Stochastic Programming, ed. by M.A.H. Dempster, Academic Press, 1980, pp. 369–385.

The STABIL Stochastic Programming Model and its Experimental Application to the Electrical Energy Sector of the Hungarian Economy

ANDRÁS PRÉKOPA
Budapest

SÁNDOR GANCZER
Budapest

ISTVÁN DEÁK
Budapest

KÁROLY PATYI
Budapest

Abstract

The stochastic programming model named STABIL is given by (1.1), where we minimize a linear or nonlinear objective function under probabilistic and some further constraints. In this paper we are concerned with a special case of this model-type, where the problem functions $g_1, \ldots, g_{m+M}$, f are linear. The model is applied to the fourth Five-Year Plan of the electrical energy sector of the Hungarian economy where the underlying deterministic model is the corresponding deterministic sector model of the fourth Five-Year Plan. This application is experimental, as the construction of the model and the computations were performed at a time when the fourth Five-Year Plan was already running. In this paper we describe the model, its solution algorithm, the computer program, the parameters of the model and the computational results.

A very interesting phenomenon is that although there is no significant difference between the optimum value of the deterministic underlying problem and that of the related STABIL model, the optimal solutions turned out to be different. Further, the reliability level of the optimal solution of the deterministic underlying problem turned out to be very low, while that of the optimal solution of the STABIL model is high. The appearance of this phenomenon makes the numerical example interesting in itself, not just as a special application.

1 Introduction

In this paper we describe in detail the probabilistically constrained stochastic programming model named STABIL. The model, its theory and solution algorithm are described briefly in [4, 5, 8]. Here we apply the model to the electrical energy sector of the Hungarian economy in the setting of the fourth Five-Year Plan. The description of the linear programming model of the fourth Five-Year Plan can be found in [1] and [3].

The name STABIL for the model is introduced in this paper. This name is not an abbreviation; we have chosen it because the model contains a probabilistic constraint which prescribes a high probability level with which the system must operate.

In this paper the computer program system for the solution algorithm of model (1.1) is also described in a short form. For further details the reader is referred to [2].

The application of our model is experimental for two reasons. Firstly, because the underlying problem for the stochastic programming model was the deterministic electrical energy sector model of the fourth Five-Year Plan – which was already in operation at the time of this work. Thus we could not consider a practical application of the numerical results. Secondly, there was a lack of information connected with the relevant probability distribution for the random variables in the model and subjective considerations were applied in order to specify this distribution.

The STABIL stochastic programming model is the following

$$
\begin{aligned}
G(x) = P(g_i(x) \geq \xi_i, \qquad & i = 1, \ldots, m) \geq p, \\
g_i(x) \geq b_i, \qquad & i = m+1, \ldots, m+M, \\
\min f(x). & \qquad (1.1)
\end{aligned}
$$

We are interested in the special case of the model when the functions $g_1, \ldots, g_{m+M}$, f are linear. The joint distribution of the random variables $\xi_1, \ldots, \xi_m$ is supposed to be continuous with logarithmic concave joint density. The case we are dealing with here is the non-degenerate, multivariate, normal distribution (which has this property). In model (1.1) p is a prescribed probability, $0 < p < 1$, chosen near unity in practice. In the electrical energy problem the values of p were chosen to be 0.9 and 0.95. In this problem, $m = 4$, $M = 106$, when the individual upper and lower bounds and the nonnegativity constraints are incorporated, and the vector x has 46 components.

The deterministic model, which is the starting point for the stochastic programming model construction, i.e. the underlying deterministic model for (1.1), is the

following

$$
\begin{aligned}
& g_i(x) \geq b_i, \qquad i = 1, \ldots, m, \\
& g_i(x) \geq b_i, \qquad i = m+1, \ldots, m+M, \\
& \min f(x).
\end{aligned} \tag{1.2}
$$

Supposing the values on the right hand side of the first row in (1.2) are random leads to Model (1.1). In model construction in practice the random variables are frequently replaced by their expectations. In such cases, in formulating the stochastic programming decision model and specifying the probability distribution of the random variables appearing in the model, the expectations can be removed from the underlying deterministic model. Following this methodology, the random variables of (1.1) will be written in the form

$$b_i + \sigma_i \beta_i, \qquad i = 1, \ldots, m,$$

where

$$E(\beta_i) = 0, \qquad i = 1, \ldots, m. \tag{1.3}$$

We may also suppose that

$$V(\beta_i) = 1, \qquad i = 1, \ldots, m. \tag{1.4}$$

The symbols E, V are used to denote expectation and variance, respectively. As mentioned above, we are interested in the special case of (1.1) when the functions $g_1, \ldots, g_{m+M}$, f are linear. Let us introduce the notation

$$
\begin{aligned}
& g_i(x) = a_i' x, \qquad i = 1, \ldots, m+M, \\
& f(x) = c' x.
\end{aligned} \tag{1.5}
$$

The mathematical model and the solution algorithms presented in this paper were constructed by A. Prékopa. The procedure for evaluating the distribution function of the multivariate normal distribution, the computer program for the solution of the problem (except for the linear programming package, which was worked out by G. Kéri), and the numerical computations were done by I. Deák. The formulation of the deterministic underlying problem and all data collection were done by S. Ganczer and K. Patyi. The application of the stochastic programming model to the electrical energy sector, i.e. the adaptation of the model was done jointly by the authors of this paper.

2 Detailed Description of the Stochastic Programming Model

We are concerned with the numerical solution of the following special case of the STABIL model

$$
\begin{aligned}
& G(x) = P\left(\frac{1}{\sigma_i}(a_i'x - b_i) \geq \beta_i, \qquad i = 1, \ldots, m\right) \geq p, \\
& a_i'x \geq b_i, \qquad i = m+1, \ldots, m+M, \\
& \min c'x,
\end{aligned}
\tag{2.1}
$$

where $x \in \mathbb{R}^n$.

A. Prékopa proved [5, 6] that if the joint distribution of the random variables $\beta_1, \ldots, \beta_m$ is continuous and the joint density is of the form

$$
e^{-Q(z)}, \qquad z \in \mathbb{R}^m,
\tag{2.2}
$$

where $Q(z)$ is a convex function on the entire space, which can take values equal to $+\infty$, then the function given by $G(x)$ is logarithmic concave on the entire space $\mathbb{R}^n$. In our model we suppose that the joint distribution of the random variables $\beta_1, \ldots, \beta_m$ is a nondegenerate normal distribution. In this case their joint density is of the form

$$
\frac{1}{(2\pi)^{m/2}\sqrt{|C|}} e^{-1/2 z'C^{-1}z},
\tag{2.3}
$$

where the matrix C is the covariance (correlation) matrix of the random variables $\beta_1, \ldots, \beta_m$, i.e.

$$
c_{ik} = E(\beta_i \beta_k), \qquad i, k = 1, \ldots, m.
\tag{2.4}
$$

Since the distribution is non-degenerate the matrix C is positive definite, so C^{-1} exists and is also positive definite. Hence, by a well known theorem, the function given by

$$
z'C^{-1}z,
\tag{2.5}
$$

is convex in the entire space $\mathbb{R}^m$. If the joint density of the random variables $\beta_1, \ldots, \beta_m$ is (2.3), then the function $G(x)$ is *logarithmic concave* on the entire space $\mathbb{R}^n$, i.e. for every pair of vectors $x_1, x_2 \in \mathbb{R}^n$ and $0 < \lambda < 1$ we have the inequality

$$
G(\lambda x_1 + (1-\lambda)x_2) \geq [G(x_1)]^{\lambda}[G(x_2)]^{1-\lambda}.
\tag{2.6}
$$

Since $G(x) > 0$ for every $x \in \mathbb{R}^n$, this implies that $\log G(x)$ is a finite valued concave function on the entire space $\mathbb{R}^n$.

Algorithm for the Solution of Problem (2.1)

The function in the first constraint of (2.1) is quasiconcave. If a nonlinear programming method converges whenever the constraints are quasi-concave and the objective function is linear, it can be applied for the solution of our problem. Such a method is Zoutendijk's *method of feasible directions*, in particular Procedure $P2$ [10, p. 74]. For the proof of the convergence in case of quasi-concave constraining functions and a convex objective function (to be minimized) see [4, 5, 7]. Before presenting the method for the solution of Problem (2.1) we describe Zoutendijk's method for the problem

$$G(x) \geq p, \qquad a_i' x \geq b_i, \qquad i \in I, \qquad \min f(x), \tag{2.7}$$

where the functions G and f are *not* specialized according to (1.1) and (1.5) and I denotes the appropriate constraint index set. We suppose that the functions $f(x)$, $G(x)$ are differentiable on the entire space $\mathbb{R}^n$. Suppose further, that the set determined by the linear constraints is nonempty and bounded. Let x_1 be an arbitrary vector satisfying the constraints of Problem (2.7). We define by induction the successive iterations; suppose we have already determined the vectors $x_1, \ldots, x_k$ and want to determine the vector x_{k+1} at the $(k+1)$th iteration. One iteration consists of two parts. In the first part we solve the following *direction finding* problem

$$\begin{aligned} G(x_k) + \nabla G(x_k)[x - x_k] + \theta y \geq p, \qquad a_i' \geq b_i, \\ i \in I, \qquad \nabla f(x_k)[x - x_k] \leq y, \qquad \min y, \end{aligned} \tag{2.8}$$

where θ is an arbitrary positive number fixed throughout the whole procedure. The number of the variables in the linear programming problem (2.8) is $n+1$, since the vector x has n components and y is also variable. There is always at least one vector satisfying the constraints of (2.8) – the $n+1$ component vector $x = x_k$, $y = 0$. Since x varies in a bounded set, the objective function is bounded from below, hence Problem (2.8) has a finite optimum. Let y_{opt} denote the optimal value of the objective function of Problem (2.8). If $y_{\text{opt}} = 0$, then the procedure terminates. If $y_{\text{opt}} \neq 0$, which means in this case $y_{\text{opt}} < 0$, then we proceed to the second part of the $(k+1)$th iteration, which is the determination of the *step length*. Let x_k^* be an optimal solution of Problem (2.8). Then we minimize the function of the variable λ given by

$$f(x_k + \lambda(x_k^* - x_k)) \tag{2.9}$$

on that part of the halfline $x_k + \lambda(x_k^* - x_k)$, $\lambda \geq 0$, which belongs to the set of feasible solutions of Problem (2.7). Under very general assumptions this minimum is obtained for some λ. If λ_k is a minimizing λ, then we define x_{k+1} as

$$x_{k+1} = x_k + \lambda_k(x_k^* - x_k) \tag{2.10}$$

Solution of Problem (2.1) in Two Phases

In the second phase we solve Problem (2.1) under the assumption that we have a vector x_1 satisfying the constraints. In the first phase our aim is to find a vector x_1 satisfying the constraints.

In the second phase we take into account the meaning of the function G and the fact that f is a linear function of the form (1.5). The gradient of the function G will be given in the next section. The gradient of the function f is the constant vector c'. Function (2.9) has the form

$$c'(x_k + \lambda(x_k^* - x_k)). \tag{2.11}$$

This is to be minimized on a bounded interval since the set of vectors x defined by the constraints of Problem (2.1) is convex and bounded. Since we have $y_{\text{opt}} < 0$, it follows that $c'x_k^* < c'x_k$. Thus $x_k^* \neq x_k$ which implies that the set of vectors $x = x_k + \lambda(x_k^* - x_k)$ for $0 \leq \lambda \leq 1$ is a nondegenerate interval, which implies that the closed interval mentioned above (which contains it) is also nondegenerate. The minimum is attained at the endpoint of this larger interval which corresponds to a positive λ value. It is easy to see that the vector x_{k+1} is a boundary point of the set defined by the constraints of the problem. Thus all the points $x_2, x_3, \ldots$ generated by the algorithm are boundary points.

In the first phase we want to find a vector x_1 satisfying the constraints of Problem (2.1). We use the method described for the solution of Problem (2.7), but now we maximize the first constraining function G of Problem (2.1) under the linear constraints of Problem (2.1). This can be considered as a special case of Problem (2.7). Thus we apply the algorithm to the problem

$$a_i'x \geq b_i, \qquad i \in I, \qquad \max G(x), \tag{2.12}$$

i.e. to the equivalent problem

$$a_i'x \geq b_i, \qquad i \in I, \qquad \min(-G(x)), \tag{2.13}$$

until we reach a vector x_1 which satisfies the inequality

$$G(x_1) \geq p. \tag{2.14}$$

This vector x_1 can be used as the starting vector for the second phase. The method applied to Problem (2.13) can be summarized as follows. We start from a vector z_1 which satisfies the constraints of the Problem (2.13); such a vector can easily be found by linear programming. If we have already determined the vectors $z_1, \ldots, z_k$ then in order to define z_{k+1} we consider the following direction finding problem

$$a_i'z \geq b_i, \quad i \in I, \quad -G(z_k) + \nabla(-G(z_k))(z - z_k) \leq y, \quad \min y \tag{2.15}$$

which can be reformulated as

$$a_i' z \geq b_i, \quad i \in I, \quad \min(\nabla G(z_k)(z - z_k)). \tag{2.16}$$

The stopping rule and the step length determination was discussed above in connection with Problem (2.7). Zoutendijk's method applied in the first phase is just the well known *gradient method.*

3 Convergence of the Procedure

As mentioned above, the convergence of the procedure applied to Problem (2.7) was considered in [4, 5, 7]. We recall the main theorem in a weaker, but for the present purpose more useful, form.

THEOREM 1. *Suppose that the following conditions hold*

(i) *The functions G and f are defined on the whole space $\mathbb{R}^n$ and have a continuous gradient there.*

(ii) *The function G is quasiconcave and the function f is convex on the entire space.*

(iii) *The set $K = \{x \mid a_i' x \geq b_i,\ i \in I\}$ is non-empty and bounded.*

(iv) *For every x satisfying the equality $G(x) = p$ there corresponds a vector y in the set of feasible solutions with the property that*

$$\nabla G(x)(y - x) > 0. \tag{3.1}$$

If the procedure terminates in a finite number of steps and the last vector is x_N, then we have

$$f(x_N) = \min_{x \in L} f(x). \tag{3.2}$$

If the sequence $x_1, x_2, \ldots$ is infinite then we have

$$\lim_{k \to \infty} f(x_k) = \min_{x \in L} f(x), \tag{3.3}$$

where L is the set of the feasible solutions of the problem.

The convergence of the procedure applied to Problem (2.1) can be proved using Theorem 1. We formulate two different theorems for the two phases. It is obvious that the first phase must terminate in a finite number of steps, while in the second phase it is enough if the procedure converges. First we consider the convergence problem of the second phase.

THEOREM 2. *Besides the assumptions formulated in connection with Problem* (2.1) *suppose that there exists a vector* $y \in L$ *for which*

$$G(y) > p. \tag{3.4}$$

Then the second phase procedure is either finite, when the last vector x_N *satisfies Relation* (3.2)*, or it is infinite and Relation* (3.3) *holds. As before,* L *denotes the set of feasible solutions of the problem.*

Proof. We have to prove that assumptions (i)–(iv) of Theorem 1 hold. Assumptions (i) and (ii) are satisfied trivially. Assumptions (iii) was introduced earlier in connection with Problem (2.1). So we only have to check the validity of Assumption (iv). The proof will be indirect. Suppose there exists an $x \in L$ for which $G(x) = p$ such that for every feasible $y \in L$ we have

$$\nabla G(x)(y - x) \leq 0. \tag{3.5}$$

The function $\log G$ is finite-valued and concave in the entire space. This implies that the following inequality holds for every $y \in \mathbb{R}^n$

$$\log G(y) - \log G(x) \leq \nabla \log G(x)(y - x) = \frac{1}{G(x)} \nabla G(x)(y - x). \tag{3.6}$$

If $y \in L$, then (3.5) and the inequality $G(x) > 0$ together imply that

$$G(y) \leq G(x) = p. \tag{3.7}$$

This contradicts to (3.4); thus the theorem is proved. □

The following theorem summarizes our statement in connection with the finiteness of the first phase.

THEOREM 3. *Besides the assumptions already introduced in connection with Problem* (2.1)*, assume that there exists a vector* $y \in K$*, for which inequality* (3.4) *holds. Then, starting from any vector* $z_1 \in K$*, we reach a vector lying in the set* L *after a finite number of steps.*

Proof. The method applied in the first phase is the classical gradient method, so we might refer to any already existing convergence theorem for the gradient method. For the sake of presenting a unified approach, however, we refer to Theorem 1. By Theorem 1 it follows that if we apply the method of the first phase not to Problem (2.13) but to the problem

$$a_i' z \geq b_i, \qquad i \in I, \qquad \min(-\log G(z)), \tag{3.8}$$

then the sequence $z_1, z_2, \ldots$ obtained is either finite, and the last vector z minimizes the objective function of Problem (3.8), or the following relation holds

$$\lim_{k\to\infty}(-\log G(z_k)) = \min_{z\in K}(-\log G(z)). \tag{3.9}$$

Now the problem of type (2.16) corresponding to (3.8) is

$$a_i'z \geq b_i, \quad i \in I, \quad \min\left(-\frac{1}{G(z_k)}\nabla G(z_k)(z - z_k)\right). \tag{3.10}$$

The objective functions of Problem (3.10) and (2.16) differ only by a positive constant factor; hence the sets of optimal solutions are the same. This is also the situation in the second part of the kth iteration in which the step length is determined, as it makes no difference whether the function $-G$ or the function $-\log G$ is minimized. Thus if we obtain a sequence $z_1, z_2, \ldots$ by the procedure applied to Problem (3.8), this sequence is also appropriate for Problem (2.13). Hence in case of a finite sequence the last vector simultaneously minimizes the functions $-\log G$ and $-G$, and in case of an infinite sequence

$$\lim_{k\to\infty}(-G(z_k)) = \min_{z\in K}(-G(z)). \tag{3.11}$$

But, in either case, since there exists a $y \in K$ for which $G(y) > p$ holds, after a finite number of steps we must reach a vector lying in the set L. Thus the theorem is proved. □

4 Evaluation of the Gradient of the Non-linear Constraining Function

In both the first and second phases of the procedure applied to the solution of Problem (2.1) we need values of the gradient of the function $G(x)$. In addition we need the function values $G(x)$. We return to the problem of determination of these latter values in Section 6. We show here that the method we use to determine values of $G(x)$ is essentially suitable for the determination of values of $\nabla G(x)$. We shall see that while the evaluation of $G(x)$ requires the evaluation of the distribution function of the m-dimensional normal distribution, the evaluation of $\nabla G(x)$ requires the evaluation of the distribution function of the $(m-1)$-dimensional normal distribution.

Denote by $\phi(z;C)$ the probability density function (2.3) and $\Phi(z;C)$ the corre-

sponding probability distribution function. We introduce the following notation.

$$L_i(x) = \frac{1}{\sigma_i}(a_i'x - b_i), \qquad i = 1, \ldots, m$$

$$L(x) = \begin{pmatrix} L_1(x) \\ \vdots \\ L_m(x) \end{pmatrix}. \tag{4.1}$$

The function G can be written now as

$$G(x) = \Phi(L(x); C). \tag{4.2}$$

It is well known in probability theory that if $F(z) = F(z_1, \ldots, z_m)$ is the absolutely continuous joint distribution function of the random variables $\xi_1, \ldots, \xi_m$, then the following relation holds between the conditional distribution function $F(z_1, \ldots, z_m \mid z_1)$ of the random variables $\xi_2, \ldots, \xi_m$ given that $\xi_1 = z_1$ and the partial derivative of the function F with respect to z_1

$$\frac{\partial F(z_1, \ldots, z_m)}{\partial z_1} = F(z_2, \ldots, z_m \mid z_1) f_1(z_1), \tag{4.3}$$

where $f_1(z)$ is the density function of the random variable ξ_1. Similar equalities hold for the derivatives with respect to the other variables. Applying formula (4.3) for the distribution function $\Phi(z; C)$ we obtain

$$\frac{\partial \Phi(z; C)}{\partial z_1} = \Phi(z_2, \ldots, z_m \mid z_1) \phi(z_1), \tag{4.4}$$

where $\phi(z)$ is the density function of the standard normal distribution. It is also well-known that

$$\Phi(z_2, \ldots, z_m \mid z_1) = \Phi\left(\frac{z_2 - r_{12} z_1}{(1 - r_{12}^2)^{1/2}}, \ldots, \frac{z_m - r_{1m} z_1}{(1 - r_{1m}^2)^{1/2}}; S^{(1)} \right), \tag{4.5}$$

where the covariance matrix $S^{(1)}$ has elements

$$S_{ik}^{(1)} = \frac{r_{ik} - r_{i1} r_{k1}}{(1 - r_{i1}^2)^{1/2} (1 - r_{k1}^2)^{1/2}} \qquad i, k = 2, \ldots, m. \tag{4.6}$$

Similar formulae hold if one of the variables $z_2, \ldots, z_m$ plays the role of z_1. The corresponding covariance matrices will be denoted by $S^{(2)}, \ldots, S^{(m)}$.

Now we give a formula for $\nabla G(x)$ in terms of the components of the vectors $a_1, \ldots, a_m$ given by $a_{j1}, \ldots, a_{jn}$, $j = 1, \ldots, m$. We describe the components of this

vector, but to avoid cumbersome notation do not display them in vector form, as follows

$$\Phi\left(\frac{L_2(x)-r_{12}L_1(x)}{(1-r_{12}^2)^{1/2}},\dots,\frac{L_m(x)-r_{1m}L_1(x)}{(1-r_{1m}^2)^{1/2}};S^{(1)}\right)\phi(L_1(x))\frac{a_{1i}}{\sigma_1}+\dots$$

$$+\Phi\left(\frac{L_1(x)-r_{m1}L_m(x)}{(1-r_{m1}^2)^{1/2}},\dots,\frac{L_{m-1}(x)-r_{mm-1}L_m(x)}{(1-r_{mm-1}^2)^{1/2}};S^{(m)}\right)\phi(L_m(x))\frac{a_{mi}}{\sigma_m}, \tag{4.7}$$

$$i=1,\dots,n.$$

The numerical determination of the values of the function ϕ is easy, and an inspection of (4.7) shows that the same method may be used for the determination of $\nabla G(x)$ and $G(x)$.

5 Formulation of the Economic Problem

At the Institute of Economic Planning of the Hungarian National Planning Office a linear programming model was developed as a part of the planning method for the fourth Five-Year Plan. It is a large-scale model comprising the interrelations of physical values and financial processes and has a decomposition structure, i.e. the model contains sectors related to the branches of the national economy. Our underlying deterministic model is the electrical energy sector model of this large-scale model. When formulating our model the remaining sectors were assumed to work at fixed levels. Special features of the electrical energy sector were taken into account such as the long time-lag to production of investments in energy, the substitution possibility of different kinds of fuels, etc. The variables of the model can be classified in the following manner: production of electrical energy by exogenous capacity (completed before the planning period), production of electrical energy by endogenous capacity (to be completed during the planning period), use of various kinds of fuels, import and export of electrical energy in relation to both Rouble and Dollar trade, investment in individual projects (new power stations) and investment variables which take into account the different financial resources, purposes and manners of use. The constraints of the model contain manpower balances, constraints on investment, foreign trade balances, balance of the state budget, constraints containing the demand for electrical energy and other financial constraints.

Numerical data for the model were obtained from the large scale model mentioned above, from coordination with works in planning and from official statistics. Since the plan indices are obtained by forecasting, we face uncertainty, so the formulation of a stochastic model is reasonable. We assumed that uncertainty in the statistical data is negligible relative to that of the plan indices. Having analysed their economic interpretation, four constraints of the deterministic model were regarded stochastic, i.e. $m=4$ in Model (2.1).

We now outline the interpretation of these four constraints and their right hand side values in the deterministic model. The value b_1 is the planned deficit of foreign trade in Roubles and the first constraint prescribes that this deficit should not exceed a certain planned level. The value b_2 and the corresponding constraint have the same interpretation for dollar trade. The underlying deterministic version of the third and fourth stochastic constraints express the relations between the electrical energy sector and the other sectors. The third constraint is essentially the row of the input-output table corresponding to the electrical energy sector. This constraint includes the requirement for the electrical energy sector which assures equilibrium of input and output of the national gross product in value terms. The value b_3 is equal to the sum of inputs of all productive sectors except electrical energy sector. Finally, the right hand side value b_4 is equal to the cumulative minimum demand for electrical energy in the productive sectors (except electrical energy) and the nonproductive sectors. The corresponding constraint is the product balance of electrical energy in natural units. The right hand side values of the stochastic constraints in the stochastic programming model are $b_1+\sigma_1\beta_1$, $b_2+\sigma_2\beta 2$, $b_3+\sigma_3\beta_3$, $b_4+\sigma_4\beta_4$. The joint distribution of these random variables was supposed to be normal. Its parameters are given in Section 7.

6 Brief Description of the Computer Program

The detailed description of the computer program of Model (2.1) can be found in [2]. Here we outline only the most important features.

The main problem was the numerical determination of the values of the joint distribution function Φ of the random variables $\beta_1, \ldots, \beta_m$. We recall that the determination of these values and of the gradient values is essentially the same problem (see (4.2) and (4.7)). The determination of the values of the function Φ is carried out by a subroutine using a modified Monte Carlo integration technique. We approximate the integral of the function $\phi(z;C)$ over the set $\{z \mid z \leq u\}$ in the following way. First we choose a vector u_0 so that the integral over the set $\{z \mid u_0 \leq z \leq u\}$ should be very near the integral over the former set. Then we choose uniformly distributed random points from the set $\{z \mid u_0 \leq z \leq u\}$ and take the arithmetic mean of the function values belonging to these points. It is possible to determine the number of random points resulting in a required precision, i.e. so that the relative error does not exceed a prescribed level. In the special model concerning us we deal with the four-dimensional normal distribution. The computation of one value of the distribution function takes 0.5 s when the prescribed upper bound for the relative error is 5%.

Special attention should be paid to the determination of the step length at each iteration of the nonlinear programming procedure of Section 2. Our procedure for

the evaluation of the distribution function of the normal distribution (in common with any other available procedure) does not give exact results. Only the expectations of the estimations are equal to the true values. This fact is very inconvenient when determining the feasible part of the ray $\{x_k + \lambda(x_k^* - x_k) \mid \lambda \geq 0\}$, i.e. the intersection of this ray with the boundary of the set of feasible solutions. We employ an iterative procedure which moves backwards and forwards along the ray and successively reduces the length between evaluation points. In this way we are lead near to the required intersection point.

We had to agree on an "optimality criterion". A criterion formulated only in terms of the value y_{opt} may have been satisfactory, but we thought it more suitable to formulate the following more rigorous stopping rule. The vector x_k is considered optimal if the difference between the values of the objective function at x_{k+1} and x_k does not exceed 1% of the latter and at the same time each individual component of $x_{k+1} - x_k$ does not exceed 2% of the corresponding component of x_k.

The computations were executed on a CDC 3300 computer of the Hungarian Academy of Sciences. The program consists of a main program and six overlays. Five overlays concern the simplex method, the sixth one contains the algorithm described in Section 2 and the evaluation procedure of the values of the multivariate normal distribution function.

The problem was solved in two very similar runs; the first executed the first phase of the procedure and the other executed its second phase.

7 Numerical Data and Results

The special model is of the following form

$$\begin{aligned} G(x) = P(a_i'x \geq b_i + \sigma_i\beta_i, \quad i = 1,2,3,4) \geq p \\ a_i'x \geq b_i, \qquad i = 5,\ldots,110, \\ \min c'x. \end{aligned} \tag{7.1}$$

The linear functions on the left hand sides of the stochastic constraints are specialized as follows:

$$\begin{aligned} a_1'x &= -25x_{25}, \\ a_2'x &= -16.67x_{26} \\ a_3'x &= 0.8696x_{24} + x_{40}, \\ a_4'x &= 0.9(x_1 + x_2 + x_6 + x_7) - 0.115x_{24}. \end{aligned} \tag{7.2}$$

Here x_1 and x_2 are the production of electrical energy by exogenous and endogenous capacity respectively, x_6 and x_7 are the imports of electrical energy in Rouble and

Dollar trade respectively, x_{24} is the value of the production of the electrical energy sector, x_{25} and x_{26} are the values of the imports of the electrical energy sector in Rouble and Dollar trade respectively, and x_{40} is the total value of the imports of the electrical energy sector.

The expectations b_1, b_2, b_3, b_4 and the standard deviations σ_1, σ_2, σ_3, σ_4 on the right hand sides of the stochastic constraints are the following

$$\begin{aligned} b_1 &= -48313, & \sigma_1 &= 483, \\ b_2 &= \quad -426, & \sigma_2 &= \quad 4, \\ b_3 &= \quad 16000, & \sigma_3 &= 160, \\ b_4 &= \quad 19000, & \sigma_4 &= 195. \end{aligned} \tag{7.3}$$

The expectations of the random variables β_1, β_2, β_3 and β_4 are equal to zero, their dispersions are equal to 1 and their covariance matrix is the following

$$C = \begin{pmatrix} 1 & -0.8 & 0.4 & 0.4 \\ -0.8 & 1 & 0.1 & 0.1 \\ 0.4 & 0.1 & 1 & 0.9 \\ 0.4 & 0.1 & 0.9 & 1 \end{pmatrix} \tag{7.4}$$

The linear functions in the second constraint block of Problem (7.1) are divided into two groups. The first group contains the linear constraints having subscripts $i = 5, \ldots, 52$; the second group contains the remaining linear constraints with subscripts $i = 53, \ldots, 110$. The latter are lower and upper bounds for some variables and nonnegativity constraints for others. These and further numerical data for Problem (7.1) can be found in [9] (under 7.5 and in Appendices 3 and 4).

The objective function, which is the profit multiplied by -1, is given by

$$c'x = x_{35} - x_{36}, \tag{7.5}$$

where x_{35} is the increase in the sector wage bill and x_{36} is enterprise profit for the sector before taxation. The prescribed probability level is $p = 0.9$

First phase. We used x_{lin}, the optimal solution of the deterministic underlying problem as the initial feasible solution. We computed the value of the function G corresponding to this vector and obtained

$$G(x_{\text{lin}}) = 0.09. \tag{7.6}$$

So the optimal solution x_{lin} guarantees only a very low reliability level for the system. Next we maximized the function $G(x)$ under the linear constraints of Problem (7.1).

Five iterations were performed and the following numbers were obtained as values of the function $G(x)$ (the first corresponds to the vector x_{lin}):

$$0.09; \quad 0.13; \quad 0.72; \quad 0.90; \quad 0.94; \quad 0.97.$$

The first phase could have been terminated upon attaining a probability greater than 0.9, so that four iterations would have been enough. However, we were interested in seeing how high a probability level could be achieved under the linear constraints of Problem (7.1). We interrupted the computations at the value 0.97 because it had already been demonstrated that this maximal probability level is very high. The first phase was executed in 19 min, consisting of 8 min 49 s of computing time and 5 min 19 s of channel time.

Second phase. As starting vector we used the final vector obtained in the first phase for which the value of the function G equals 0.97. The program ran 46 min, of which the computing time was 25 min and channel time was 12 min.

Detailed description of the 46 minutes total running time of the Second Phase[†]

Preparing the data for the simplex method	3 min 10 s
Running the simplex method	30 min 39 s
Finding the step length	6 min 34 s
Checking the optimality criteria and other computations	2 min

The optimization was performed in nine steps. The values of the objective function at each iteration were the following

$$\begin{array}{lllll} -4033; & -4101; & -4366.9 & -4367; & -4367.32; \\ -4367.48; & -4367.84; & -4367.9; & -4369.71; & -4369.86. \end{array}$$

It is surprising that at the optimal solution x_{stoch} of the stochastic programming model the value of the objective function is equal to the value of the objective function at x_{lin}. As for the values of the function G, we have $G(x_{\text{stoch}}) = 0.9$ and we recall that $G(x_{\text{lin}}) = 0.09$. Thus it is possible to achieve the same objective by a vector representing a considerably greater reliability level than x_{lin}. This phenomenon is remarkable apart from the special context of the numerical model.

Below we give those components of the vectors x_{lin} and x_{stoch} whose relative differences are more than 10%.

[†]Since the preparation of this paper much better running times have been achieved.

Component subscript	x_{lin}	$x_{\text{stoch}} = x^{(1)}_{\text{stoch}}$ $(p = 0.9)$
20	0	1233.9
21	994	13.7
22	1950	714.4
23	517	1586.2
43	2370	1655.8
46	2407	1007.3

The economic meanings of these components are described below.

Economic meanings of components of the optimal solutions involved in this section.

Component subscript	Meaning of the component
5	Production of electrical energy by endogenous capacity in nuclear power stations.
10	Individual investment project "Dunamenti II" to be completed in 1977.
12	Individual investment project of a nuclear power station at Paks, to be completed in 1978.
13	Individual investment project of an oil operated power station, to be completed in 1978.
20	Consumption of natural gas in exogenous power stations.
21	Consumption of fuel oil in exogenous power stations.
22	Consumption of natural gas in endogenous power stations.
23	Consumption of fuel oil in endogenous power stations.
41	Total machinery requirement of investments in the electrical energy sector.
43	Investment surplus in 1972.
45	Investment surplus in 1974.
46	Investment surplus in 1975.

The complete computer program was run with two more data-sets as an experiment. In (a) and (b) we give a brief account of the results obtained.

(a) Except for the probability p the data remained unchanged. The new value was somewhat greater than the old namely $p = 0.95$. The value of the objective function corresponding to the optimal solution of the stochastic programming model is -4365.8. Thus there is no great difference relative to the former optimal value of the objective function. The optimal solution differs considerably from both former vectors, x_{lin} and $x^{(1)}_{\text{stoch}}$. Here we list those components in which the relative is at least 10%.

Component subscript	x_{lin}	$x^{(2)}_{\text{stoch}}$ $(p = 0.95)$
5	0	0.353
10	0.1	0.003
13	0.37	0.44
20	0	11.1
21	994	1080
41	830	938
46	2407	1.24

(b) In this case the reliability level is again $p = 0.9$ and except for the correlation matrix the data remained unchanged. We have the following new correlation matrix

$$C_1 = \begin{pmatrix} 1 & -0.7 & 0.3 & 0.3 \\ -0.7 & 1 & 0.1 & 0.1 \\ 0.3 & 0.1 & 1 & 0.9 \\ 0.3 & 0.1 & 0.9 & 1 \end{pmatrix} \tag{7.7}$$

The optimal value of the objective function is -4292 which differs considerably from the former optimal values of the objective function. The components in which the relative deviation exceeds 10% are the following:

Component subscript	x_{lin}	$x^{(3)}_{\text{stoch}}$ $(p = 0.9, C_1)$
5	0	10.1
12	0.23	0.11
13	0.37	0.5
20	0	341
21	994	879
22	1950	1602
23	524	720
41	830	930
45	2962	2429
46	2407	38

The detailed economic analysis of the vectors x_{lin}, $x^{(1)}_{\text{stoch}}$, $x^{(2)}_{\text{stoch}}$, $x^{(3)}_{\text{stoch}}$ is not our aim in this paper. The only thing that we mention is that the plans with high reliability levels propose the consumption of more coal and less fuel oil in endogenous power stations.

References

[1] Bager, F., Morva, T. and Szabó, L. (1971). A model of the 4th five-year plan, which comprises the natural, value and financial processes. *Országos Tervhivatal, Tervgazdasági Intézetének Közleményei* **6**, 55–130 (in Hungarian).

[2] Deák, I. (1972). Computer evaluation of a stochastic programming model. *MTA Számítástechnikai Központ Közleményei* **9**, 33–49 (in Hungarian).

[3] Ganczer, S. (ed.) (1973). "National Economic Planning and Programming". Publishing House of Economics and Law, Budapest (in Hungarian).

[4] Prékopa, A. (1970). On probabilistic constrained programming. In "Proceedings of the Princeton Symposium on Mathematical Programming". Princeton University Press, Princeton, New Jersey, 113–138.

[5] Prékopa, A. (1970). "On Optimization Problems of Stochastic Systems". Hungarian Academy of Sciences, Budapest (in Hungarian).

[6] Prékopa, A. (1971). Logarithmic concave measures with applications to stochastic programming. *Acta Sci. Math.* **32**, 301–316.

[7] Prékopa, A. (1972). Extension of the method of feasible directions for quasiconcave constraining functions. *MTA Számítástechnikai Központ Közleményei* **9**, 3–16 (in Hungarian). *Math. Operationsforsch. Statist.* **5**, 281–293 (in German) (1974).

[8] Prékopa, A. (1973). Contributions to the theory of stochastic programming. *Math. Programming* **4**, 202–221.

[9] Prékopa, A., Ganczer, S., Deák, I. and Patyi, K. (1975). The STABIL stochastic programming model and its experimental application to the electrical energy sector of the Hungarian economy. *Alkalmazott Matematikai Lapok* **1**, 3–22 (in Hungarian).

[10] Zoutendijk, G. (1960). "Methods of Feasible Directions". Elsevier Publishing Co., Amstardam and New York.