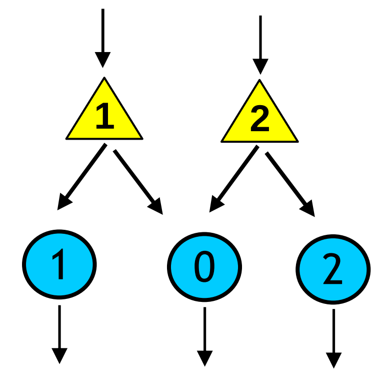
A Stochastic Programming Based Approach to Assemble-to-Order Inventory Systems



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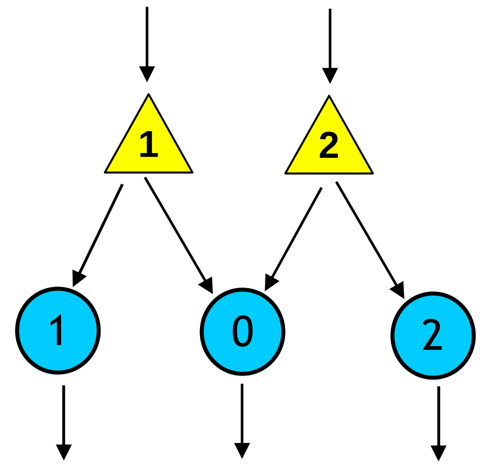
(Based on joint work with Mustafa Dogru and Qiong Wang)

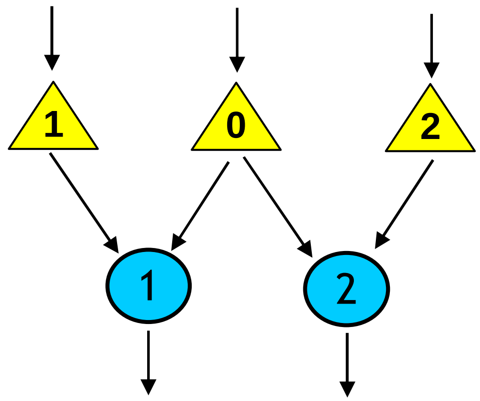
Talk Outline

- The Assemble-to-Order (ATO) inventory system
- The inventory control problem
- The stochastic programming (SP) based approach
- An SP lower bound for identical lead times
- An SP lower bound for non-identical lead times
- SP solution for W model & translation into control policy
- Some examples where SP yields optimal policy
- SP solution for M model & translation into control policy

The Assemble-to-Order (ATO) Inventory System

- *m* products assembled from *n* components
- Product i requires a_{ij} units of component j
- Procurement lead time *of* L_j for component j *(suppliers are uncapacitated)*
- Stochastic demand for products (i.i.d. or compound Poisson)
- Assembly time is negligible
- Only component inventories are kept
- Backlogging: unit backlog cost b_i
- Unit inventory holding cost h_j





Goal: Find replenishment & allocation policies to minimize the long run average expected cost

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The (Continuous Review) Demand Process

Let

$$\mathcal{D}_i(t) = \text{demand for product } i \text{ during } [0,t],\ 1 \le i \le m,$$

and $\mathcal{D}(t) \equiv (\mathcal{D}_1(t), \ldots, \mathcal{D}_m(t))$.

Assume that $\{\mathcal{D}(t), t \ge 0\}$ is a compound Poisson process.

Let

$$\Delta_i \equiv \mathsf{E}[(\mathcal{D}_i(1)],\ 1 \le i \le m, \quad \Delta \equiv (\Delta_1, \ldots, \Delta_m),$$

and

$$\sigma_{ij}^2 \equiv \mathsf{E}[(\mathcal{D}_i(1) - \Delta_i)(\mathcal{D}_j(1) - \Delta_j)].$$

Assume that

$$0 < \Delta_i < \infty \text{ and } 0 < \sigma_{ii}^2 < \infty, 1 \le i \le m.$$

The (Continuous Review) ATO Inventory Control Problem

- We want to choose a replenishment policy γ and an allocation policy p to minimize

$$C^{\gamma,p} \equiv \limsup_{T\to\infty} \frac{1}{T}\mathsf{E}\left[\int_0^T \left\{\sum_{i=1}^m b_i B_i(t) + \sum_{j=1}^n h_j I_j(t)\right\} dt\right]$$

where $\mathbf{B}_i$(t) is the product i backlog at time t and $\mathbf{I}_j$(t) is the component j inventory at time t.

- Feasible policy:

1. $B_i(t) \geq 0$
2. $I_j(t) \geq 0$
3. Decisions cannot be based on future demand information

Some Prior Efforts on ATO

- Due to the need to keep track of the items in the 'pipeline', formulating the ATO inventory control problem as a Markov decision process (MDP) would require an enormous number of states.

- 1 Period ATO models: Baker, Magazine and Nuttle (1986), Gerchak, Magazine and Gamble (1988), **Song and Zipkin (2003)**

- Periodic review ATO: Zhang (1997), Hausman et. al. (1998), Agrawal and Cohen (2001) , Akcay and Xu (2004)

- Continuous review ATO: Song (1998, 2002), Song, Xu and Liu (1999), Song and Yao (2002), Lu, Song and Yao (2003), **Lu and Song (2005),** Song and Zhao (2009), Lu, Song and Zhao(2010)

- The exact solution in the multi-period setting has been developed only for a variant of first-come-first-served allocation

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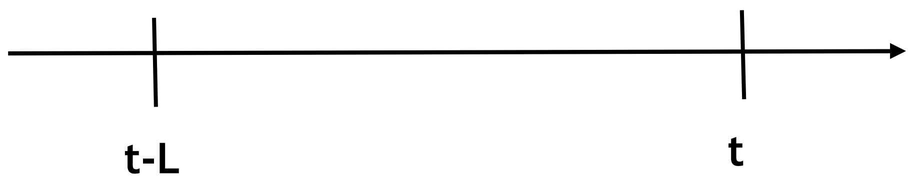
The Stochastic Programming Based Approach

1. Introduce a stochastic (linear) program (with complete recourse) whose solution provides a lower bound on the achievable cost in the inventory control system
2. Solve the stochastic program (SP)
3. Translate the SP solution into a control policy for the inventory system
4. Prove asymptotic optimality

This is similar to the approach introduced in Harrison (1988) for ‘stochastic processing networks’ in heavy traffic and has led to many insightful results. (But: There are no capacitated resources in this model, so there is no notion of traffic intensity.)

The Idea Behind the Stochastic Program Lower Bound



- Assume that L_i=L for all i
- Focus (myopically) on the cost rate at time t. This is affected by replenishment decisions made up to time t-L, and allocation decisions made up to time t.
- For the lower bound:
  - Assume no inventory and an empty pipeline at time t-L. (This is without loss of optimality.)
  - Assume backlogs $B_i(t-L) = \alpha_i \geq 0, 1 \leq i \leq m.$
  - Place order for components at time t-L to show up at time t.
  - After observing $\bar{\mathrm{D}} = \mathcal{D}(t) - \mathcal{D}(t-L)$ make allocations at t to (myopically) minimize the cost rate at time t.

The Lower Bound Stochastic Program for Identical Lead Times

Let $\bar{\mathbf{D}} \stackrel{d}{=} \mathcal{D}(L)$ $(\stackrel{d}{=} \mathcal{D}(t) - \mathcal{D}(t-L))$ denote the demand over a lead time. Let $c_i \equiv b_i + \sum_{j=1}^{n} a_{ij} h_j$.

For $\mathbf{y} \in \mathbf{R}_+^n$ and $\mathbf{x} \in \mathbf{R}_+^m$, let

Let

$$\Phi^0(\mathrm{y}, \mathrm{x}) = -\max_{\mathrm{z} \geq 0} \{\mathrm{c}'\mathrm{z} | \mathrm{z} \leq \mathrm{x},\ A'\mathrm{z} \leq \mathrm{y}'\},$$

and

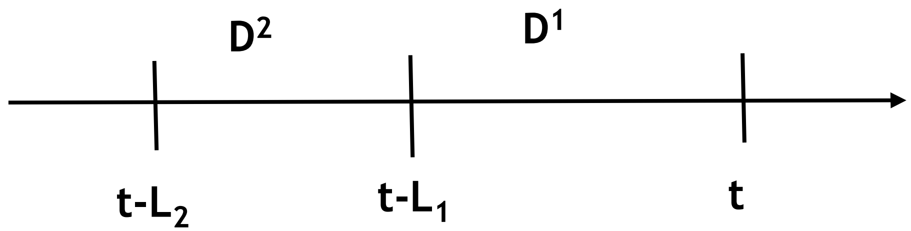
$$\Phi^1(\mathrm{x}) = \inf_{\mathrm{y} \geq 0} \left\{ \mathrm{h}'\mathrm{y} + \mathbf{E}[\Phi^0(\mathrm{y}, \mathrm{x} + \bar{\mathrm{D}})] \right\}.$$

Theorem (Dogru, R., Wang, 2010). Let (γ, p) be any feasible policy, and $C^{\gamma,p}$ be the corresponding cost. Then

$$C^{\gamma,p} \geq \inf_{\alpha \geq 0} \left\{ \mathrm{b}'(\mathbf{E}[\bar{\mathrm{D}}] + \alpha) + \Phi^1(\alpha) \right\}.$$

Two Distinct Lead Times

Suppose that $0 < L_1 < L_2 < \infty$, $1 \le n_1 < n$, and components $1, \ldots, n_1$ have lead time L_1 while components $n_1 + 1, \ldots, n$ have lead time L_2.



- Focus (myopically) on the cost rate at time t.
- For the lower bound:
  - Assume no inventory and empty pipeline at time t-L_2. (This is without loss of optimality.)
  - Assume backlogs $B_i(t - L_2) = \alpha_i \ge 0,\ 1 \le i \le m.$
  - Place order for long lead time components at time t-L_2
  - After observing $D^2 = \mathcal{D}(t - L_1) - \mathcal{D}(t - L_2)$ place order for short lead time components at time t-L_2
  - After observing $D^1 = \mathcal{D}(t) - \mathcal{D}(t - L_1)$ make allocations at t to (myopically) minimize the cost rate at time t.

The Lower Bound SP for Two Distinct Lead Times

Let $\mathbf{h}^1 = (h_1, \ldots, h_{n_1})$ and $\mathbf{h}^2 = (h_{n_1+1}, \ldots, h_n)$.

Let $\mathbf{D}^1$ and $\mathbf{D}^2$ be independent, with $\mathbf{D}^1 \stackrel{d}{=} \mathcal{D}(L_2) - \mathcal{D}(L_2 - L_1)$, $\mathbf{D}^2 \stackrel{d}{=} \mathcal{D}(L_2 - L_1)$, and $\bar{\mathbf{D}} = \mathbf{D}^1 + \mathbf{D}^2$.

Let $\mathbf{y}^1 \in \mathbf{R}_+^{n_1}$, $\mathbf{y}^2 \in \mathbf{R}_+^{n-n_1}$ and $\mathbf{x} \in \mathbf{R}_+^m$.

Let

$$\Phi^0(\mathbf{y}^1, \mathbf{y}^2, \mathbf{x}) = -\max_{\mathbf{z} \geq 0}\{\mathbf{c}'\mathbf{z} | \mathbf{z} \leq \mathbf{x},\ A'\mathbf{z} \leq (\mathbf{y}^1, \mathbf{y}^2)'\},$$

$$\Phi^1(\mathbf{y}^2, \mathbf{x}) = \inf_{\mathbf{y}^1 \geq 0}\left\{(\mathbf{h}^1)'\mathbf{y}^1 + \mathbf{E}[\Phi^0(\mathbf{y}^1, \mathbf{y}^2, \mathbf{x} + \mathbf{D}^1)]\right\},$$

and

$$\Phi^2(\mathbf{x}) = \inf_{\mathbf{y}^2 \geq 0}\left\{(\mathbf{h}^2)'\mathbf{y}^2 + \mathbf{E}[\Phi^1(\mathbf{y}^2, \mathbf{x} + \mathbf{D}^2)]\right\}.$$

Theorem (R. and Wang, 2011). Let (γ, p) be a feasible policy and $C^{\gamma,p}$ be the corresponding cost. Then

$$C^{\gamma,p} \geq \inf_{\alpha \geq 0}\left\{\mathbf{b}'(\mathbf{E}[\bar{\mathbf{D}}] + \alpha) + \Phi^2(\alpha)\right\}.$$

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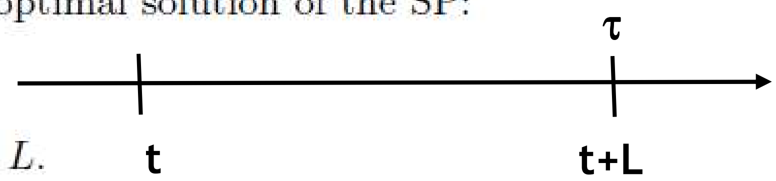
Replenishment Policy

Construct a replenishment policy that 'mimics' the optimal solution of the SP:

With $K = 1$, at time t, solve $\Phi^1(\mathbf{B}(t))$.

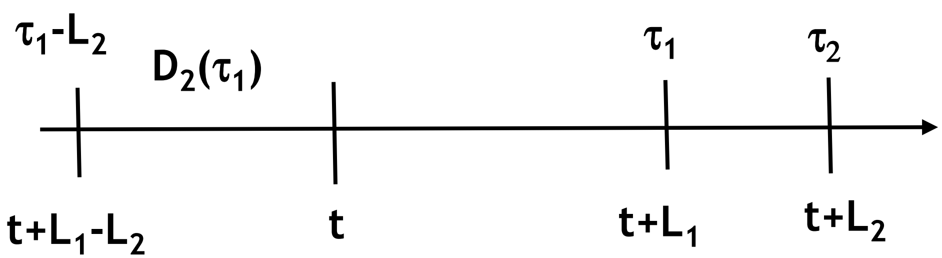


Denote the optimal solution by $\mathbf{y}^*(\tau)$, where $\tau = t + L$.

Let $\mathbf{Y}(t)$ = inventory level (on-hand + in-transit) of parts at t.

At time t order $[\mathbf{y}^*(\tau) - \mathbf{Y}(t-)]^+$.

Base-stock policy: solve $\Phi^1(0)$, and denote the solution by $\mathbf{y}^*$. At time t, order $[\mathbf{y}^* + A'\mathbf{B}(t) - \mathbf{Y}(t-)]^+$.



With $K = 2$, at time t, solve $\Phi^2(\mathbf{B}(t))$ and $\Phi^1(\mathbf{y}_2^*(\tau_1), \mathbf{B}(\tau_1 - L_2) + \mathbf{D}^2(\tau_1))$, where $\tau_1 = t + L_1$

and $\mathbf{y}_2^*(\tau_1)$ is an optimal solution of Φ^2 at $t = \tau_1 - L_2 = t + L_1 - L_2$.

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Solution of SP and Translation into Control Policy for W Model

- The SP for the W Model can be solved exactly. (There is no need for sampling.)
- Use a base stock policy for replenishment

Here $c_1 = b_1 + h_0 + h_1$ and $c_2 = b_2 + h_0 + h_2$.

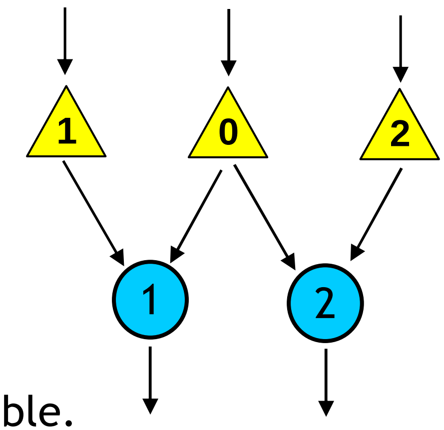
Given $y \geq 0$ the recourse LP for the W system is

$$\max_{z \geq 0} \left\{ c_1 z_1 + c_2 z_2 \middle| z_i \leq D_i,\ z_i \leq y_i, i = 1, 2,\ z_1 + z_2 \leq y_0 \right\}.$$

Assume (without loss of generality) that $c_1 \geq c_2$.

The *recourse* LP has solution

$$z_1^* = D_1 \wedge y_1, \quad z_2^* = D_2 \wedge y_2 \wedge (y_0 - z_1^*).$$



This motivates priority to product 1 in allocation:

- A demand is served as long as all required components are available.
- If both products have backlogs due to the lack of the common component, when a replenishment arrives, then all product 1 backlogs are cleared first (as long as there are enough unique components) before serving product 2 demand

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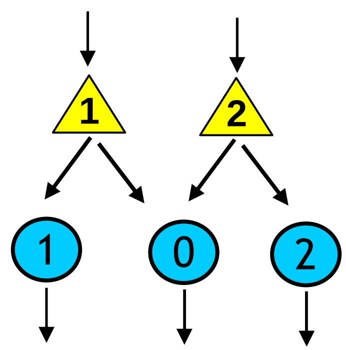
Sometimes the SP-Based Policy is Optimal

- We have developed sufficient conditions under which the SP-based policy is optimal.
- Some examples:

1. 1 product system. (The optimal policy was already known.)
2. Generalized W (to more than 2 products), where all c_i are equal, the common part has lead time L_1, all unique parts have lead time L_2, and $L_1 \leq L_2$.
3. W system with identical lead times where $\Phi^1(0)$ has an optimal solution with $y_0^* = y_1^* + y_2^*$

Solution of SP and Translation into Control Policy for M Model

- The SP for the M Model can be solved exactly. (There is no need for sampling.)
- Use a base-stock policy for replenishment.
- There are 4 cost parameter regions, with different recourse LP solutions.
- In one region the recourse LP solution motivates a priority policy for allocation.
- In another region the recourse LP solution motivates a state-dependent priority policy for allocation.



Some Next Steps/ Open Problems

- Prove equivalence of an alternate SP, which has fewer variables and attains minimum. (The lower bound SP has an infimum.)
- Reduce the frequency of re-solving the SP without sacrificing too much in cost.
- Examine the asymptotic (as L grows large) behavior of the SP and ATO inventory system.
  - **Conjecture:** The SP provides an asymptotically optimal inventory control policy