbf{y}$ is a q-component vector. Let further $\boldsymbol{\xi}$ be a q-component random vector the probability distribution of which is logarithmic concave in $\mathbb{R}^q$. We consider constraints of the following type

$$\mathsf{P}\{g_i(\mathbf{x}, \boldsymbol{\xi}) \geq 0,\ i = 1, \ldots, r\} \geq p. \tag{2.1}$$

The most important fact concerning this is expressed by the following

Theorem *Under the conditions mentioned above, the* $\mathbf{x}$*-function standing on the left hand side of* (2.1) *is logarithmic concave in the entire space* $\mathbb{R}^n$*.*

Proof. Consider the sets depending on the n-component parameter $\mathbf{x}$:

$$H(\mathbf{x}) = \{\mathbf{y} \mid g_i(\mathbf{x}, \mathbf{y}) \geq 0, \quad i = 1, \ldots, r\}. \tag{2.2}$$

Let L be the set of those $\mathbf{x}$ vectors for which $H(\mathbf{x})$ is not empty. We prove that L is convex and $H(\mathbf{x})$, $\mathbf{x} \in L$ is a concave family of sets i.e. if $\mathbf{x}_1$, $\mathbf{x}_1 \in L$ and $0 < \lambda < 1$, then

$$H(\lambda \mathbf{x}_1 + (1-\lambda)\mathbf{x}_2) \supset \lambda H(\mathbf{x}_1) + (1-\lambda) H(\mathbf{x}_2). \tag{2.3}$$

In fact if $\mathbf{y}_1 \in H(\mathbf{x}_1)$, $\mathbf{y}_2 \in H(\mathbf{x}_2)$, then

$$\begin{aligned} g_i(\mathbf{x}_1, \mathbf{y}_1) \geq 0, \\ g_i(\mathbf{x}_2, \mathbf{y}_2) \geq 0, \end{aligned} \qquad i = 1, \dots, r. \tag{2.4}$$

Taking into account the concavity of the functions $g_1, \dots, g_r$, we obtain

$$\begin{aligned} & g_i(\lambda \mathbf{x}_1 + (1-\lambda)\mathbf{x}_2, \lambda \mathbf{y}_1 + (1-\lambda)\mathbf{y}_2) \\ & \quad \geq \lambda g_i(\mathbf{x}_1, \mathbf{y}_1) + (1-\lambda) g_i(\mathbf{x}_2, \mathbf{y}_2) \geq 0, \qquad i = 1, \dots, r. \end{aligned} \tag{2.5}$$

This implies the convexity of L and the inequality (2.3). We remark that the sets $H(\mathbf{x})$ are convex. This is a consequence of the inequality (2.3).

The function standing on the left hand side of (2.1) can be expressed in the following way

$$\mathsf{P}\{g_i(\mathbf{x}, \boldsymbol{\xi}) \geq 0,\ i = 1, \dots, r\} = \mathsf{P}\{\boldsymbol{\xi} \in H(\mathbf{x})\}, \quad \text{if} \quad \mathbf{x} \in L. \tag{2.6}$$

Let $\mathbf{x}_1, \mathbf{x}_2 \in L$ and $0 < \lambda < 1$. We have by (2.3) and the logarithmic concavity of the probability distribution of $\boldsymbol{\xi}$ that

$$\begin{aligned} & \mathsf{P}\{\boldsymbol{\xi} \in H(\lambda \mathbf{x}_1 + (1-\lambda)\mathbf{x}_2)\} \\ & \quad \geq \mathsf{P}\{\boldsymbol{\xi} \in \lambda H(\mathbf{x}_1) + (1-\lambda) H(\mathbf{x}_2)\} \\ & \quad \geq (\mathsf{P}\{\boldsymbol{\xi} \in H(\mathbf{x}_1)\})^{\lambda} (\mathbf{P}\{\boldsymbol{\xi} \in H(\mathbf{x}_2)\})^{1-\lambda}. \end{aligned} \tag{2.7}$$

Thus the function is logarithmic concave in the convex set L. On the other hand our function is equal to zero outside L hence it is logarithmic concave in the entire space $\mathbb{R}^n$. Thus the Theorem is proved. $\square$

Now we consider the following problem

$$\left.\begin{aligned} & h_0(\mathbf{x}) = \mathsf{P}\{g_i(\mathbf{x}, \boldsymbol{\xi}) \geq 0,\ i = 1, \dots, r\} \geq p, \\ & h_i(\mathbf{x}) \geq 0, \qquad i = 1, \dots, m, \\ & \min G(\mathbf{x}), \end{aligned}\right\} \tag{2.8}$$

where $h_1, \dots, h_m$ are concave or logarithmic concave in $\mathbb{R}^n$, $G(\mathbf{x})$ is convex in $\mathbb{R}^n$, p is a prescribed probability for which $0 < p < 1$ and $g_1, \dots, g_r$, have the property described in the beginning of this section. Problem (2.8) is equivalent to a convex programming problem since we can take the logarithm of each constraint function which is only logarithmic concave and not necessarity concave without changing the problem at all. Remarkable is the fact that in the first constraint we may have inequalities for nonlinear functions inside the parantheses. Special examples of problem (2.8) are given below. In the first two examples we pay attention only to the probabilistic constraint.

EXAMPLE 1 Let $g_1(\mathbf{x}),\ldots,g_r(\mathbf{x})$ be concave functions in $\mathbb{R}^n$ and define the functions $g_1(\mathbf{x},\mathbf{y}),\ldots,g_r(\mathbf{x},\mathbf{y})$ as follows

$$g_i(\mathbf{x},\mathbf{y}) = g_i(\mathbf{x}) - y_i, \qquad i=1,\ldots,r. \tag{2.9}$$

In this case our probabilistic constraint reduces to the following

$$\begin{aligned}&\mathsf{P}\{g_i(\mathbf{x},\boldsymbol{\xi}) \ge 0,\ i=1,\ldots,r\}\\ &\quad = \mathsf{P}\{g_i(\mathbf{x}) \ge \xi_i,\ i=1,\ldots,r\} \ge p.\end{aligned} \tag{2.10}$$

Problems containing such constraint are considered in [4] and [5].

EXAMPLE 2 Suppose that the random variables $\eta_1,\ldots,\eta_r$ can be expressed as the mixed quadratic function of the deterministic n-vector $\mathbf{x}$ and the stochastic q-vector $\boldsymbol{\xi}$ i.e. we can represent them as follows

$$\eta_i = \begin{pmatrix}\mathbf{x}\\ \boldsymbol{\xi}\end{pmatrix}' A_i \begin{pmatrix}\mathbf{x}\\ \boldsymbol{\xi}\end{pmatrix} + \mathbf{b}_i'\mathbf{x} + \mathbf{d}_i'\boldsymbol{\xi} + k_i + \varepsilon_i, \qquad i=1,\ldots,r, \tag{2.11}$$

where $A_1,\ldots,A_r$ are $(n+q)x(n+q)$ negative semi-definite matrices, $\mathbf{b}_1,\ldots,\mathbf{b}_r,\mathbf{d}_1,\ldots,\mathbf{d}_r$ are constant vectors, $k_1,\ldots,k_r$ are constants, $\varepsilon_1,\ldots,\varepsilon_r$ are random variables independent of each other and of the random vector $\boldsymbol{\xi}$. We introduce the constraint

$$\mathsf{P}\{\eta_i - t_i \ge 0,\ i=1,\ldots,r\} \ge p, \tag{2.12}$$

where $t_1,\ldots,t_r$ are certain prescribed constants and p is a prescribed probability. Since all functions of the variables $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$:

$$\left.\begin{aligned}&g_i(\mathbf{x},\mathbf{y},\mathbf{z}) = \begin{pmatrix}\mathbf{x}\\ \mathbf{y}\end{pmatrix}' A_i \begin{pmatrix}\mathbf{x}\\ \mathbf{y}\end{pmatrix} + \mathbf{b}_i'\mathbf{x} + \mathbf{d}_i'\mathbf{y} + k_i - t_i + z_i,\\ &i=1,\ldots,r\end{aligned}\right\} \tag{2.13}$$

are concave, it follows that the $\mathbf{x}$-function on the left hand side of (2.12) is logarithmic concave in the entire n-dimensional space provided $\boldsymbol{\xi}$ has a continuous distribution with logarithmic concave density in $\mathbb{R}^q$ further the residual random variables $\varepsilon_1,\ldots,\varepsilon_r$ have continuous distributions with logarithmic concave densities. In fact in this case the components of $\boldsymbol{\xi}$ together with the components of $\boldsymbol{\varepsilon}' = (\varepsilon_1,\ldots,\varepsilon_r)$ have a continuous joint density in $\mathbb{R}^{q+r}$. Furthermore

$$\eta_i - t_i = g_i(\mathbf{x},\boldsymbol{\xi},\boldsymbol{\varepsilon}), \qquad i=1,\ldots,r, \tag{2.14}$$

thus our statement is implied by the Theorem of this paper.

In the practical application $\boldsymbol{\xi}$ is frequently supposed to be a normally distributed random vector and the residual variables are also supposed to be normally distributed in which case the densities are logarithmic concave.

EXAMPLE 3 As the third and last example we consider a water reservoir system design problem. We restrict ourselves to a relatively simple situation because the main purpose here is to show the importance of our class of stochastic programming decision problems. Suppose we have two possible sites for building reservoirs the purpose of which is to catch the flood once in each year. This will therefore be a single period problem. We suppose that the reservoirs are practically empty beginning of the floody period in each year. The possible sites and the rivers are illustrated in Figure 1. Denote by K_1 and K_2 the unknown capacities of the reservoirs, ξ_1 and ξ_2 the direct flood inflows into Reservoirs 1 and 2, respectively. The flood will be caught by the reservoirs if and only if the following inequality holds

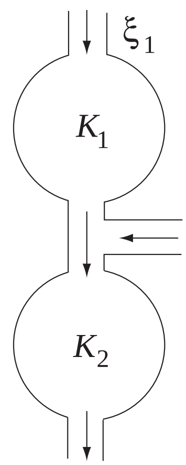

Figure 1:

$$\xi_1 - \min(\xi_1, K_1) + \xi_2 \leq K_2. \tag{2.15}$$

Let us introduce the following function of the four variables K_1, K_2, z_1, z_2:

$$g(\mathbf{K}, \mathbf{z}) = K_2 + \min(z_1, K_1) - z_1 - z_2, \tag{2.16}$$

where $\mathbf{K}' = (K_1, K_2)$, $\mathbf{z}' = (z_1, z_2)$. Then (2.15) can be expressed as

$$g(\mathbf{K}, \boldsymbol{\xi}) \geq 0, \tag{2.17}$$

where $\boldsymbol{\xi}' = (\xi_1, \xi_2)$. Formulating not only the probabilistic constraint but the complete problem, we may write the following

$$\begin{aligned} &\mathsf{P}\{g(\mathbf{K}, \boldsymbol{\xi}) \geq 0\} \geq p, \\ &0 \leq K_1 \leq L_1, \\ &0 \leq K_2 \leq L_2, \\ &\min[G_1(K_1) + G_2(K_2)] \end{aligned} \tag{2.18}$$

Here L_1, L_2 are some fixed upper bounds, $G_1(K_1)$ and $G_2(K_2)$ are the reservoir building cost functions and p is a fixed probability prescribed by ourselves somewhere near unity. Since the function (2.16) is concave in the four variables, the $\mathbf{K}$-function on the left hand side in the first constraint is logarithmic concave provided ξ_1, ξ_2 have a continuous joint distribution with logarithmic concave density. This is an immediate consequence of our Theorem. If G_1 and G_2 are convex functions then (2.18) is a convex (or quasi-convex if we do not take the logarithm in the first constraint) programming problem. More sophisticated reservoir system design problems are discussed in [6].

3 On the algorithmic solution of problem (2.8)

Under the conditions introduced in Section 2, the function $h_0(\mathbf{x})$ standing on the left hand side in the first constraint is logarithmic concave in the entire space $\mathbb{R}^n$. It seems therefore reasonable to apply the SUMT method with logarithmic penalty function for the solution of Problem (2.8). The interior point SUMT method proceeds as follows. Take any decreasing sequence r_k for which $r_k > 0$, $k = 1, 2, \ldots$ and $\lim\limits_{k\to\infty} r_k = 0$. Solve the unconstrained minimization problem

$$\text{minimize } T(\mathbf{x}, r_k) = G(\mathbf{x}) - r_k \sum_{i=0}^{m} \log h_i(\mathbf{x}). \tag{3.1}$$

Denote by $\mathbf{x}(r_k)$ the optimal solution of (3.1). Then under some regularity conditions we have

$$\lim_{k\to\infty} G(\mathbf{x}(d_k)) = \min_{x\in S} G(\mathbf{x}), \tag{3.2}$$

where S is the set of feasible solutions of problem (2.8).

Denote by S_0 the set of those $\mathbf{x}$ vectors for which the inequalities in (2.8) hold strictly. Maintaining the conditions introduced in Section 2,we introduce also the following ones:

a) S_0 is not empty,

b) the constraining functions are continuous,

c) the set $\{\mathbf{x} \mid G(\mathbf{x}) \le \alpha,\ \mathbf{x} \in S\}$ is bounded for every finite value of α.

Under these conditions relation (3.2) holds true. In fact the functions $G(\mathbf{x})$, $-\log h_i(\mathbf{x})$, $i = 0, \ldots, m$ are convex by the assumptions and results of the previous section. Thus, applying the theory presented in [1], our assertion follows.

When solving (3.1) we frequently have to obtain values of $h_0(\mathbf{x})$. This can be done by simulation. Sometimes it is possible to evaluate the gradient of $h_0(\mathbf{x})$ too (see e.g. [4]). There are, however, sophisticated cases where the evaluation of $\nabla h_0(\mathbf{x})$ is very difficult, like in Example 2 of the previous section. In such cases methods without derivatives are advised (see [2]).

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