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ON THE NUMBER OF VERTICES OF RANDOM CONVEX POLYHEDRA[1]

by

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To the memory of A. Rényi

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1 Introduction

A convex polyhedron is defined as the intersection of a finite number of closed half spaces. If the boundary hyperplanes contain random variables in the expression of their analytic definition then we have a random convex polyhedron. Thus we fix the number of half spaces but allow them to be random. The number of vertices ν of a random convex polyhedron is a random variable which we define to be equal to zero if the intersection of half spaces is empty. We are interested in the probabilistic behaviour of ν in particular to find the expectation $\mathsf{E}(\nu)$ under various assumptions.

The problem arose in connection with linear programming therefore we formulate the problem so that the results should have immediate application or interpretation to this field. Thus we consider the following equality, inequality systems

$$\begin{aligned} a_{11}x_1 + \cdots + a_{1n}x_n &\le b_1, \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n &\le b_m, \\ x_j \ge 0, \quad j \in J, \end{aligned} \tag{1.1}$$

[1]The results of Sections 3–5 were presented by the author at the 5[th] International Symposium on Mathematical Programming, London, 1964.

where J is a set of subscripts $J \subset \{1, \dots, n\}$,

$$\begin{aligned} a_{11}x_1 + \cdots + a_{1n}x_n \ &= b_1, \\ \dots\dots\dots\dots\dots\dots\dots \\ a_{m1}x_1 + \cdots + a_{mn}x_n &= b_m, \\ x_j \geq 0, \quad j \in J. \end{aligned} \tag{1.2}$$

In both (1.1) and (1.2) the positive integers m, n are fixed while some or all of the numbers a_{ik}, b_i are random.

If we are only interested in the number of vertices then instead of (1.1) we may consider the following equality-inequality system

$$\begin{aligned} a_{11}x_1 + \cdots + a_{1n}x_n \ + y_1 &= b_1, \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ a_{m1}x_1 + \cdots + a_{mn}x_n + y_m &= b_m, \\ x_j \geq 0, \quad k \in J, \quad y_1 \geq 0, \dots, y_m \geq 0. \end{aligned} \tag{1.3}$$

In fact keeping fixed for a moment the entries a_{ik}, b_i, it can be shown that the set defined by (1.1) is nonempty if and only if the same holds for the set (1.3) and the convex polyhedra (1.1), (1.3) have the same number of vertices. We remark that (1.1) is a subset of R^n while (1.3) is a subset of R^{n+m}.

2 Necessary and sufficient condition that a vector x is a vertex of the convex polyhedron (1.2)

The equality system (1.2) can be written in the matrix form $A\mathbf{x} = \mathbf{b}$ and also in the form

$$\mathbf{a}_1 x_1 + \cdots + \mathbf{a}_n x_n = \mathbf{b}, \tag{2.1}$$

where $\mathbf{a}_1, \dots, \mathbf{a}_n$ are the columns of the matrix A. A system of linearly independent vectors $\mathbf{a}_{i_1}, \dots, \mathbf{a}_{i_r}$ will be called a basis of the vectors $\mathbf{a}_1, \dots, \mathbf{a}_n$ if every $\mathbf{a}_k$ $(k = 1, ..., n)$ can be expressed as a linear combination of vectors belonging to the system. Now we prove the following

THEOREM 1 [2] *Suppose that the set K determined by the constraints* (1.2) *is not empty. A vector $\mathbf{x}$ is a vertex of the convex polyhedron K if and only if there exists a basis $\mathbf{a}_{i_1}, \dots, \mathbf{a}_{i_r}$ of the set of vectors $\mathbf{a}_1, \dots, \mathbf{a}_n$ such that*

$$x_j = 0 \quad \textit{for} \quad j \notin \{i_1, \dots, i_r\} = I \tag{2.2}$$

and

$$\{1, \dots, n\} - J \subset I. \tag{2.3}$$

[2] This theorem was published first in [1].

PROOF. Let $\mathbf{x}$ be an element of K having the mentioned property. We prove that $\mathbf{x}$ is a vertex. Indirect proof is used. Thus we assume that there exist vectors $\mathbf{y}, \mathbf{z} \in K$, and a λ such that $\mathbf{y} \neq \mathbf{z}$, $0 < \lambda < 1$ and

$$\mathbf{x} = \lambda \mathbf{y} + (1-\lambda)\mathbf{z}. \tag{2.4}$$

Since $x_j = 0$ for $j \notin I$ and by (2.3) $y_j \geq 0$, $z_j \geq 0$ for $j \notin I$, it follows from (2.4) that

$$y_j = z_j = 0 \text{ if } j \notin I. \tag{2.5}$$

This is, however, impossible because (2.5) implies

$$\begin{aligned} \sum_{j \in I} y_j \mathbf{a}_j &= \mathbf{b}, \\ \sum_{j \in I} z_j \mathbf{a}_j &= \mathbf{b}, \end{aligned} \tag{2.6}$$

the vectors $\mathbf{a}_j$, $j \in I$ are linearly independent and $\mathbf{y} = \mathbf{z}$. This proves the first part of the theorem.

To prove the second part let $\mathbf{x} \in K$ be a vertex. First we show that there exists a basis such that (2.2) holds. We may suppose that the non-zero components of $\mathbf{x}$ are those which come first i.e.

$$x_1 \neq 0, \ldots, x_k \neq 0, \quad x_{k+1} = 0, \ldots, x_n = 0. \tag{2.7}$$

It is not necessary that there exist any non-zero or any zero component. The proof allows both cases. Now it is clear that if $\mathbf{x} = \mathbf{0}$ then a basis exists with the required property. If on the other hand $\mathbf{x} \neq \mathbf{0}$, then such a basis exists if and only if

$$\mathbf{a}_1, \ldots, \mathbf{a}_k \tag{2.8}$$

are linearly independent. Using indirect proof we assume that the vectors (2.8) are linearly dependent. This means that there exist numbers $\mu_1, \ldots, \mu_k$ such that

$$\begin{gathered} \mu_1 \mathbf{a}_1 + \cdots + \mu_k \mathbf{a}_k = \mathbf{0}, \\ |\mu_1| + \cdots + |\mu_k| > 0. \end{gathered} \tag{2.9}$$

Let $\varepsilon > 0$ be so small that in

$$\mathbf{y} = \begin{pmatrix} x_1 + \varepsilon\mu_1 \\ \vdots \\ x_k + \varepsilon\mu_k \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \qquad \mathbf{z} = \begin{pmatrix} x_1 - \varepsilon\mu_1 \\ \vdots \\ x_k - \varepsilon\mu_k \\ 0 \\ \vdots \\ 0 \end{pmatrix} \tag{2.10}$$

we have in each case where $x_i > 0$,

$$x_i + \varepsilon\mu_i > 0, \qquad x_i - \varepsilon\mu_i > 0. \tag{2.11}$$

The vectors $\mathbf{y}, \mathbf{z}$ are elements of K, $\mathbf{y} \neq \mathbf{z}$, and

$$\mathbf{x} = \frac{1}{2}\mathbf{y} + \frac{1}{2}\mathbf{z}.$$

This is a contradiction because $\mathbf{x}$ is a vertex of K. Thus there exists a basis satisfying (2.2).

Let us add vectors to the set $\mathbf{a}_1, \dots, \mathbf{a}_k$ to obtain a basis. We may suppose that these additional vectors are $\mathbf{a}_{k+1}, \dots, \mathbf{a}_r$. Now we subdivide the set of subscripts $1, \dots, n$ into three categories in the following manner

$$\begin{array}{ccc} 1. & 2. & 3. \\ 1, \dots, k; & k+1, \dots, r; & r+1, \dots, n. \end{array}$$

We shall consider the logically possible following cases:

α. $\{r+1, \dots, n\} \subset J$,

β. there exists a j, $r+1 \leq j \leq n$ such that $j \notin J$ and $i \notin J$ for $i = k+1, \dots, r$,

γ. there exists a j, $r+1 \leq j \leq n$ such that $j \notin J$ and there exists an i, $k+1 \leq i \leq r$ such that $i \in J$.

In case of α the proof is ready. In case of β we do the following. We choose an $\mathbf{a}_j$ for which $r+1 \leq j \leq n$ and $j \notin J$. The vector $\mathbf{a}_j$ is a linear combination of the basis vectors i.e. $\mathbf{a}_j$ can be written in the form

$$\mathbf{a}_j = \sum_{p=1}^{r} d_p \mathbf{a}_p. \tag{2.12}$$

If $\varepsilon > 0$ is small enough then in the relations

$$\begin{aligned} \mathbf{b} &= \sum_{p=1}^{r} x_p \mathbf{a}_p - \varepsilon \mathbf{a}_j + \varepsilon \mathbf{a}_j = \sum_{p=1}^{r} (x_p - \varepsilon d_p)\mathbf{a}_p + \varepsilon \mathbf{a}_j, \\ \mathbf{b} &= \sum_{p=1}^{r} x_p \mathbf{a}_p + \varepsilon \mathbf{a}_j - \varepsilon \mathbf{a}_j = \sum_{p=1}^{r} (x_p + \varepsilon d_p)\mathbf{a}_p - \varepsilon \mathbf{a}_j, \end{aligned} \tag{2.13}$$

$x_p - \varepsilon d_p > 0$, $x_p + \varepsilon d_p > 0$ whenever $x_p > 0$. This means that the vectors

$$\begin{aligned} &(x_1 - \varepsilon d_1, \dots, x_r - \varepsilon d_r, 0, \dots, 0, \varepsilon, 0, \dots, 0), \\ &(x_1 + \varepsilon d_1, \dots, x_r + \varepsilon d_r, 0, \dots, 0 - \varepsilon, 0, \dots, 0) \end{aligned} \tag{2.14}$$

are elements of K. Denoting these by $\mathbf{y}$ and $\mathbf{z}$, respectively, we have

$$\mathbf{x} = \frac{1}{2}\mathbf{y} + \frac{1}{2}\mathbf{z}, \tag{2.15}$$

contradicting to the supposition that $\mathbf{x}$ is a vertex of K.

In case of γ we start to interchange the vectors $\mathbf{a}_j$ for which $r+1 \le j \le n$ and $j \notin J$ with the vectors $\mathbf{a}_i$ for which $k+1 \le i \le r$ and $i \in J$, maintaining the condition that the first r vectors are linearly independent. If such an interchange is not possible because the first r vectors would be linearly dependent then every $\mathbf{a}_j$, $r+1 \le j \le n$, $j \notin J$ would be an element of the subspace spanned by the vectors $\mathbf{a}_1, \dots \mathbf{a}_k$, $\mathbf{a}_i$, $k+1 \le i \le r$, $i \in J$. Repeating the procedure applied in the case β, we are lead to a contradiction. Thus the interchange is possible and we go ahead step by step performing the interchange of one-one vector each time. The procedure ends with the situation that there are no more vectors among $\mathbf{a}_{r+1}, \dots, \mathbf{a}_n$ to be interchanged with which we are ready. In fact we cannot arrive at a situation where there is at least one among the vectors $\mathbf{a}_j$, $r+1 \le j \le n$ to be interchanged and none among the vectors $\mathbf{a}_i$, $k+1 \le i \le r$ with which the interchange is possible because this is just the case β, which was shown to be impossible. Thus the theorem is proved. □

Corollary If $J = \{1, \dots, n\}$ then the set of vertices is identical to the set of $\mathbf{x}$ vectors to which we can find basises $\mathbf{a}_{i_1}, \dots, \mathbf{a}_{i_r}$ such that

$$x_j = 0 \text{ for } j \notin \{i_1, \dots, i_r\}.$$

An element of K will be called a solution while an element $\mathbf{x}$ of the sort just mentioned in the Corollary will be called a basic solution.

We remark that the convex polyhedron (1.1) has the same number of vertices as the convex polyhedron (1.3). In fact there is a one to one correspondence between the feasible solutions of (1.1) and (1.3), namely if $\mathbf{x}$ is feasible in (1.1) then

$$\mathbf{x} \Leftrightarrow (\mathbf{x}, \mathbf{y}), \quad \mathbf{y} = \mathbf{b} - A\mathbf{x}. \tag{2.16}$$

Now let $\mathbf{x}$ be a vertex in case of (1.1) and suppose that it is not a vertex in (1.3). Then there exist vectors $(\mathbf{x}_1, \mathbf{y}_1) \neq (\mathbf{x}_2, \mathbf{y_2})$, feasible in (1.3) such that with suitable λ, $0 < \lambda < 1$, we have

$$(\mathbf{x}, \mathbf{y}) = \lambda(\mathbf{x}_1, \mathbf{y}_1) + (1-\lambda)(\mathbf{x}_2, \mathbf{y_2}). \tag{2.17}$$

Since

$$\mathbf{y}_1 = \mathbf{b} - A\mathbf{x}_1, \qquad \mathbf{y}_2 = \mathbf{b} - A\mathbf{x}_2,$$

it follows that $\mathbf{x}_1 \neq \mathbf{x}_2$ and according to (2.17)

$$\mathbf{x} = \lambda\mathbf{x}_1 + (1-\lambda)\mathbf{x}_2. \tag{2.18}$$

This contradicts to the supposition that $\mathbf{x}$ is a vertex in case of (1.1). Similar argument shows that if $\mathbf{x}$ is not a vertex in case of (1.1) then the corresponding $(\mathbf{x}, \mathbf{y})$ is also not a vertex in case of (1.3). Hence the assertion.

3 The case of symmetrical distributions

Let us introduce the notations

$$\mathbf{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}, \quad \mathbf{a}_i = \begin{pmatrix} a_{1i} \\ \vdots \\ a_{mi} \end{pmatrix}, \quad i = 1, \dots, n. \tag{3.1}$$

Now we can write (1.1) in the concise form

$$\begin{gathered} \mathbf{a}_1 x_1 + \cdots + \mathbf{a}_n x_n \leq \mathbf{b}, \\ x_j \geq 0, \quad j \in J. \end{gathered} \tag{3.2}$$

Similarly the concise version of (1.3) is the following

$$\begin{gathered} \mathbf{a}_1 x_1 + \cdots + \mathbf{a}_n x_n + \mathbf{e}_1 y_1 + \cdots + \mathbf{e}_n y_m = \mathbf{b}, \\ x_j \geq 0, \quad j \in J, \quad y_1 \geq 0, \dots, y_m \geq 0, \end{gathered} \tag{3.3}$$

where $\mathbf{e}_1, \dots, \mathbf{e}_m$ are the m-dimensional unit vectors. First we prove

THEOREM 2 *We consider the random convex polyhedron* (3.2) *where we suppose that m, n are fixed, $m \leq n$, the set J has r elements and $r \geq n - m$. Let $\mathbf{a}_1, \dots, \mathbf{a}_n$, $\mathbf{b}$ be stochastically independent random vectors where $\mathbf{a}_1, \dots, \mathbf{a}_n$ are continuously and symmetrically distributed. Under these conditions*

$$\mathsf{E}(\nu) = \binom{r}{r-(n-m)} \frac{1}{2^{r-(n-m)}}, \tag{3.4}$$

where ν is the number of vertices of the random convex polyhedron (3.2).

PROOF. We may suppose that $x_1, \dots, x_{n-r}$ are those variables which are not constrained by non-negativity condition. Consider the equation

$$\mathbf{a}_1 x_1 + \cdots + \mathbf{a}_{n-r} x_{n-r} + \mathbf{a}_{n-r+1} x_{n-r+1} + \cdots + \mathbf{a}_m x_m = \mathbf{b}, \tag{3.5}$$

where $\mathbf{b}$ is fixed at some of its realizations. Since $\mathbf{a}_1, \dots, \mathbf{a}_m$, are independently and continuously distributed, the joint distribution of the altogether m^2 components is continuous and from this it follows immediately that equation (3.5) has a unique solution with probability 1 and

$$\mathsf{P}(x_1 \neq 0, \dots, x_m \neq 0) = 1. \tag{3.6}$$

Out of the r vectors, $\mathbf{a}_{n-r+1}, \dots, \mathbf{a}_n$ we may choose $m - (n - r) = r - (n - m)$ in

$$\binom{r}{r-(n-m)} = s \tag{3.7}$$

different ways. Thus we have exactly s different vector systems containing always $\mathbf{a}_1, \ldots,$ $\mathbf{a}_{n-r}$ with which we can write a (3.5) type equation. Let $B_1, \ldots, B_s$ denote these vector systems and let

$$\nu(B_1), \ldots, \nu(B_s) \tag{3.8}$$

denote random variables taking on values 0 or 1, defined in the following manner

$$\nu(B_i) = \begin{cases} 1, & \text{if equation (3.5) written with the vectors} \\ & \text{belonging to } B_i \text{ has a solution with } x_j \geq 0, \\ & j > n-r, \\ 0 & \text{otherwise,} \end{cases} \qquad i = 1, \ldots, s. \tag{3.9}$$

Obviously

$$\nu = \nu(B_1) + \cdots + \nu(B_s). \tag{3.10}$$

It follows from this that

$$\mathsf{E}(\nu) = \sum_{i=1}^{s} \mathsf{E}(\nu(B_i)) = \sum_{i=1}^{s} \mathsf{P}(\nu(B_i) = 1). \tag{3.11}$$

We shall see that each probability standing on the right hand side is equal to

$$\frac{1}{2^{r-(n-m)}}.$$

It is enough to consider equation (3.5) and prove that

$$\mathsf{P}(x_{n-r+1} > 0, \ldots, x_m > 0) = \frac{1}{2^{r-(n-m)}}. \tag{3.12}$$

This follows from the fact that $\mathbf{a}_{n-r+1}, \ldots, \mathbf{a}_m$ are independent, continuously and symmetrically distributed. Symmetrical distribution means that $\mathbf{a}_i$ has the same distribution as $-\mathbf{a}_i$. In fact the event

$$x_{n-r+1} > 0, \ldots, x_{m-1} > 0, \quad x_m > 0 \tag{3.13}$$

has the same probability as the event

$$x_{n-r+1} > 0, \ldots, x_{m-1} > 0, \quad x_m < 0 \tag{3.14}$$

because if we use $-\mathbf{a}_m$ instead of $\mathbf{a}_m$ in (3.5), then the probability that $x_j > 0$, $j > n-r$ remains the same. This can be done with any of the random variables $x_{n-r+1}, \ldots, x_m$.

As we may prescribe positivity and negativity of these in $2^{r-(n-m)}$ different ways and obtain always the same probability, moreover

$$\mathsf{P}(x_j = 0) = 0, \qquad j = n-r+1, \ldots, m, \tag{3.15}$$

it follows that (3.12) holds true. We proved somewhat more then what was wanted namely that the events

$$x_{n-r+1} > 0, \dots, x_m > 0 \tag{3.16}$$

are independent and each of them has the probability 1/2. We proved that the conditional expectation $\mathsf{E}(\nu \mid b)$ is equal to the right hand side of (3.4). Thus (3.4) itself holds too and our theorem is proved. □

Theorem 3 *We consider the random convex polyhedron* (3.3) *where m and n are arbitrary but fixed positive integers. Let r denote the number of elements of J and suppose that $r \geq n - m$. We suppose furthermore that all the $m(n+1)$ components of the random variables $\mathbf{a}_1, \dots, \mathbf{a}_n$, $\mathbf{b}$ are independent, continuously and (with respect to zero) symmetrically distributed. If ν denotes the number of vertices of the random convex polyhedron* (3.3) *then we have*

$$\mathsf{E}(\nu) = \binom{r+m}{r-(n-m)} \frac{1}{2^{r-(n-m)}}. \tag{3.17}$$

Proof. Let $x_1, \dots, x_{n-r}$ be those variables which are not restricted by non-negativity constraints. We add to these some of the variables $x_{n-r+1}, \dots, x_n, y_1, \dots, y_m$ in order to complete their number to m. There are two cases: a) there are no y variables among the additional ones, b) there is at least one y variable among the additional ones. Case a) requires only such investigations which are contained in the proof of Theorem 1. We refer to the equation (3.5) and call the attention for (3.12). In connection with Case b) we shall prove that the probability that in the solution of the equation

$$\begin{gathered}\mathbf{a}_1 x_1 + \cdots + \mathbf{a}_{n-r} x_{n-r} + \mathbf{a}_{n-r+1} x_{n-r+1} + \cdots + \\ + \mathbf{a}_{n-r+t} x_{n-r+t} + \mathbf{e}_1 y_1 + \cdots + \mathbf{e}_v y_v = \mathbf{b}, \\ t + v = m - (n-r), \quad t \geq 0, \quad v > 0,\end{gathered} \tag{3.18}$$

we have

$$x_{n-r+1} > 0, \dots, x_{n-r+t} > 0, \; y_1 > 0, \dots, y_v > 0, \tag{3.19}$$

is again equal to

$$\frac{1}{2^{r-(n-m)}}.$$

We remark that zero is the probability that at least one of the unknowns in (3.18) is equal to zero. By the rule of Cramer we have

$$\begin{aligned} x_{n-r+j} &= \frac{(\mathbf{a}_1, \dots, \mathbf{a}_{n-r}, \mathbf{a}_{n-r+1}, \dots, \mathbf{b}, \dots, \mathbf{a}_{n-r+t}, \mathbf{e}_1, \dots, \mathbf{e}_v)}{(\mathbf{a}_1, \dots, \mathbf{a}_{n-r}, \mathbf{a}_{n-r+1}, \dots, \mathbf{a}_{n-r+j}, \dots, \mathbf{a}_{n-r+t}, \mathbf{e}_1, \dots, \mathbf{e}_v)}, \quad j = 1, \dots, t, \\ y_j &= \frac{(\mathbf{a}_1, \dots, \mathbf{a}_{n-r}, \mathbf{a}_{n-r+1}, \dots, \mathbf{a}_{n-r+t}, \mathbf{e}_1, \dots, \mathbf{b}, \dots, \mathbf{e}_v)}{(\mathbf{a}_1, \dots, \mathbf{a}_{n-r}, \mathbf{a}_{n-r+1}, \dots, \mathbf{a}_{n-r+t}, \mathbf{e}_1, \dots, \mathbf{e}_j, \dots \mathbf{e}_v)}, \quad j = 1, \dots, v, \end{aligned} \tag{3.20}$$

where vectors in parentheses means determinant.

We see from (3.20) that if we replace $\mathbf{a}_{n-r+j}$ by $-\mathbf{a}_{n-r+j}$ then x_{n-r+j} changes and only this changes the sign. Therefore the probability of (3.19) is the same as the probability of the event

$$\begin{aligned} &x_{n-r+1} > 0, \dots, x_{n-r+j-1} > 0,\ x_{n-r+j} < 0, \\ &x_{n-r+j+1} > 0, \dots, x_{n-r+t} > 0,\ y_1 > 0, \dots, y_v > 0. \end{aligned} \tag{3.21}$$

This can be done successively hence we may impose positivity or negativity conditions on $x_{n-r+1}, \dots, x_{n-r+t}$ and the probability will be always the same as the probability of (3.19). Consider now a y variable, y_1, say. We have

$$y_1 = \frac{(\mathbf{a}_1, \dots, \mathbf{a}_{n-r+t}, \mathbf{b}, \mathbf{e}_2, \dots, \mathbf{e}_v)}{(\mathbf{a}_1, \dots, \mathbf{a}_{n-r+t}, \mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_v)}. \tag{3.22}$$

If we replace $a_{11}, \dots, a_{1,n-r+t}, b_1$ by $-a_{11}, \dots, -a_{1,n-r+t}, -b_1$ then $x_{n-r+j},\ j = 1, \dots, t$ and $y_j,\ j = 2, \dots, t$ remain unchanged while y_1 goes over to $-y_1$. This can also be done successively proving that the probability of (3.19) and any other inequality system obtained from that by changing some signs is the same. The sum of these probabilities is equal to 1 hence the probability of (3.19) is again equal to

$$\frac{1}{2^{r-(n-m)}}.$$

Enumerating the possible cases we obtain the required result.

Much more difficult is to obtain higher order moments of ν. This is explained by the fact that the all order moments determine the distribution of ν and determine in particular the largest possible value of ν. Consider e.g. the problem to find the maximum number of vertices of convex polyhedra of the type

$$\begin{aligned} \mathbf{a}_1 x_1 + \cdots + \mathbf{a}_n x_n &= \mathbf{b} \\ x_1 \geq 0, \dots, x_n &\geq 0, \end{aligned} \tag{3.23}$$

where m and n are fixed but $\mathbf{a}_1, \dots, \mathbf{a}_n$, $\mathbf{b}$ vary. Let N be this maximum number of vertices. Now let us randomize $\mathbf{a}_1, \dots, \mathbf{a}_n$, $\mathbf{b}$ so that all possible nunbers of vertices occur with a positive probability. Essentially we are interested in having N vertices with a positive probability but we do not know N in advance. We know, however, that if all components in $\mathbf{a}_1, \dots, \mathbf{a}_n$, $\mathbf{b}$ are independently and normally distributed, then every possible number of vertices occurs with a positive probability (there are other distributions too, of course, having this property). Let p_k be the probability that (3.23) has exactly k vertices, $k = 0, 1, \dots, N$. The M-th moment of ν is given by

$$\mathsf{E}(\nu^M) = \sum_{k=0}^{N} k^M p_k. \tag{3.24}$$

It follows from this that

$$N = \lim_{M\to\infty} [\mathsf{E}(\nu^M)]^{\frac{1}{M}}. \tag{3.25}$$

Thus if we know $\mathsf{E}(\nu^M)$ for every $M = 1, 2, ...$, then we also know N and this would solve a hard problem unsolved until now: what is the maximum number of vertices of the convex polyhedron (3.23) if m, n are fixed and $\mathbf{a}_1, \dots, \mathbf{a}_n$, $\mathbf{b}$ vary arbitrarily in the m-dimensional space. We only prove the following

THEOREM 4 *If $m = 2$, $\mathbf{b} \neq \mathbf{0}$ is any fixed vector and all the $2n$ components of the random variables $\mathbf{a}_1, \dots, \mathbf{a}_n$ are independent, normally distributed with expectations* 0 *and variances* 1, *further ν is the number of vertices of the random convex polyhedron* (3.23), *then*

$$\mathsf{E}(\nu^2) = \frac{1}{4}\left[\binom{n}{2} + 2\binom{n}{3} + \frac{3}{2}\binom{n}{4}\right]. \tag{3.26}$$

PROOF. We may suppose that the length of $\mathbf{b}$ is equal to $\mathbf{1}$. Consider the equations

$$\begin{aligned} \mathbf{a}_1 x_1 + \mathbf{a}_2 x_2 + \qquad &= \mathbf{b}, \\ \mathbf{a}_2 y_2 + \mathbf{a}_3 y_3 &= \mathbf{b}. \end{aligned} \tag{3.27}$$

We find the probability that $x_1 > 0$, $x_2 > 0$, $y_2 > 0$, $y_3 > 0$ simultaneously holds. This will be shown to be equal to $\frac{1}{12}$. First we remark that $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$, $\mathbf{a}_4$ can be supposed to be unit vectors, independently and uniformly distributed on the unit circle with center in the origin. In fact the random vectors

$$\frac{1}{\sqrt{a_{11}^2 + a_{21}^2}}\mathbf{a}_1, \quad \frac{1}{\sqrt{a_{12}^2 + a_{22}^2}}\mathbf{a}_2, \quad \frac{1}{\sqrt{a_{13}^2 + a_{23}^2}}\mathbf{a}_3 \tag{3.28}$$

are independent of each other and of the random variables

$$\sqrt{a_{11}^2 + a_{21}^2}, \quad \sqrt{a_{12}^2 + a_{22}^2}, \quad \sqrt{a_{13}^2 + a_{23}^2}, \tag{3.29}$$

which are also independent of each other. Defining the random variables

$$\begin{aligned} x_1\sqrt{a_{11}^2 + a_{21}^2} &= v_1, \qquad & x_2\sqrt{a_{12}^2 + a_{22}^2} &= v_2, \\ y_2\sqrt{a_{12}^2 + a_{22}^2} &= z_2, \qquad & y_3\sqrt{a_{13}^2 + a_{23}^2} &= z_3, \end{aligned}$$

we see that with probability 1, $x_1 > 0$, $x_2 > 0$, $y_2 > 0$, $y_3 > 0$ if and only if $v_1 > 0$, $v_2 > 0$, $z_2 > 0$, $z_3 > 0$. The unit vectors (3.28) are uniformly distributed on the circumference of the unit circle.

From (3.27) we obtain the formulae for the solutions

$$\begin{aligned} x_1 &= \frac{(\mathbf{b}, \mathbf{a}_2)}{(\mathbf{a}_1, \mathbf{a}_2)}, \qquad & x_2 &= \frac{(\mathbf{a}_1, \mathbf{b})}{(\mathbf{a}_1, \mathbf{a}_2)}, \\ y_2 &= \frac{(\mathbf{b}, \mathbf{a}_3)}{(\mathbf{a}_2, \mathbf{a}_3)}, \qquad & y_3 &= \frac{(\mathbf{a}_2, \mathbf{b})}{(\mathbf{a}_2, \mathbf{a}_3)}. \end{aligned} \tag{3.30}$$

Considering the probability that they are all positive we immediately see that

$$\mathsf{P}(x_1 > 0,\ x_2 > 0,\ y_2 > 0,\ y_3 > 0) = \mathsf{P}(x_1 < 0,\ x_2 > 0,\ y_2 > 0,\ y_3 > 0) \tag{3.31}$$

because if we replace $\mathbf{a}_1$ by $-\mathbf{a}_1$, then the probability that $x_1 > 0$, $x_2 > 0$. $y_2 > 0$, $y_3 > 0$ remains the same. Similarly we can write

$$\mathsf{P}(x_2 > 0,\ y_2 > 0,\ y_3 > 0) = \mathsf{P}(x_2 > 0,\ y_2 > 0,\ y_3 < 0). \tag{3.32}$$

(3.31) and (3.32) together imply

$$\begin{aligned}
&\mathsf{P}(x_1 > 0,\ x_2 > 0,\ y_2 > 0,\ y_3 > 0)\\
&\quad = \mathsf{P}(x_2 > 0,\ y_2 > 0)\mathsf{P}(x_1 > 0)\mathsf{P}(y_3 > 0)\\
&\quad = \frac{1}{4}\mathsf{P}(x_2 > 0,\ y_2 > 0).
\end{aligned} \tag{3.33}$$

The probability in the last row is equal to the following

$$\begin{aligned}
&\mathsf{P}(x_2 > 0,\ y_2 > 0)\\
&\quad = 4\mathsf{P}((\mathbf{b}, \mathbf{a}_3) > 0,\ (\mathbf{a}_2, \mathbf{a}_3) > 0,\ (\mathbf{a}_1, \mathbf{b}) > 0,\ (\mathbf{a}_1, \mathbf{a}_2) > 0).
\end{aligned} \tag{3.34}$$

This probability is the same for every $\mathbf{b} \neq 0$. Thus we may suppose

$$\mathbf{b} = \mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Our problem can be formulated in the following manner: what is the probability that $\mathbf{a}_3$ is on the upper, $\mathbf{a}_1$ is on the lower hemicircle and the angle between $\mathbf{a}_1$ and $\mathbf{a}_2$ as well as between $\mathbf{a}_2$ and $\mathbf{a}_3$ is positive but less than π.

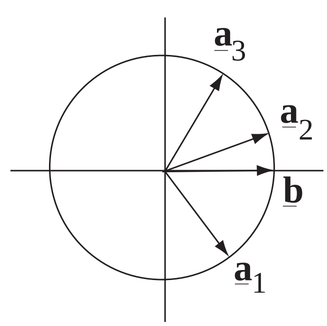

Fig. 1:

The point $\mathbf{a}_2$ can be supposed to be on the upper hemicircle. If its position is x $(0 < x < \pi)$, then the probability of the event just mentioned is

$$\frac{1}{4}\frac{(\pi - x)^2}{\pi^2}$$

hence

$$\mathsf{P}(x_2 > 0,\ y_2 > 0) = 4\int_0^\pi \frac{1}{4}\frac{(\pi - x)^2}{\pi^2}\frac{\mathrm{d}x}{\pi} = \frac{1}{3}. \tag{3.35}$$

Let us consider the random variable

$$\nu = \nu_1 + \cdots + \nu_s, \quad s = \binom{n}{2}. \tag{3.36}$$

If ν_i, ν_j belong to disjoint pairs of vectors then ν_i, ν_j are independent. If these two pairs have one vector in common then by (3.33) and (3.35) we have

$$\mathsf{P}(\nu_i = 1,\ \nu_j = 1) = \frac{1}{3}\cdot\frac{1}{4}. \tag{3.37}$$

For every i we have $\mathsf{P}(\nu_i = 1) = \frac{1}{4}$. Thus

$$\mathsf{E}(\nu_2) = s\mathsf{E}\left(\sum_{j=2}^{s}\nu_1\nu_j\right) + \frac{s}{4} = \frac{1}{4}\left[\binom{n}{2} + 2\binom{n}{3} + \frac{3}{2}\binom{n}{4}\right]. \tag{3.38}$$

This proves the theorem. □

4 The case of non-negative random variables

We consider the system of constraints

$$\begin{gathered}\mathbf{a}_1 x_1 + \cdots + \mathbf{a}_n x_n = \mathbf{b}\\ x_1 \geq 0, \ldots, x_n \geq 0,\end{gathered} \tag{4.1}$$

where the dimension of the vectors $\mathbf{a}_i, \mathbf{b}$ will be denoted by m and we suppose m and n to be fixed, $n \geq m$.

Theorem 5 *Suppose that the $m(n+1)$ components of the vectors $\mathbf{a}_1, \ldots, \mathbf{a}_n$, $\mathbf{b}$ are independent and exponentially distributed with the same parameter $\lambda > 0$. Let ν denote the number of vertices of the random convex polyhedron* (4.1). *Then*

$$\mathsf{E}(\nu) < \binom{n}{m}\frac{\sqrt{(m+1)!}}{m^m}. \tag{4.2}$$

PROOF. We may suppose that $\lambda = 1$. Consider the first m vectors and the equation

$$\begin{aligned}
&\mathbf{a}_1 x_1 + \cdots + \mathbf{a}_m x_m \\
&\quad = \frac{1}{\sum_{i=1}^m a_{i1}} \mathbf{a}_1 \left(\sum_{i=1}^m a_{i1} x_1 \right) + \cdots + \frac{1}{\sum_{i=1}^m a_{im}} \mathbf{a}_m \left(\sum_{i=1}^m a_{im} x_m \right) \\
&\quad = \frac{1}{\sum_{i=1}^m b_i} \mathbf{b} \left(\sum_{i=1}^m b_i \right).
\end{aligned} \tag{4.3}$$

The vectors

$$\mathbf{d}_1 = \frac{1}{\sum_{i=1}^m a_{i1}} \mathbf{a}_1, \ldots, \mathbf{d}_m = \frac{1}{\sum_{i=1}^m a_{im}} \mathbf{a}_m, \quad \mathbf{d} = \frac{1}{\sum_{i=1}^m b_i} \mathbf{b} \tag{4.4}$$

are independent and uniformly distributed over the simplex in the m-dimensional space

$$\begin{aligned}
z_1 + \cdots + z_m &= 1 \\
z_1 \geq 0, \ldots, z_m &\geq 0.
\end{aligned} \tag{4.5}$$

We have to find the probability that the convex polyhedral cone generated by $\mathbf{d}_1, \ldots, \mathbf{d}_m$ contains the vector $\mathbf{d}$. This probability p is equal to the espectation of the $m-1$-dimensional volume t of the intersection of this convex polyhedral cone and the simplex (4.5) divided by the $(m-1)$-dimensional volume of the simplex (4.5). The distance of the simplex (4.5) from the origin is $\frac{1}{\sqrt{m}}$, the $(m-1)$-dimensional volume of the simplex (4.5) is $\frac{\sqrt{m}}{(m-1)!}$ hence

$$\frac{1}{m!} |(\mathbf{d}_1, \ldots, \mathbf{d}_m)| = t \cdot \frac{1}{\sqrt{m}} \cdot \frac{1}{m} \tag{4.6}$$

furthermore

$$p = \mathsf{E} \left\{ \frac{t}{\frac{\sqrt{m}}{(m-1)!}} \right\} = \mathsf{E}\{|(\mathbf{d}_1, \ldots, \mathbf{d}_m)|\}. \tag{4.7}$$

The random variables $\sum_{i=1}^m a_{ik}$, $k = 1, \ldots, n$ are independent of each other and of the random vectors $\mathbf{d}_1, \ldots, \mathbf{d}_m$. Thus

$$\mathsf{E}\{|(\mathbf{a}_1, \ldots, \mathbf{a}_m)|\} = \mathsf{E} \left\{ \prod_{k=1}^m \sum_{k=1}^m a_{ik} \right\} \mathsf{E}\{|(\mathbf{d}_1, \ldots, \mathbf{d}_m)|\} = m^m \mathsf{E}\{|(\mathbf{d}_1, \ldots, \mathbf{d}_m)|\}. \tag{4.8}$$

We anticipate that

$$\mathsf{E}\{(\mathbf{a}_1,\dots,\mathbf{a}_m)^2\} = (m+1)!. \tag{4.9}$$

Taking this into account we obtain

$$\mathsf{E}\{|(\mathbf{a}_1,\dots,\mathbf{a}_m)|\} < \sqrt{(m+1)}!. \tag{4.10}$$

Combining (4.7), (4.8) and (4.10) we obtain the inequality (4.2). It remains to prove (4.9).

LEMMA 1 *Suppose that the elements of the random determinant*

$$\Delta_m = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ . & . & \dots & . \\ a_{m1} & a_{m2} & \dots & a_{mm} \end{vmatrix} \tag{4.11}$$

are independent, exponentially distributed random variables with the same parameter $\lambda = 1$*. Then*

$$\mathsf{E}(\Delta_m^2) = (m+1)!. \tag{4.12}$$

PROOF. Let us develope Δ according to the first row. We obtain

$$\Delta_m = a_{11}A_{11} + a_{12}A_{12} + \ldots + a_{1m}A_{1m}, \tag{4.13}$$

furthermore

$$\mathsf{E}(\Delta_m^2) = \mathsf{E}\left(\sum_{i,j=1}^{m} a_{1i}a_{1j}A_{1i}A_{1j}\right) = 2\sum_{i=1}^{m}\mathsf{E}(A_{1i}^2) + \sum_{i\neq j}\mathsf{E}(A_{1i}A_{1j}). \tag{4.14}$$

Here we took into account that

$$\mathsf{E}(a_{ij}) = 1, \quad \mathsf{E}(a_{ij}^2) = 2, \quad i,j = 1,\dots,m. \tag{4.15}$$

Introduce the notations

$$\begin{aligned} \mathsf{E}(\Delta_m^2) &= D_n, \\ -\mathsf{E}(A_{11}A_{12}) &= B_{m-1}. \end{aligned} \tag{4.16}$$

It follows from (4.14) that

$$D_m = 2mD_{m-1} + 2\binom{m}{2}B_{m-1}. \tag{4.17}$$

The determinants A_{11}, $-A_{12}$ differ only in the first column. If we develope both according to the first columns then we obtain

$$B_{m-1} = -\mathsf{E}(A_{11}A_{12}) = (m-1)D_{m-2} - 2\binom{m-1}{2}B_{m-2}. \tag{4.18}$$

We can express B_{m-1} and B_{m-2} from (4.17) and substitute them into (4.18). We conclude

$$D_m = -m(m-3)D_{m-1} + m(m-1)^2 D_{m-2}, \qquad m = 3, 4, \ldots. \tag{4.19}$$

It is easy to see that $D_1 = 2$, $D_2 = 6$, further $D_m = (m+1)!$ satisfies (4.19). Thus the lemma is proved and with this the proof of our theorem is also complete. □

THEOREM 6 *Let $\mathbf{a}_1, \ldots, \mathbf{a}_n$, $\mathbf{b}$ be m-dimensional random vectors where all the $(n+1)m$ components are independent, exponentially distributed with the same parameter $\lambda > 0$. Let ν denote the number of vertices of the random convex polyhedron*

$$\begin{aligned} \mathbf{a}_1 x_1 + \cdots + \mathbf{a}_n x_n &\le \mathbf{b}, \\ x_1 \ge 0, \ldots, x_n &\ge 0, \end{aligned} \tag{4.20}$$

where m and n are fixed. Then we have

$$\begin{aligned} \mathsf{E}(\nu) < \binom{n}{m}\frac{\sqrt{(m+1)!}}{m^m} + \binom{n}{m-1}\binom{m}{1}\frac{\sqrt{m!}}{m^{m-1}} \\ + \binom{n}{m-2}\binom{m}{2}\frac{\sqrt{(m-1)!}}{m^{m-2}} + \cdots + \binom{n}{2}\binom{m}{m-2}\frac{\sqrt{3!}}{m^2} + n + 1. \end{aligned} \tag{4.21}$$

PROOF. We have seen that the convex polyhedron (4.20) has the same number of vertices as the convex polyhedron

$$\begin{aligned} \mathbf{a}_1 x_1 + \cdots + \mathbf{a}_n x_n + \mathbf{e}_1 y_1 + \cdots + \mathbf{e}_m y_m = \mathbf{b}, \\ x_1 \ge 0, \ldots, x_n \ge 0, \quad y_1 \ge 0, \ldots, y_m \ge 0. \end{aligned} \tag{4.22}$$

To form a basis we may choose a certain number of vectors among $\mathbf{a}_1, \ldots, \mathbf{a}_n$ and a certain number among $\mathbf{e}_1, \ldots, \mathbf{e}_m$. This selected system consists of altogether m vectors. If we choose

$$\mathbf{a}_1, \ldots, \mathbf{a}_k, \quad \mathbf{e}_1, \ldots, \mathbf{e}_{m-k}, \tag{4.23}$$

then we have to find the probability that the solutions of the equation

$$\mathbf{a}_1 x_1 + \cdots + \mathbf{a}_k x_k + \mathbf{e}_1 y_1 + \cdots + \mathbf{e}_{m-k} y_{m-k} = \mathbf{b} \tag{4.24}$$

are positive. The same reasoning can be applied as what was applied in the proof of Theorem 5. We only have to recognize that

$$\mathsf{E}\{(\mathbf{a}_1, \ldots, \mathbf{a}_k, \mathbf{e}_1, \ldots, \mathbf{e}_{m-k})^2\} = (k+1)! \tag{4.25}$$

i.e. the same as $\mathsf{E}(\Delta_k^2)$. But it is obvious since $\mathbf{e}_1, \ldots, \mathbf{e}_{m-k}$ are unit vectors and Lemma 1 applies. We may select a system of type (4.23) in $\binom{n}{k}\binom{m}{m-k}$ different ways, $k = m, m-1, \ldots, 1, 0$. If $k = 1$ then instead of the estimation

$$\mathsf{E}\{|(\mathbf{a}_1, \mathbf{e}_1, \ldots, \mathbf{e}_{m-1})|\} < \sqrt{2}$$

the exact value of the left hand side is used which is equal to 1. If $k = 0$ then (4.24) has positive solution with probability 1. Thus Theorem 6 is proved. □

5 Some exact results for non-negative random variables

In this section we consider random convex polyhedra of the type

$$\begin{aligned} a_{11}x_1 + \cdots + a_{1n}x_n &= 1, \\ a_{21}x_1 + \cdots + a_{2n}x_n &= 1, \\ a_{31}x_1 + \cdots + a_{3n}x_n &= 1, \\ x_1 \geq 0, \ldots, x_n &\geq 0, \end{aligned} \tag{5.1}$$

where the a_{ik}'s are independent non-negative random variables depending on the same probability distribution, which is supposed to be continuous. Let ν denote the number of vertices of this random convex polyhedron. We want to find the exact formulae of $\mathsf{E}(\nu)$ in some particular cases. What we have to do is to find the probability that

$$\mathsf{P}(x_1 > 0, x_2 > 0, x_3 > 0), \tag{5.2}$$

where the random variables x_1, x_2, x_3 are defined as the unique (with probability 1) solutions of the system of random equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= 1, \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= 1, \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= 1. \end{aligned} \tag{5.3}$$

Let us define the vectors

$$\mathbf{d}_i = \frac{1}{a_{1i} + a_{2i} + a_{3i}} \begin{pmatrix} a_{1i} \\ a_{2i} \\ a_{3i} \end{pmatrix}. \tag{5.4}$$

Considering the random equation

$$\mathbf{d}_1 x_1 + \mathbf{d}_2 x_2 + \mathbf{d}_3 x_3 = \mathbf{1}, \tag{5.5}$$

where $\mathbf{1}$ is the vector all components of which are equal to 1, it is true that the probability of $x_1 > 0, x_2 > 0, x_3 > 0$ here is the same as the probability (5.2). The vectors $\mathbf{d}_i$ are elements of the simplex in the three-dimensional space:

$$\begin{gathered} z_1 + z_2 + z_3 = 1, \\ z_1 \geq 0,\ \ z_2 \geq 0,\ \ z_3 \geq 0, \end{gathered} \tag{5.6}$$

which is a regular triangle. Thus our problem can be formulated so that we choose three independently and identically distributed random points in a regular triangle, what is the probability that the random triangle the vertices of which are these random points, covers the center of gravity C of the regular triangle (see Fig. 2)? The probability distribution

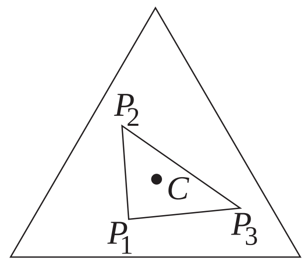

Fig. 2:

of these random points can be determined by the probability distribution of the random variables a_{ik}. If e.g. this is exponential then the random points are uniformly distributed in the regular triangle. Two special cases will be investigated, the case of the exponential and the case of the uniform distribution. First a general formula is derived for the probability in question and this will be specialized. We remark that the random triangle with vertices P_1, P_2, P_3 covers the center of gravity if and only if their projections Q_1, Q_2, Q_3 from the center of gravity onto the boundary of the regular triangle have the property that the new triangle belonging to Q_1, Q_2, Q_3 covers the center of gravity (see Fig. 3).

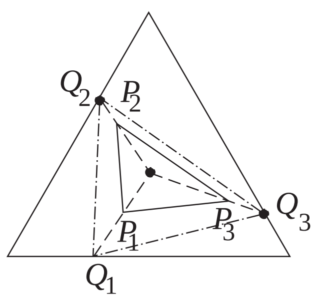

Fig. 3:

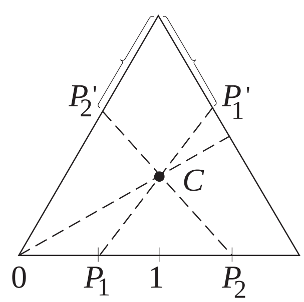

Fig. 4:

The probability distribution of a random point P in this triangle uniquely determines the probability distribution of the projected point Q on the boundary. In the course of the calculation the length of the boundary will be supposed to be equal to 6. One point, P_1 say, can be supposed to be placed on one half of a side (see Fig. 4) where the cumulative probability between 0 and x will be denoted by $F(x)$ with $F(0) = 0$, $F(1) = 1/6$. Thus the conditional probability distribution function of the point P_1 on this section is equal to $6F(x)$, $0 \leq x \leq 1$. The point P_2 will run around the circumference of the triangle. To each fixed positions of P_1 and P_2 there corresponds a probability that P_3 has a position such that the triangle with vertices P_1, P_2, P_3 covers the center of gravity C. This favourable part of the boundary for P_3 is indicated by the parentheses in Fig. 4. P_2 starts at 0 and goes to the right. Six half sides are on the way and we denote by $p_1, p_2, p_3, p_4, p_5, p_6$, the probabilities belonging to the half sides, that the small triangle covers C. Before giving the formulae for p_i, $i = 1, 2, 3, 4, 5, 6$, we remark that if P_1 is at the point x, $0 \le x \le 1$ then its transformed P_1' via C onto the other side of the regular triangle is at

$$x' = \frac{4(1-x)}{4-3x}, \tag{5.7}$$

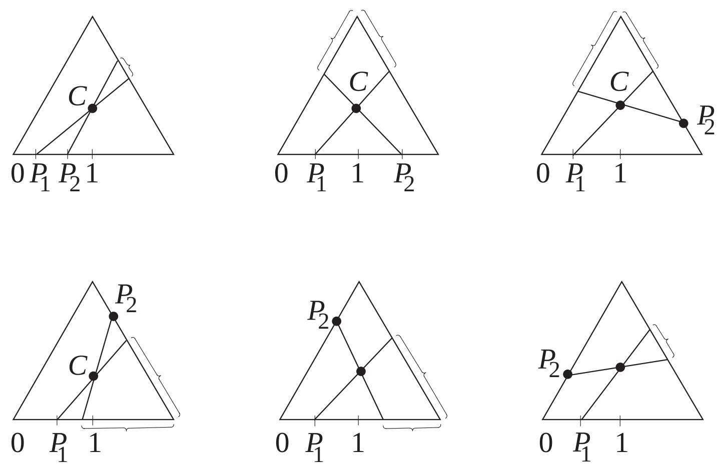

Fig. 5:

where $x' = 0$ at the top vertex of the triangle on Fig. 4 and $x' = 1$ at the half of the side coming to the right from the top vertex. Figure 5 shows the six cases for P_2 and illustrates the calculation of the probabilities p_i, $i = 1, 2, 3.4, 5, 6$. From Fig. 5 we see that

$$\begin{aligned}
p_1 &= 6\int_0^1 \left\{ \int_0^x \left[F\left(\frac{4(1-y)}{4-3y}\right) - F\left(\frac{4(1-x)}{4-3x}\right)\right] \mathrm{d}F(y) \right. \\
&\qquad \left. + \int_x^1 \left[F\left(\frac{4(1-x)}{4-3x}\right) - F\left(\frac{4(1-y)}{4-3y}\right)\right] \mathrm{d}F(y) \right\} \mathrm{d}F(x), \\
p_2 &= 6\int_0^1 \int_0^1 \left[F\left(\frac{4(1-x)}{4-3x}\right) + F\left(\frac{4(1-y)}{4-3y}\right)\right] \mathrm{d}F(y)\,\mathrm{d}F(x), \\
p_3 &= 6\int_0^1 \int_0^1 \left[F\left(\frac{4(1-x)}{4-3x}\right) + \frac{1}{3} - F\left(\frac{4(1-y)}{4-3y}\right)\right] \mathrm{d}F(y)\,\mathrm{d}F(x), \\
p_4 &= 6\int_0^1 \left[\int_0^{\frac{(4(1-x)}{4-3x}} \left(\frac{1}{3} + \frac{1}{6} - F\left(\frac{4(1-y)}{4-3y}\right) + \frac{1}{6} - F\left(\frac{4(1-x)}{4-3x}\right)\right) \mathrm{d}F(y) \right.
\end{aligned} \tag{5.8}$$

$$+\int_{\frac{4(1-x)}{4-3x}}^{1}\left(\frac{1}{3}+F\left(\frac{4(1-y)}{4-3y}\right)+F\left(\frac{4(1-x)}{4-3x}\right)\right)\,\mathrm{d}F(y)\Bigg]\,\mathrm{d}F(x),$$

$$p_5=p_3,$$

$$p_6=6\int_0^1\int_0^1\left[\frac{1}{6}-F\left(\frac{4(1-y)}{4-3y}\right)+\frac{1}{6}-F\left(\frac{4(1-x)}{4-3x}\right)\right]\,\mathrm{d}F(y)\,\mathrm{d}F(x).$$

Using the notations

$$x'=\frac{4(1-x)}{3-4x},\qquad y'=\frac{4(1-y)}{3-4x}, \tag{5.9}$$

we obtain further

$$\begin{aligned}
p_1&=12\int_0^1\int_0^x[F(y')-F(x')]\,\mathrm{d}F(y)\,\mathrm{d}F(x)-6\int_0^1\int_0^1[F(y')-F(x')]\,\mathrm{d}F(y)\,\mathrm{d}F(x)\\
&=12\int_0^1\int_0^x F(y')\,\mathrm{d}F(y)\,\mathrm{d}F(x)-12\int_0^1 F(x)F(x')\,\mathrm{d}F(x)\\
&=12\int_0^1\int_y^1 F(y')\,\mathrm{d}F(x)\,\mathrm{d}F(y)-12\int_0^1 F(x)F(x')\,\mathrm{d}F(x) \qquad (5.10)\\
&=12\int_0^1 F(y')\left[\frac{1}{6}-F(y)\right]\,\mathrm{d}F(y)-12\int_0^1 F(x)F(x')\,\mathrm{d}F(x)\\
&=2\int_0^1 F(y')\,\mathrm{d}F(y)-24\int_0^1 F(x)F(x')\,\mathrm{d}F(x).
\end{aligned}$$

Let us introduce the notation

$$\alpha=\int_0^1 F(x')\,\mathrm{d}F(x). \tag{5.11}$$

We see easily that

$$p_2=2\alpha, \tag{5.12}$$

$$p_3=p_5=\frac{1}{18}, \tag{5.13}$$

$$\begin{aligned}
p_4=\frac{1}{18}&+2\int_0^1 F(x')\,\mathrm{d}F(x)+6\int_0^1\int_0^1[F(x')+F(y')]\,\mathrm{d}F(x)\,\mathrm{d}F(y)\\
&-12\int_0^1\int_0^{x'}[F(y')+F(x')]\,\mathrm{d}F(y)\,\mathrm{d}F(x) \qquad (5.14)\\
=\frac{1}{18}&+4\alpha-12\int_0^1 F^2(x')\,\mathrm{d}F(x)-12\int_0^1\int_0^{x'}F(y')\,\mathrm{d}F(y)\,\mathrm{d}F(x).
\end{aligned}$$

For the last term we derive

$$\int_0^1 \int_0^{x'} F(y')\,\mathrm{d}F(y)\,\mathrm{d}F(x) = \int_0^1 \int_0^{y'} F(y')\,\mathrm{d}F(x)\,\mathrm{d}F(y) = \int_0^1 F^2(y')\,\mathrm{d}F(y). \tag{5.15}$$

Thus

$$p_4 = \frac{1}{18} + 4\alpha - 24 \int_0^1 F^2(x')\,\mathrm{d}F(x). \tag{5.16}$$

Finally it is easy to see that

$$p_6 = \frac{1}{18} - 2\alpha. \tag{5.17}$$

Some further simplifications are possible. In fact

$$\begin{aligned} \int_0^1 F^2(x')\,\mathrm{d}F(x) &= [F^2(x')F(x)]_0^1 - 2\int_0^1 F(x)\,F(x')\,\mathrm{d}F(x') \\ &= 2\int_0^1 F(x)F(x')\,\mathrm{d}F(x) = 2\beta. \end{aligned} \tag{5.18}$$

Summing up the probabilities we conclude

$$p = p_1 + p_2 + p_3 + p_4 + p_5 + p_6 = \frac{2}{9} + 6\alpha - 72\beta. \tag{5.19}$$

Before specializing this to particular distributions we mention the following

Lemma 2 *Consider a regular n-dimensional simplex and let P be a random point uniformly distributed in the simplex. Let C be the center of gravity and consider the random point Q on the boundary of the simplex which is defined so that P is projected from C onto the boundary. Then Q is uniformly distributed on the boundary.*

Proof. Let H be a measurable set on the boundary. The $(n-1)$-dimensional measure of this set and the n-dimensional measure of the set

$$\{Z \mid Z = \lambda C + (1-\lambda)Q, \quad Q \in H, \quad 0 \le \lambda \le 1\} \tag{5.20}$$

differ from each other only by a constant. Thus the probability of $Q \in H$ is proportional to the $(n-1)$-dimensional measure of H which proves the lemma. □

The probability p in (5.19) equals the probability (5.2). It remains for us to obtain the function $F(x)$ for special probability distributions concerning the random variables a_{ik}. First we consider the exponential distribution. In this case the vectors $\mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3$ defined by (5.4) are uniformly distributed on the regular triangle (5.6) the side-length of which is equal to $\sqrt{2}$. By the previous lemma we can find the probability (5.2) so that we substitute in (5.19) the function

$$F(x) = \frac{x}{6}, \qquad 0 \le x \le 1. \tag{5.21}$$

For α and β we obtain

$$\alpha = \int_0^1 F\left(\frac{4(1-x)}{4-3x}\right) \mathrm{d}F(x) = \frac{1}{36}\int_0^1 \frac{4(1-x)}{4-3x}\,\mathrm{d}x = \frac{1}{27}\left(1 - \frac{2}{3}\log 2\right), \tag{5.22}$$

$$\begin{aligned}\beta &= \int_0^1 F\left(\frac{4(1-x)}{4-3x}\right) F(x)\,\mathrm{d}F(x) = \frac{1}{216}\int_0^1 \frac{4(1-x)}{4-3x} x\,\mathrm{d}x \\ &= \frac{1}{108} - \frac{15 - 16\log 2}{27}.\end{aligned} \tag{5.23}$$

Thus we have for the required probability

$$p = \frac{2}{9} + 6\alpha - 72\beta = \frac{2}{27}\left(1 + \frac{10}{3}\log 2\right) \approx 0.2452. \tag{5.24}$$

We have proved the following

Theorem 7 *The expectation of the number of vertices of the random convex polyhedron* (5.1) *in case of independently and exponentially distributed coefficients a_{ik} having the same parameter $\lambda > 0$, is given by the formula*

$$E(\nu) = \binom{n}{3}\frac{2}{27}\left(1 + \frac{10}{3}\log 2\right). \tag{5.25}$$

If instead of (5.1) we consider the random convex polyhedron

$$\begin{aligned} a_{11}x_1 + \cdots + a_{1n}x_n &\le 1, \\ a_{21}x_1 + \cdots + a_{2n}x_n &\le 1, \\ a_{31}x_1 + \cdots + a_{3n}x_n &\le 1, \\ x_1 \ge 0, \ldots, x_n &\ge 0, \end{aligned} \tag{5.26}$$

then beyond the probability (5.24) we have to find the probability

$$\mathsf{P}(x_1 > 0,\ x_2 > 0,\ y_1 > 0), \tag{5.27}$$

where

$$\mathbf{a}_1 x_1 + \mathbf{a}_2 x_2 + \mathbf{e}_1 y_1 = \mathbf{1}, \tag{5.28}$$

and also the probability

$$\mathsf{P}(x_1 > 0,\ y_1 > 0,\ y_2 > 0) \tag{5.29}$$

where

$$\mathbf{a}_1 x_1 + \mathbf{e}_1 y_1 + \mathbf{e}_2 y_2 = \mathbf{1}. \tag{5.30}$$

We suppose again that the coefficients are independently and exponentially distributed with the same parameter. The probability (5.29) will be equal to 1/3 while for the probability (5.27) we get

$$p = \frac{5}{18} - 2\alpha = \frac{1}{162}[33 + 8\log 2]. \tag{5.31}$$

These can be obtained similarly as we obtained (5.24), the difference is that one, resp. two points on the boundary of the triangle are fixed at vertices. From these immediately follows

THEOREM 8 *If the a_{ik}'s are independently and exponentially distributed random variables having the same parameter then the expected number of vertices of the random convex polyhedron* (5.26) *is given by*

$$\mathsf{E}(\nu) = \binom{n}{3}\frac{2}{81}(3 + 10\log 2) + 3\binom{n}{2}\frac{1}{162}(33 + 8\log 2) + n + 1. \tag{5.32}$$

Now we turn to the case where the a_{ik}'s are uniformly distributed in the interval $(0, a)$, where $a > 0$. As $\mathsf{E}(\nu)$ is independent of how large a is choosen, let us choose $a = \sqrt{2}$. It can be shown that on the boundary of the triangle with side length 2 we now have

$$F(x) = \frac{1}{6}\frac{\sqrt{x}}{2 - \sqrt{x}}, \qquad 0 \le x \le 1. \tag{5.33}$$

In fact Figure 6 shows a part of the cube with side length 2 which has a probability equal to the probability that $Q \in (0, x)$ where $0 \le x \le 1$, Q is the projection onto the boundary of the point P which is the intersection of the random ray varying in the cube and the triangle having vertices $(\sqrt{2}, 0, 0)$, $(0, \sqrt{2}, 0)$, $(0, 0, \sqrt{2})$. The part of the cube in question is between the origin and the triangle on the upper face. The value of z depends on x so that

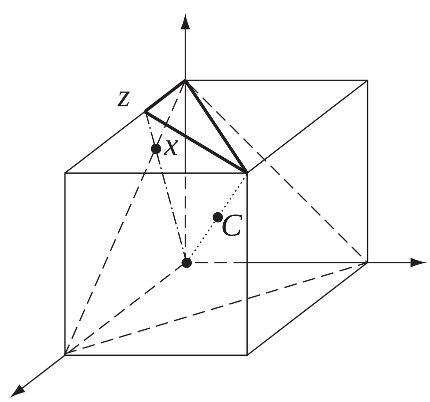

Fig. 6:

$$z = \frac{\sqrt{x}}{\sqrt{2} - \sqrt{\frac{x}{2}}}, \tag{5.34}$$

thus the probability of that part of the cube is

$$\frac{z}{3}\Big/(\sqrt{2})^3 = \frac{1}{6}\frac{\sqrt{x}}{2-\sqrt{x}} \tag{5.35}$$

and this equals $F(x)$, $0 \le x \le 1$. Using this, we have

$$\alpha = \int_0^1 F\left(\frac{4(1-x)}{4-3x}\right) \mathrm{d}F(x) = \frac{1}{36}\int_0^1 \frac{\sqrt{\frac{4(1-x)}{4-3x}}}{2-\sqrt{\frac{4(1-x)}{4-3x}}}\left(\frac{\sqrt{x}}{2-\sqrt{x}}\right)' \mathrm{d}x \approx 0.021, \tag{5.36}$$

$$\begin{aligned}\beta &= \int_0^1 F\left(\frac{4(1-x)}{4-3x}\right) F(x)\,\mathrm{d}F(x) \\ &= \frac{1}{216}\int_0^1 \frac{\sqrt{\frac{4(1-x)}{4-3x}}}{2-\sqrt{\frac{4(1-x)}{4-3x}}} \cdot \frac{\sqrt{x}}{2-\sqrt{x}}\left(\frac{\sqrt{x}}{2-\sqrt{x}}\right)' \mathrm{d}x \approx 0.00146.\end{aligned} \tag{5.37}$$

THEOREM 9 *If the a_{ik}'s are independently and uniformly distributed in $(0, a)$ where $a > 0$, then the expectation of the number of vertices of the random convex polyhedron* (5.1) *is given by*

$$\mathsf{E}(\nu) = \binom{n}{3}\left(\frac{2}{9} + 6\alpha - 72\beta\right), \tag{5.38}$$

where α and β are defined by (5.36) *and* (5.37), *respectively.*

The proof is already given. We call the attention to the fact that the probability (5.29) is equal to 1/3 i.e. it does not depend on F while the probability (5.27) is equal to (as it stands in (5.31)) $\frac{5}{18} - 2\alpha$. Thus we have proved also

THEOREM 10 *If the a_{ik}'s are independently and uniformly distributed in $(0, a)$ where $a > 0$, then the expectation of the number of vertices of the random convex polyhedron* (5.26) *is equal to*

$$\mathsf{E}(\nu) = \binom{n}{3}\left(\frac{2}{9} + 6\alpha - 72\beta\right) + 3\binom{n}{2}\left(\frac{5}{18} - 2\alpha\right) + n + 1, \tag{5.39}$$

where α and β are given by (5.36) *and* (5.37), *respectively.*

References

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[2] PRÉKOPA, A. (1964). Some distribution problems in stochastic programming, *Abstracta of the Fifth Internat. Sympos. Math. Programming*, London.