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Generalizations of the Theorems of Smirnov with Application to a Reliability Type Inventory Problem

András Prékopa

Technological University, Budapest, XI. Stoczek u. 2–4
and
Computing Center of the HAS, Budapest I. Uri u. 49, Hungary

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Abstract

Let τ be the number of elements of a sample taken from a population uniformly distributed in $[0, 1]$. Let $\alpha \geq 0$ be a number such that $\lambda = n\ \alpha \leq 1$. Subdivide an interval of length $1 - \lambda$ into n parts by $n - 1$ independently and uniformly distributed points. Denote $\delta_1, \dots, \delta_n$ the lengths of these subintervals. Using the notations $F_n(t, \lambda) = \delta_1 + \dots + \delta_\tau + \tau\alpha$, the generalizations of the theorems of Smirnov are expressed by (4.14), (4.15), where $G_m(t, \mu)$ is defined similarly to $F_n(t, \lambda)$ and these two stochastic processes are supposed to be independent. These theorems were already published in [7], the proofs are given here. Applications to inventory problems are also mentioned.

1 Introduction

Let A and B be two factories which agree that in the course of the time interval $[0, T]$ factory B transports an amount of a certain material for the production at factory A. Let η_t denote the *amount of material transported up to time* t and ξ_t denote the *amount of the same material used by factory* A *up to* t provided there is no shortage between 0 and t where $0 \leq t \leq T$. Both η_t and ξ_t are stochastic processes. At time $t = 0$ factory A has an *initial stock* which we denote by M. We wish to determine this initial stock by the following principle

$$\begin{aligned}&\text{minimize } M\\ &\text{subject to the constraint}\\ &\mathsf{P}\left\{\inf_{0\leq t\leq T}(M + \eta_t - \xi_t) > 0\right\} \geq 1 - \varepsilon,\end{aligned} \tag{1.1}$$

where $\varepsilon > 0$ is a fixed probability near zero in practice. The probability on the left hand side in (1.1) is obviously a monotonically non-decreasing function of the variable M. If this function is continuous and increasing, then the solution of problem (1.1), i.e. the optimal M is given by the only one solution of the following equation

$$\mathsf{P}\left\{\inf_{0\le t\le T}(M+\eta_t-\xi_t)>0\right\}=1-\varepsilon. \tag{1.2}$$

Equation (1.2) was called in [7] the "*reliability equation*" of the above-mentioned inventory problem. This inventory problem does not use costs with respect to which a minimization is carried out in conventional inventory models but the probability of the no shortage case is prescribed. The reason of the construction of this model was the difficulty of the determination of the costs in very many practical cases.

In [7] we considered the solution of the reliability equation (1.2) under some assumptions in connection with the stochastic processes ξ_t, η_t. First we suppose that factory A uses this material with constant intensity i.e. the consumption is ct during the time interval $[0,t]$ where c is a positive constant and that $\eta_T=cT$ i.e. the total input is equal to the total consumption provided there is no shortage during the time interval $[0,T]$. We suppose also that the input process η_t is generated in the following way: the material is transported at n occasions which are the elements of the ordered sample $t_1^*\le\cdots\le t_n^*$ where $t_1,\dots,t_n$ i.e. the original sample is taken from the population uniformly distributed in the interval $[0,T]$. The subdivision of the material among these occasions is stochastically independent of the sample $t_1,\dots,t_n$ and is done so that there exists a constant $\alpha\ge 0$ denoting the amount what is surely transported at any of the occasions t_i^*, $i=1,\dots,n$ further we subdivide an interval with length $cT-n\alpha$ (this number is supposed to be non-negative) by $n-1$ independent and uniformly distributed random points into n random subintervals with lengths $\delta_1,\dots,\delta_n$, finally we assign to $t_1^*,\dots,t_n^*$ as arriving amounts the following:

$$\alpha+\delta_1,\dots,\alpha+\delta_n. \tag{1.3}$$

The sum of the numbers (1.3) is equal to cT. We may suppose that $T=1$ and $c=1$, since this can always be attained by a suitable choice of the units. Let further λ denote the value $n\alpha$ which is the rate of uniformity in the transported amounts. We have $0\le\lambda\le 1$. Instead of the notation η_t we shall use the notation $F_n(t,\lambda)$ as well which contains n and λ and is in accordance with the usual notation of the sampling probability distribution function to which $F_n(t,\lambda)$ reduces in case of $\lambda=1$. The reliability equation (1.2) can be written in the following equivalent form

$$\mathsf{P}\left\{\sup_{0\le t\le 1}(t-F_n(t,\lambda))<M\right\}=1-\varepsilon. \tag{1.4}$$

The *consumption process* can also be modelized in a similar way as we modelized the input process. In this case we denote it by $G_m(t,\mu)$ instead of ξ_t where the roles of m and μ are similar to those of n and λ. In this case the reliability equation is the following

$$\mathsf{P}\left\{\sup_{0\le t\le 1}(G_m(t,\mu)-F_n(t,\lambda))<M\right\}=1-\varepsilon. \tag{1.5}$$

We remark that the functions of M on the left hand sides in (1.4) and (1.5) are continuous, increasing, equal to 0 at $M=0$ and to 1 at $M=1$. Thus in both cases there are unique solutions with respect to M if $0<\varepsilon<1$.

In [7] we gave asymptotic solutions for the equations (1.4), (1.5), which are based on some generalizations of the order statistical theorems of Smirnov. We published only the statements of the generalizations, not their proofs and the purpose of the present paper is to complete this insufficiency. The proofs use known technique in stochastic processes, this was the reason why they were left out. However, taking into account the fact that these models became widely used, the necessity of the publication of the proofs was emphasized by several experts.

The first version of this model was that special case of (1.4) when $\lambda = 1$. This was considered by the author of this paper and M. Ziermann. For this special case classical order statistical results could be applied to obtain the solution of the reliability equation. This model formulation and the application of the classical results were published separately in English and in Hungarian in [7] and [10]. For the sake of completeness we recall these here too. If $\lambda = 1$ then we have

$$\begin{aligned}&\mathsf{P}\left\{\sup_{0\le t\le 1}(t - F_n(t,1)) < M\right\}\\ &\quad = 1 - M\sum_{j=0}^{[n(1-M)]}\binom{n}{j}\left(1-\mu-\frac{j}{n}\right)^{n-j}\left(\mu+\frac{j}{n}\right)^{j-1},\end{aligned}\tag{1.6}$$

as it is proved e.g. in [1] and [2]. Thus the solution of the reliability equation (1.4) is given by the solution (with respect to M) of the following equation

$$M\sum_{j=0}^{[n(1-M)]}\binom{n}{j}\left(1-\mu-\frac{j}{n}\right)^{n-j}\left(\mu+\frac{j}{n}\right)^{j-1} = \varepsilon.\tag{1.7}$$

An asymptotic solution of (1.4) can be obtained by the

First theorem of Smirnov. *If $F_n(t)$ is the sampling probability distribution function of a sample of size n taken from a continuously distributed population with probability distribution function $F(t)$, then*

$$\begin{aligned}&\lim_{n\to\infty}\mathsf{P}\left\{\sqrt{n}\,\sup_t(F(t)-F_n(t)) < y\right\}\\ &\quad = \lim_{n\to\infty}\mathsf{P}\left\{\sqrt{n}\,\sup_t(F_n(t)-F(t)) < y\right\} = \begin{cases}1-e^{-2y^2}, & \textit{if}\quad y>0,\\ 0, & \textit{if}\quad y\le 0.\end{cases}\end{aligned}\tag{1.8}$$

Applying this theorem to the probability distribution function

$$F(t) = \begin{cases}0, & \text{if}\quad t\le 0,\\ t, & \text{if}\quad 0\le t\le 1,\\ 1, & \text{if}\quad t\ge 1,\end{cases}\tag{1.9}$$

we see that $F_n(t,1)$ is the same as $F_n(t)$ in (1.8) and we can write

$$\lim_{n\to\infty}\mathsf{P}\left\{\sqrt{n}\sup_{0\le t\le 1}(t-F_n(t,1)) < y\right\} = 1-e^{-2y^2}\ \text{if}\ y>0.\tag{1.10}$$

For a finite n supposing approximate equality in (1.10) and putting $y = \sqrt{n}\, M$, we obtain from the approximate equality

$$1 - e^{-2(\sqrt{n}M)^2} \approx 1 - \varepsilon,$$

the approximate solution

$$M \approx \left(\frac{1}{2n} \log \frac{1}{\varepsilon} \right)^{\frac{1}{2}}. \tag{1.11}$$

Finally we formulate the second theorem of Smirnov which will also be generalized for the sake of the solution of the reliability equation (1.5).

Second theorem of Smirnov. *Let $F_n(t)$ and $G_m(t)$ be two sampling probability distribution functions of two independent samples taken from the same continuously distributed population. Then we have*

$$\lim_{\substack{m\to\infty \\ n\to\infty}} \mathsf{P} \left\{ \sqrt{\frac{mn}{m+n}} \sup_t \left(G_m(t) - F_n(t)\right) < y \right\} = \begin{cases} 1 - e^{2y^2}, & \text{if} \quad y > 0, \\ 0, & \text{if} \quad y \le 0. \end{cases} \tag{1.12}$$

In Section 2 we deduce the covariance functions of the stochastic processes $F_n(t,\lambda) - t$ and $G_m(t,\mu) - F_n(t,\lambda)$. In Section 3 the limits of the finite dimensional distributions are given for the above-mentioned stochastic processes provided $m, n \to \infty$. In Section 4 the theorems of Smirnov are generalized and in Section 5 our theorems are applied for the reliability equations (1.4), (1.5).

2 The covariance functions of the stochastic processes $F_n(t,\lambda) - t = u(t)$ and $G_m(t,\mu) - F_n(t,\lambda) = v(t)$

Consider $n-1$ independently and in $[0,1]$ uniformly distributed random points. The probability density of the jth point to the left is equal to the following

$$\frac{(n-1)!}{(j-1)!(n-1-j)!} x^{j-1}(1-x)^{n-1-j}, \quad 0 \le x \le 1, \quad j = 1, \ldots, n-1. \tag{2.1}$$

The *expectation* belonging to this equals

$$\frac{(n-1)!}{(j-1)!(n-1-j)!} \int_0^1 x^j (1-x)^{n-1-j}\, \mathrm{d}x = \frac{j}{n}. \tag{2.2}$$

The *joint density* of the jth and kth random points ($j < k$) is the following

$$\frac{(n-1)!}{(j-1)!(k-j-1)!(n-1-k)!} y_1^{j-1}(y_2 - y_1)^{k-j-1}(1-y_2)^{n-1-k}, \quad 0 \le y_1 \le y_2 \le 1. \tag{2.3}$$

A simple integration shows that the covariance of the jth and kth random points equals

$$\frac{1}{n+1} \frac{j}{n} \left(1 - \frac{k}{n}\right). \tag{2.4}$$

Using these facts we show now that

$$\mathsf{E}\ \{F_n(t,\lambda)\} = t, \tag{2.5}$$

or what is the same, the expectation of $u(t)$ is equal to zero. In fact we have

$$F_n(t,\lambda) = \delta_1 + \cdots + \delta_\tau + \tau\alpha, \tag{2.6}$$

where τ is a random variable taking on values $0, 1, \ldots, n$ with probabilities[1]

$$\mathsf{P}\ \{\tau = k\} = \binom{n}{k} t^k (1-t)^{n-k} \tag{2.7}$$

and τ is independent of $\delta_1, \ldots, \delta_n$. The random variable

$$\frac{1}{1-n\alpha}(\delta_1 + \cdots + \delta_k) \tag{2.8}$$

has the probability density (2.1) (with $j = k$) if $n\ \alpha < 1$. Thus

$$\begin{aligned}\mathsf{E}\,\{F_n(t,\lambda)\} &= \mathsf{E}\,\{\delta_1 + \cdots + \delta_\tau + \tau\alpha\} \\ &= \sum_{k=1}^{n} \mathsf{E}\,\{\delta_1 + \cdots + \delta_\tau + \tau\alpha \mid \tau = k\}\mathsf{P}\,\{\tau = k\} \\ &= \sum_{k=1}^{n} \mathsf{E}\,\{\delta_1 + \cdots + \delta_k + k\alpha\}\binom{n}{k} t^k (1-t)^{n-k} \\ &= \sum_{k=1}^{n} \left(\frac{k}{n}(1-n\alpha) + k\alpha\right)\binom{n}{k} t^k (1-t)^{n-k} = t\end{aligned} \tag{2.9}$$

which applies to the case $n\ \alpha < 1$ and also to the case $n\ \alpha = 1$.

Next we calculate the *covariance* of the random variables $u(s)$, $u(t)$ where

$$0 \le s \le t \le 1.$$

We use (2.6) and the following equality

$$F_n(s,\lambda) = \delta_1 + \cdots + \delta_\sigma + \sigma\alpha, \tag{2.10}$$

where σ has the probability distribution

$$\mathsf{P}\ \{\sigma = j\} = \binom{n}{j} s^j (1-s)^{n-j}, \tag{2.11}$$

[1] If $\tau = 0$ then $F_n(t,\lambda) = 0$ by definition.

further σ and τ are independent of $\delta_1, \dots, \delta_n$. Thus

$$\begin{aligned}
&\mathsf{E}\{F_n(s,\lambda)F_n(t,\lambda)\} \\
&\quad = \mathsf{E}\{(\delta_1 + \cdots + \delta_\sigma + \sigma\alpha)(\delta_1 + \cdots + \delta_\tau + \tau\alpha)\} \\
&\quad = \sum_{1\le j\le k\le n} \mathsf{E}\{(\delta_1 + \cdots + \delta_j + j\alpha)(\delta_1 + \cdots + \delta_k + k\alpha)\} \\
&\qquad \times \mathsf{P}\{\sigma = j,\ \tau = k\} \\
&\quad = \sum_{1\le j\le k\le n} \mathsf{E}\left\{\left(\delta_1 + \cdots + \delta_j - (1-n\alpha)\frac{j}{n}\right)\left(\delta_1 + \cdots + \delta_k - (1-n\alpha)\frac{k}{n}\right)\right\} \\
&\qquad \times \mathsf{P}\{\sigma = j,\ \tau = k\} + \sum_{1\le j\le k\le n} \frac{jk}{n^2}\mathsf{P}\{\sigma = j,\ \tau = k\}.
\end{aligned} \tag{2.12}$$

We remark that in (2.12)

$$\mathsf{P}\{\sigma = j,\ \tau = k\} = \frac{n!}{j!(k-j)!(n-j)!} s^j (t-s)^{k-j}(1-t)^{n-k}. \tag{2.13}$$

The second to the last row in (2.12) contains a covariance. By (2.4) this is equal to the following

$$\begin{aligned}
&\mathsf{E}\left\{\left(\delta_1 + \cdots + \delta_j - (1-n\alpha)\frac{j}{n}\right)\left(\delta_1 + \cdots + \delta_k - (1-n\alpha)\frac{k}{n}\right)\right\} \\
&\quad = (1-n\alpha)^2 \frac{1}{n+1}\frac{j}{n}\left(1 - \frac{k}{n}\right)
\end{aligned} \tag{2.14}$$

hence (2.12), (2.13) and (2.14) imply

$$\begin{aligned}
&\mathsf{E}\{F_n(s,\lambda)F_n(t,\lambda)\} \\
&\quad = \sum_{1\le j\le k\le n} \left[(1-n\alpha)^2 \frac{1}{n+1}\frac{j}{n}\left(1 - \frac{k}{n}\right) + \frac{jk}{n^2}\right] \mathsf{P}\{\sigma = j,\ \tau = k\}.
\end{aligned} \tag{2.15}$$

Incidentally we mention the following equalities

$$\sum_{k_1+k_2+k_3=n} k_1 k_2 \frac{n!}{k_1!k_2!k_3!} p_1^{k_1} p_2^{k_2} p_3^{k_3} = n(n-1)p_1 p_2, \tag{2.16}$$

$$\sum_{k_1+k_2+k_3=n} k_1^2 \frac{n!}{k_1!k_2!k_3!} p_1^{k_1} p_2^{k_2} p_3^{k_3} = np_1(1-p_1) + np_1^2. \tag{2.17}$$

These can be regarded well-known because the covariances and the variances of random variables having polynomial joint distribution appear in them provided $p_1 \ge 0$, $p_2 \ge 0$, $p_3 \ge 0$, $p_1 + p_2 + p_3 = 1$.

If we apply (2.16) for $p_1 = s$, $p_2 = 1-t$, then we obtain

$$\sum_{1\le j\le k\le n} j(n-k) \frac{n!}{j!(k-j)!(n-k)!} s^j (t-s)^{k-j}(1-t)^{n-k} = n(n-1)s(1-t). \tag{2.18}$$

We further apply (2.17) for $p_1 = s$, $p_2 = t - s$. Then we receive

$$\begin{aligned}
&\sum_{1\le j\le k\le n} jk\frac{n!}{j!(k-j)!(n-k)!}s^j(t-s)^{k-j}(1-t)^{n-k} \\
&\quad = \sum_{1\le j\le k\le n} j(k-j)\frac{n!}{j!(k-j)!(n-k)!}s^j(t-s)^{k-j}(1-t)^{n-k} \\
&\qquad + \sum_{1\le j\le k\le n} j^2\frac{n!}{j!(k-j)!(n-k)!}s^j(t-s)^{k-j}(1-t)^{n-k} \\
&\quad = n(n-1)s(t-s) + ns(1-s) + n^2s^2. \qquad (2.19)
\end{aligned}$$

From (2.15), (2.18) and (2.19) we conclude

$$\begin{aligned}
\mathsf{E}\left\{F_n(s,\lambda)F_n(t,\lambda)\right\} &= (1-n\alpha)^2\frac{1}{(n+1)n^2}n(n-1)s(1-t) \\
&\quad + \frac{1}{n^2}[n(n-1)s(t-s) + ns(1-s) + n^2s^2] \qquad (2.20) \\
&= \frac{1}{n}\left[1 + \frac{n-1}{n+1}(1-n\alpha)^2\right]s(1-t) + st.
\end{aligned}$$

Taking into account (2.5), we have proved the following

Theorem 1. *The covariance of the random variables $F_n(s,\lambda)$ and $F_n(t,\lambda)$ is equal to*

$$\frac{1}{n}\left[1 + \frac{n-1}{n+1}(1-\lambda)^2\right]s(1-t), \qquad (2.21)$$

where $0 \le s \le t \le 1$.

The same formula holds for the stochastic process $G_m(t,\mu)$, we only have to replace n and λ by m and μ in (2.21). Since the stochastic processes $G_m(t,\mu)$ and $F_n(t,\lambda)$ are totally independent, we have also proved

Theorem 2. *The covariance of the random variables $G_m(s,\mu)-F_n(s,\lambda)$ and $G_m(t,\mu)-F_n(t,\lambda)$ is equal to*

$$\left\{\frac{1}{m}\left[1 + \frac{m-1}{m+1}(1-\mu)^2\right] + \frac{1}{n}\left[1 + \frac{n-1}{n+1}(1-\lambda)^2\right]\right\}s(1-t), \qquad (2.22)$$

where $0 \le s \le t \le 1$.

3 The finite dimensional distributions of the stochastic processes $F_n(t,\lambda) - t = u(t)$, $G_m(t,\mu) - F_n(t,\lambda) = v(t)$

The purpose of this section is to prove the following

THEOREM 3. *For every positive integer r, every $t_1, \ldots, t_r$ satisfying*

$$0 < t_1 < \cdots < t_r < 1$$

and every real $x_1, \ldots, x_r$ the following limit relations hold

$$\begin{aligned}
&\lim_{n\to\infty} \mathsf{P}\left\{\sqrt{\frac{n}{1+(1-\lambda)^2}}\frac{u(t_i)}{\sqrt{t_i(1-t_i)}} < x_i,\ i=1,\ldots,r\right\} \\
&\qquad = \Phi(x_1,\ldots,x_r;R), \qquad (3.1)\\
&\lim_{\substack{m\to\infty\\ n\to\infty}} \mathsf{P}\left\{\sqrt{\frac{mn}{m+n+m(1-\lambda)^2+n(1-\mu)^2}}\frac{v(t_i)}{\sqrt{t_i(1-t_i)}} < x_i,\ i=1,\ldots,r\right\} \\
&\qquad = \Phi(x_1,\ldots,x_r;R), \qquad (3.2)
\end{aligned}$$

where $\lambda = \lambda_n$, $\mu = \mu_n$ are (not necessarily convergent) sequences of numbers defined in Section 1, $\Phi(x_1,\ldots,x_r;R)$ denotes the multivariate normal probability distribution function with standardized marginal distributions and correlation matrix R, which in this special case has the following elements

$$r_{ij} = \frac{t_i(1-t_j)}{\sqrt{t_i(1-t_i)}\sqrt{t_j(1-t_j)}} = \sqrt{\frac{t_i}{t_j}\frac{1-t_j}{1-t_i}}, \quad 1 \le i \le j \le r. \qquad (3.3)$$

Proof. For the sake of simplicity the proof will be restricted to the case $r = 2$. This covers the general case with a trivial modification. First we prove the relation (3.1). The two time points in the interval $[0,1]$ will be denoted by s and t instead of t_1, t_2.

The random variables $\sqrt{n}\, u(s)$, $\sqrt{n}\, u(t)$ can be written in the following form

$$\begin{aligned}
\sqrt{n}\, u(s) &= \sqrt{n}\,(F_n(s,\lambda) - s) = \sqrt{n}\,(\delta_1 + \cdots + \delta_\sigma + \sigma\alpha - s) \\
&= (1-\lambda)\sqrt{n}\left(\frac{\delta_1+\cdots+\delta_\sigma}{1-\lambda} - \frac{\sigma}{n}\right) + \sqrt{n}\left(\frac{\sigma}{n} - s\right), \\
\sqrt{n}\, u(t) &= \sqrt{n}\,(F_n(t,\lambda) - s) = \sqrt{n}\,(\delta_1 + \cdots + \delta_\tau + \tau\alpha - t) \\
&= (1-\lambda)\sqrt{n}\left(\frac{\delta_1+\cdots+\delta_\tau}{1-\lambda} - \frac{\tau}{n}\right) + \sqrt{n}\left(\frac{\tau}{n} - t\right).
\end{aligned} \qquad (3.4)$$

It is well-known that the random variables

$$\sqrt{n}\left(\frac{\sigma}{n} - s\right), \quad \sqrt{n}\left(\frac{\tau}{n} - t\right) \qquad (3.5)$$

have a joint normal limit distribution with covariance matrix

$$\begin{pmatrix} s(1-s) & s(1-t) \\ s(1-t) & t(1-t) \end{pmatrix}. \qquad (3.6)$$

Let us consider the random variables

$$\sqrt{n}\left(\frac{\delta_1+\cdots+\delta_\sigma}{1-\lambda} - \frac{\sigma}{n}\right), \quad \sqrt{n}\left(\frac{\delta_1+\cdots+\delta_\tau}{1-\lambda} - \frac{\tau}{n}\right). \qquad (3.7)$$

We show that if we substitute j, k in the place of σ and τ, where $j = j_n$, $k = k_n$ are non-random sequences with

$$\lim_{n\to\infty} \frac{j}{n} = s, \quad \lim_{n\to\infty} \frac{k}{n} = t, \tag{3.8}$$

then the random variables have a joint normal limit distribution with covariance matrix (3.6). Since $(1-\lambda)^{-1}(\delta_1 + \cdots + \delta_j)$ and $(1-\lambda)^{-1}(\delta_1 + \cdots + \delta_k)$ can be considered as the jth and kth elements of an ordered sample of size $n-1$ taken from a population uniformly distributed in the interval $[0,1]$, it follows that their $\sqrt{n}$ multiples have the same joint distribution as the random variables

$$\sqrt{n}\,\frac{\zeta_1 + \cdots + \zeta_j}{\zeta_1 + \cdots + \zeta_n}, \quad \sqrt{n}\,\frac{\zeta_1 + \cdots + \zeta_k}{\zeta_1 + \cdots + \zeta_n}, \tag{3.9}$$

where $\zeta_1, \zeta_2, \ldots$ is a sequence of independent, exponentially distributed random variables with $\mathsf{E}\{\zeta_i\} = 1$, $i = 1, 2, \ldots$.

The random variables (3.9) have expectations $j/\sqrt{n}$ and $k/\sqrt{n}$, respectively. Taking this into account, we write

$$\begin{aligned}
&\sqrt{n}\left(\frac{\zeta_1 + \cdots + \zeta_j}{\zeta_1 + \cdots + \zeta_n} - \frac{j}{n}\right) \\
&\quad = \frac{1}{\dfrac{\zeta_1 + \ldots + \zeta_n}{n}}\,\frac{j}{n}\left(\sqrt{\frac{n}{j}}\,\frac{\zeta_1 - 1 + \ldots + \zeta_j - 1}{\sqrt{j}} - \frac{\zeta_1 - 1 + \ldots + \zeta_n - 1}{\sqrt{n}}\right), \\
&\sqrt{n}\left(\frac{\zeta_1 + \cdots + \zeta_k}{\zeta_1 + \cdots + \zeta_n} - \frac{k}{n}\right) \\
&\quad = \frac{1}{\dfrac{\zeta_1 + \ldots + \zeta_n}{n}}\,\frac{k}{n}\left(\sqrt{\frac{n}{k}}\,\frac{\zeta_1 - 1 + \ldots + \zeta_k - 1}{\sqrt{k}} - \frac{\zeta_1 - 1 + \ldots + \zeta_n - 1}{\sqrt{n}}\right).
\end{aligned} \tag{3.10}$$

Consider the random variables

$$\frac{\zeta_1 - 1 + \cdots + \zeta_j - 1}{\sqrt{j}}, \quad \frac{\zeta_1 - 1 + \cdots + \zeta_k - 1}{\sqrt{k}}, \quad \frac{\zeta_1 - 1 + \cdots + \zeta_n - 1}{\sqrt{n}}. \tag{3.11}$$

An elementary argument shows that they have a joint normal limit distribution. Thus the random variables (3.10) also have a joint normal limit distribution. By (2.4) the covariance matrix of the random variables (3.9) equals

$$\frac{n}{n+1}\begin{pmatrix} \dfrac{j}{n}\left(1 - \dfrac{j}{n}\right) & \dfrac{j}{n}\left(1 - \dfrac{k}{n}\right) \\ \dfrac{j}{n}\left(1 - \dfrac{k}{n}\right) & \dfrac{k}{n}\left(1 - \dfrac{k}{n}\right) \end{pmatrix}, \tag{3.12}$$

hence the assertion concerning the random variables (3.7) is proved.

The next step in the proof of Theorem 3 will be the proof of the fact that the following two random vectors are independent in the limit:

$$\left(\sqrt{n}\left(\frac{\delta_1+\cdots+\delta_\sigma}{1-\lambda}-\frac{\sigma}{n}\right),\ \sqrt{n}\left(\frac{\delta_1+\cdots+\delta_\tau}{1-\lambda}-\frac{\tau}{n}\right)\right),$$
$$\left(\sqrt{n}\left(\frac{\sigma}{n}-s\right),\ \sqrt{n}\left(\frac{\tau}{n}-t\right)\right). \tag{3.13}$$

Let $[a,b]$, $[c,d]$ be two fixed intervals with $a<b$, $c<d$ and x, y two fixed real numbers. Then we have

$$\mathsf{P}\left\{\sqrt{n}\left(\frac{\delta_1+\cdots+\delta_\sigma}{1-\lambda}-\frac{\sigma}{n}\right)<x,\ \sqrt{n}\left(\frac{\delta_1+\cdots+\delta_\tau}{1-\lambda}-\frac{\tau}{n}\right)<y,\right.$$
$$\left.a\le\sqrt{n}\left(\frac{\sigma}{n}-s\right)\le b,\ c\le\sqrt{n}\left(\frac{\tau}{n}-t\right)\le d\right\}$$

$$=\sum_{\substack{ns+a\sqrt{n}\le j\le ns+b\sqrt{n}\\ nt+c\sqrt{n}\le k\le nt+d\sqrt{n}}}\mathsf{P}\left\{\sqrt{n}\left(\frac{\delta_1+\cdots+\delta_j}{1-\lambda}-\frac{j}{n}\right)<x,\right.$$
$$\left.\sqrt{n}\left(\frac{\delta_1+\cdots+\delta_k}{1-\lambda}-\frac{k}{n}\right)<y\right\}\mathsf{P}\{\sigma=j,\ \tau=k\}$$

$$=\sum_{\substack{ns+a\sqrt{n}\le j\le ns+b\sqrt{n}\\ nt+c\sqrt{n}\le k\le nt+d\sqrt{n}}}\left[\mathsf{P}\left\{\sqrt{n}\left(\frac{\delta_1+\cdots+\delta_j}{1-\lambda}-\frac{j}{n}\right)<x,\right.\right. \tag{3.14}$$
$$\left.\sqrt{n}\left(\frac{\delta_1+\cdots+\delta_k}{1-\lambda}-\frac{k}{n}\right)<y\right\}$$
$$\left.-\Phi\left(\frac{x}{\sqrt{s(1-s)}},\frac{y}{\sqrt{t(1-t)}};\sqrt{\frac{s(1-t)}{t(1-s)}}\right)\right]\mathsf{P}\{\sigma=j,\ \tau=k\}$$
$$+\Phi\left(\frac{x}{\sqrt{s(1-s)}},\frac{y}{\sqrt{t(1-t)}};\sqrt{\frac{s(1-t)}{t(1-s)}}\right)$$
$$\times\mathsf{P}\left\{a\le\sqrt{n}\left(\frac{\sigma}{n}-s\right)\le b,\ c\le\sqrt{n}\left(\frac{\tau}{n}-t\right)\le d\right\}.$$

If $n\to\infty$, then for any $j=j_n$ and $k=k_n$ satisfying

$$ns+a\sqrt{n}\le j\le ns+b\sqrt{n},\quad nt+c\sqrt{n}\le k\le nt+d\sqrt{n},$$

respectively, the limit relations (3.8) hold. Consequently

$$\lim_{n\to\infty}\sum_{\substack{ns+a\sqrt{n}\le j\le ns+b\sqrt{n}\\ nt+c\sqrt{n}\le k\le nt+d\sqrt{n}}}\left[\mathsf{P}\left\{\sqrt{n}\left(\frac{\delta_1+\cdots+\delta_j}{1-\lambda}-\frac{j}{n}\right)<x,\right.\right.$$

$$\sqrt{n}\left(\frac{\delta_1+\cdots+\delta_k}{1-\lambda}-\frac{k}{n}\right)<y\Bigg\} \tag{3.15}$$

$$-\Phi\left(\frac{x}{s(1-s)},\frac{y}{t(1-t)};\sqrt{\frac{s(1-t)}{t(1-s)}}\right)\Bigg]\ \mathsf{P}\left\{\sigma=j,\ \tau=k\right\}=0.$$

This proves that the random vectors (3.13) are independent in the limit.

Since the random vectors (3.13) have normal limit distributions and are independent in the limit, it follows that the limit joint distribution of the random variables

$$\begin{aligned}&\sqrt{n}\left(\frac{\delta_1+\cdots+\delta_\sigma}{1-\lambda}-\frac{\sigma}{n}\right)+\sqrt{n}\left(\frac{\sigma}{n}-s\right),\\&\sqrt{n}\left(\frac{\delta_1+\cdots+\delta_\tau}{1-\lambda}-\frac{\tau}{n}\right)+\sqrt{n}\left(\frac{\tau}{n}-t\right)\end{aligned} \tag{3.16}$$

is a normal distribution.

If λ_n is convergent, then similarly as the random variables (3.16), the random variables (3.4) also have a normal limit joint distribution. In the general case we divide both random variables in (3.4) by $\sqrt{1+(1-\lambda)^2}$. The limit joint distribution of

$$\sqrt{\frac{n}{1+(1-\lambda)^2}}\frac{u(s)}{\sqrt{s(1-s)}},\quad \sqrt{\frac{n}{1+(1-\lambda)^2}}\frac{u(t)}{\sqrt{t(1-t)}} \tag{3.17}$$

is a normal distribution with standardized marginal distributions and with correlation coefficient

$$\sqrt{\frac{s(1-t)}{t(1-s)}} \tag{3.18}$$

for any convergent λ_n. Now if λ_n is an arbitrary sequence then consider the random variables (3.17) and suppose that for some subsequence λ_{n_i} we have

$$\mathsf{P}\left\{\sqrt{\frac{n_i}{1+(1-\lambda_{n_i})^2}}\frac{u(s)}{\sqrt{s(1-s)}}<x,\ \sqrt{\frac{n_i}{1+(1-\lambda_{n_i})^2}}\frac{u(t)}{\sqrt{t(1-t)}}<y\right\}\to A, \tag{3.19}$$

where

$$A\neq\Phi\left(x,y;\ \sqrt{\frac{s(1-t)}{t(1-s)}}\right). \tag{3.20}$$

This is, however, a contradiction because λ_{n_i} is a bounded sequence hence it always contains a convergent subsequence. Thus Theorem 3 is proved. □

4 Generalizations of the theorems of Smirnov

This section is based on the theory of weak convergence of probability measures in the space $D\,[0,1]$ of functions defined in the interval $[0,1]$, having right and left hand limits at every point and satisfying the condition that each function is either right or left continuous at every point. This theory is developed in [3], [5], [8] and is reproduced in [6]. First we prove the following

THEOREM 4. *The probability distribution of the stochastic process*

$$\sqrt{\frac{n}{1+(1-\lambda)^2}}u(t) \tag{4.1}$$

converges weakly to the probability distribution of the GAUSS*ian process $U(t)$ defined in $[0,1]$ having continuous sample functions with probability 1 and satisfying the property*

$$\begin{aligned} &\mathsf{E}\{U(t)\}=0, \qquad 0\le t\le 1,\\ &\mathsf{E}\{U(s)U(t)\}=s(1-t), \qquad 0\le s\le t\le 1. \end{aligned} \tag{4.2}$$

Proof. In view of Theorem 3, we only have to prove the conditional compactness of the distributions of the stochastic processes (4.1). In connection with a sequence of stochastic processes $\zeta_t^{(n)}$ a sufficient condition for the conditional compactness is expressed by the following relation

$$\lim_{\gamma\to 0}\overline{\lim_{n\to\infty}}\,\mathsf{P}\left\{\sup_{|s-t|\le\gamma}\left|\zeta_s^{(n)}-\zeta_t^{(n)}\right|>\varepsilon\right\}=0, \tag{4.3}$$

where $\gamma>0$, $\varepsilon>0$. It is easy to see that if (4.3) holds for two stochastic processes then it also holds for their sum. Thus in order to check (4.3) in connection with the stochastic process (4.1), we can use the representations (3.4) and since $0\le\lambda\le 1$, we only have to prove the fulfilment of (4.3) in connection with the following stochastic processes

$$\sqrt{n}\left(\frac{\delta_1+\cdots+\delta_\tau}{1-\lambda}-\frac{\tau}{n}\right), \tag{4.4}$$

$$\sqrt{n}\left(\frac{\tau}{n}-t\right). \tag{4.5}$$

The fulfilment of (4.3) in case of the stochastic process (4.5) is a well-known classical result. In connection with (5.4) the proof presented in [5], [6] can be applied. In fact it is obvious from this proof that (4.3) is fulfilled if for every small but fixed h values the following inequality holds

$$\mathsf{E}\left\{\left(\zeta_{t+h}^{(n)}-\zeta_t^{(n)}\right)^4\right\}\le K\left(\frac{h}{n}+h^2\right) \tag{4.6}$$

provided n is large enough. Now if we consider the increment of the stochastic process (4.5) relative to an interval of length h, then its distribution equals the distribution of

$$\sqrt{n}\left(\beta_\nu-\frac{\nu}{n}\right), \tag{4.7}$$

where $\beta_0=0$, β_i $(1\le i\le n)$ is a random variable with the density (2.1), ν is independent of the random variables $\beta_1,\dots,\beta_n$ and has the following binomial distribution:

$$\mathsf{P}\{\nu=i\}=\binom{n}{i}h^i(1-h)^{n-i}, \qquad i=0,1,\dots,n. \tag{4.8}$$

It is easy to see that for any positive integer p

$$\mathsf{E}\,\{\beta_i^p\} = \frac{i}{n}\frac{i+1}{n+1}\cdots\frac{i+p-1}{n+p-1}, \tag{4.9}$$

hence the fourth moment of (4.7) equals

$$\begin{aligned} &n^2\,\mathsf{E}\left\{\left(\beta_\nu - \frac{\nu}{n}\right)^4\right\} \\ &\quad = n^2\,\mathsf{E}\left\{\beta_\nu^4 - 4\beta_\nu^3\frac{\nu}{n} + 6\beta_\nu^2\left(\frac{\nu}{n}\right)^2 - 4\beta_\nu\left(\frac{\nu}{n}\right)^3 + \left(\frac{\nu}{n}\right)^4\right\} \\ &\quad = n^2\sum_{k=0}^{n}\left[\frac{k}{n}\frac{k+1}{n+1}\frac{k+2}{n+2}\frac{k+3}{n+3} - 4\left(\frac{k}{n}\right)^2\frac{k+1}{n+1}\frac{k+2}{n+2}\right. \\ &\qquad \left. +6\left(\frac{k}{n}\right)^3\frac{k+1}{n+1} - 3\left(\frac{k}{n}\right)^4\right]\binom{n}{k}h^k(1-h)^{n-k}. \end{aligned} \tag{4.10}$$

After a long but elementary calculation we obtain the following result

$$\begin{aligned} &n^2\,\mathsf{E}\left\{\left(\beta_\nu - \frac{\nu}{n}\right)^4\right\} \\ &\quad = \frac{3h(1-h)}{n(n+1)(n+2)(n+3)}\{h(1-h)[n^4 - 12n^3 + 47n^2 - 72n + 36] \\ &\qquad\qquad +3n^3 - 10n^2 + 13n - 6\}. \end{aligned} \tag{4.11}$$

We see from here that the inequality (4.6) is satisfied also in our case i.e. the right hand side of (4.11) can be majorated by the right hand side of (4.6) with a universal constant K. Thus Theorem 4 is proved. □

THEOREM 5. *The probability distribution of the stochastic process*

$$\sqrt{\frac{mn}{m[1+(1-\lambda)^2] + n[1+(1-\mu)^2]}}\,v(t) \tag{4.12}$$

converges weakly to the probability distribution of the GAUSS*ian process mentioned in Theorem 4.*

Proof. We already know by Theorem 4 that

$$\sqrt{m}\,(G_m(t,\mu) - t) \text{ and } \sqrt{n}\,(F_n(t,\lambda) - t)$$

satisfy separately (4.3). Since (4.12) can be rewritten as

$$\begin{aligned} &\sqrt{\frac{n}{m[1+(1-\lambda)^2] + n[1+(1-\mu)^2]}}\sqrt{m}\,(G_m(t,\mu) - t) \\ &\quad -\sqrt{\frac{m}{m[1+(1-\lambda)^2] + n[1+(1-\mu)^2]}}\sqrt{n}\,(F_n(t,\lambda) - t) \end{aligned} \tag{4.13}$$

and the factors of $\sqrt{m}(G_m(t,\mu)-t)$, $\sqrt{n}(F_n(t,\lambda)-t)$ are smaller than 1, it follows that the process (4.12) also satisfies the relation (4.3). Thus Theorem 5 is proved. □

Now we formulate the generalizations of the theorems of Smirnov in the following theorems.

Theorem 6. *The following relation holds*

$$\lim_{n\to\infty} \mathsf{P}\left\{\sqrt{\frac{n}{1+(1-\lambda)^2}}\sup_{0\le t\le 1}(F_n(t,\lambda)-t)<y\right\} = \begin{cases} 1-e^{-2y^2}, & \text{if } \ y>0,\\ 0, & \text{if } \ y\le 0.\end{cases} \tag{4.14}$$

Theorem 7. *The following relation holds*

$$\begin{aligned}&\lim_{\substack{m\to\infty\\ n\to\infty}} \mathsf{P}\left\{\sqrt{\frac{mn}{m[1+(1-\lambda)^2]+n[1+(1-\mu)^2]}}\sup_{0\le t\le 1}(G_m(t,\mu)-F_n(t,\lambda))<y\right\}\\ &=\begin{cases} 1-e^{-2y^2}, & \text{if } \ y>0,\\ 0, & \text{if } \ y\le 0.\end{cases}\end{aligned} \tag{4.15}$$

Proof. Let $U(t)$ be the Gaussian process mentioned in Theorem 4. Since the supremum functional is continuous in $D\,[0,1]$, it follows that

$$\begin{aligned}&\lim_{n\to\infty} \mathsf{P}\left\{\sqrt{\frac{n}{1+(1-\lambda)^2}}\sup_{0\le t\le 1}(F_n(t,\lambda)-t)<y\right\}\\ &=\mathsf{P}\left\{\max_{0\le t\le 1}U(t)<y\right\},\end{aligned} \tag{4.16}$$

$$\begin{aligned}&\lim_{\substack{m\to\infty\\ n\to\infty}} \mathsf{P}\left\{\sqrt{\frac{mn}{m[1+(1-\lambda)^2]+n[1+(1-\mu)^2]}}\sup_{0\le t\le 1}(G_m(t,\mu)-F_n(t,\lambda))<y\right\}\\ &=\mathsf{P}\left\{\max_{0\le t\le 1}U(t)<y\right\}.\end{aligned} \tag{4.17}$$

On the other hand we know that ([4])

$$\mathsf{P}\left\{\max_{0\le t\le 1}U(t)<y\right\} = \begin{cases} 1-e^{-2y^2}, & \text{if } \ y>0,\\ 0, & \text{if } \ y\le 0.\end{cases} \tag{4.18}$$

Thus Theorems 6 and 7 are proved. □

5 Application to the solution of the reliability equation

The application of Theorems 6 and 7 was already given in [7]. For the sake of the completeness of the present paper we repeat it briefly. We are interested in the solutions relative to M of the reliability equations (1.4) and (1.5). Equation (1.4) can be rewritten as

$$\mathsf{P}\left\{\sqrt{\frac{n}{1+(1-\lambda)^2}}\sup_{0\le t\le 1}(t-F_n(t,\lambda))<M\sqrt{\frac{n}{1+(1-\lambda)^2}}\right\}=1-\varepsilon. \tag{5.1}$$

Substituting $y = M(n/[1+(1-\lambda)^2])^{\frac{1}{2}}$ on the right hand side in the parantheses in (5.1) and taking into acooount the limit relation (4.14), we obtain

$$1 - e^{-2y^2} \approx 1 - \varepsilon \tag{5.2}$$

from where we derive $y \approx \left(\frac{1}{2}\log\frac{1}{\varepsilon}\right)^{\frac{1}{2}}$, hence

$$M(\lambda) \approx \sqrt{1+(1-\lambda)^2}\sqrt{\frac{1}{2n}\log\frac{1}{\varepsilon}}. \tag{5.3}$$

Similarly, if $M = M(\lambda, \mu)$ is the solution of the reliability equation (1.5), we obtain

$$M(\lambda, \mu) \approx \sqrt{\frac{1+(1-\lambda)^2}{n} + \frac{1+(1-\mu)^2}{m}}\sqrt{\frac{1}{2}\log\frac{1}{\varepsilon}}. \tag{5.4}$$

References

[1] Bernstejn, Sz. H. (1946). *Teorija verojatnosztej.* Gosztehizdat, Moszkva.

[2] Birnbaum, Z. W. and F.H. Tingey (1951). One sided confidence contours for probability distribution functions. *Ann. Math. Statist.* **22**, 592–596.

[3] Donsker, M. D. (1952). Justification and extension of Doob's heuristic approach to the Kolmogorov–Smirnov theorems. *Ann. Math. Statist.* **23**, 277–281.

[4] Doob, J. L. (1949). Heuristic approach to the Kolmogorov–Smirnov theorems. *Ann. Math. Statist.* **20**, 393–403.

[5] Gihman, I. I. – Szkorohod, A. V. (1965). *VVedenie v teoriju szlucsajnüh processzov.* Izdatel'sztvo Nauka, Moszkva.

[6] Parthasarathy, K. R. (1967). *Probability measures on metric spaces.* Academic Press, New York, London.

[7] Prékopa, A. (1963). Reliability equation for an inventory problem and its asymptotic solutions. *Colloquium on Appl. of Math. to Economics.* Budapest, 1963, *Publ. House, Hungarian Acad. Sci.* 1965, 317–327.

[8] Prohorov, Ju. V. (1956). Szhodimoszt' szlucsajnüh processzov i predel'nüe teoremü teorii verojatnosztej. *Teorija Verojatn. Primen.* **1**, 177–238.

[9] Smirnov, N. V. (1949). Priblizsenie zakonov raszpredelenija szlucsajnüh velicsin po empiricseszkim dannüm. *Uszpehi Mat. Nauk* **10**, 179–206.

[10] Ziermann, M. (1964). Anwendung des Smirnovschen Satzes auf ein Lagerhaltungsproblem. *Publ. Math. Inst. Hungar. Acad. Sci. Ser. B* **8**, 509–518, (In Hungarian).