Some Issues in Stochastic Variational Problems

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Dedicated to Professor András Prékopa on his 80th Birthday
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Outline of Topics

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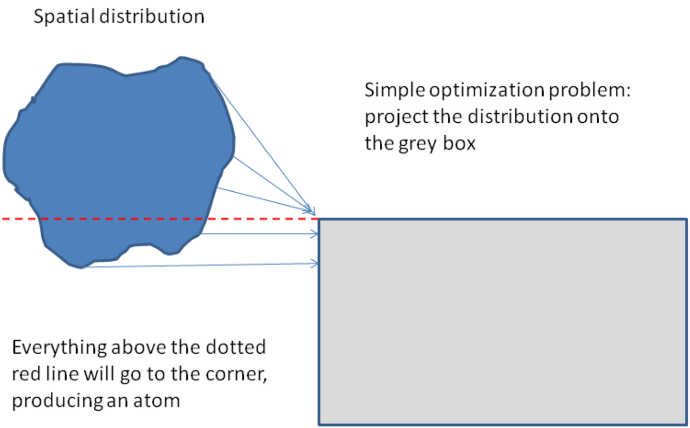
- At the end, suggest how these tools can be used for analysis of stochastic variational problems
- And encourage those who know more than I do to explore some of those possibilities

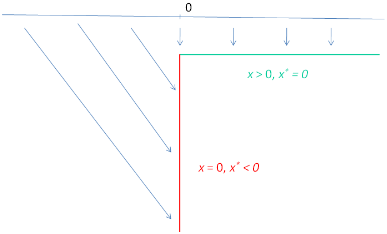
Example: Projecting a Cloud



(King & Rockafellar, 1993)

Simpler Example: What's Happening?

Projecting from the real line to a halfline



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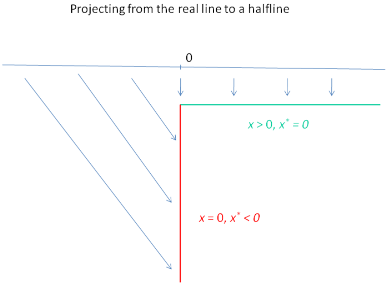

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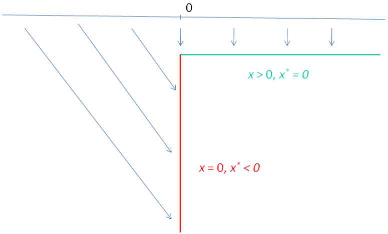

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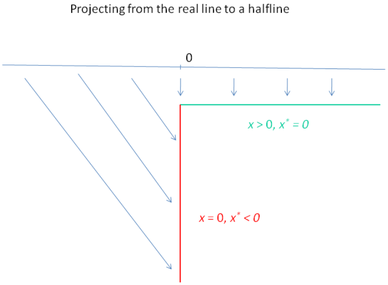

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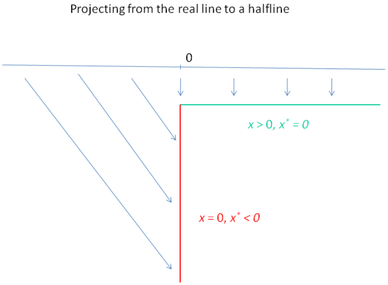

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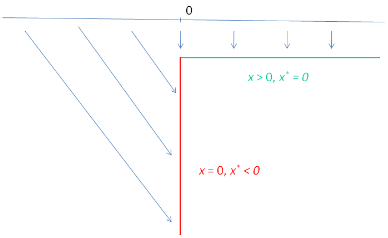

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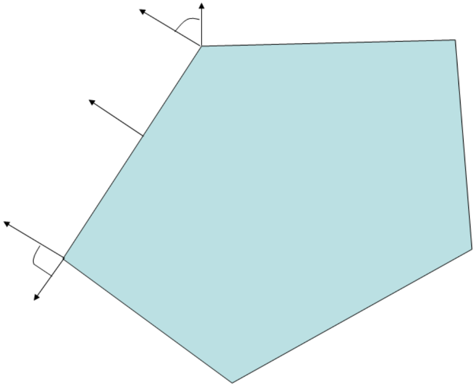
- That's what caused our difficulty with the cloud

Some Perspective

- First, this projection problem is part of a general class of *variational conditions.*
- In this case, it's a *variational inequality:* we have a closed convex set C (here, $\mathbb{R}^1_+$) and a function $f : \mathbb{R}^1 \to \mathbb{R}^1$ (here, $f(x) = x - r$), and we want to find a point x_* such that $-f(x_*)$ is (outwardly) normal to C.
- A convenient way to express this: find a solution of $0 \in f(x) + N_C(x)$, where

$$N_C(x) = \begin{cases} \{x^* \mid \text{For each } c \in C,\ \langle c - x, x^* \rangle \leq 0\} & \text{if } x \in C, \\ \emptyset & \text{if } x \notin C. \end{cases}$$

Normal Cones Illustrated



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- It does: in fact, it has a very rich structure

This formulation includes a wide class of problems

- First-order optimality conditions for nonlinear programming problems (with multipliers, if needed, to accommodate nonlinear constraints)
- Linear and nonlinear complementarity problems
- Traffic equilibrium problems
- Stationarity conditions for other Nash equilibrium problems, including those from some games
- Equilibrium problems from computational economics

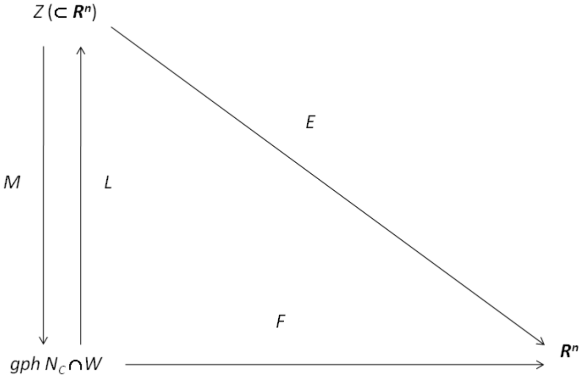
The first difficulty

- Again:

$$F(x, x^*) = f(x) + x^* = 0, \qquad (x, x^*) \in \operatorname{gph} N_C$$

- We seem to have too many variables to solve for (x, x^*): we're sending $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$
- But there's another constraint: $(x, x^*) \in \operatorname{gph} N_C$
- We have to combine these in order to do the analysis
- Next slide gives a picture of the combination

Lifting F to a nonsmooth function E



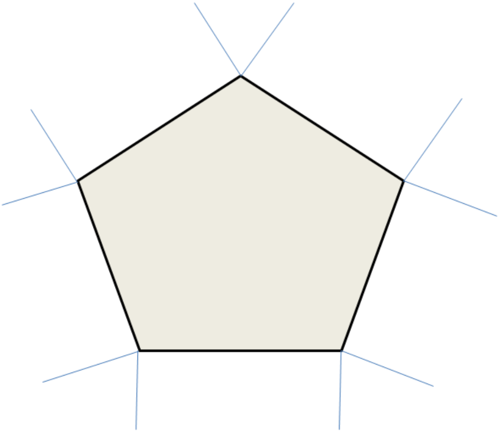
Finding Lipschitz Homeomorphisms L and M

- For $z \in \mathbb{R}^n$ let $\Pi_C(z)$ be the Euclidean projector on C, and define

$$M(z) = [\Pi_C(z), I - \Pi_C(z)], \qquad L(x, x^*) = x + x^*$$

- Minty's theorem says M is a Lipschitz homeomorphism of $\mathbb{R}^n$ onto gph N_C, with inverse L
- When C is polyhedral convex, the map M is *piecewise affine*
- The subsets on which M is affine form a polyhedral subdivision of $\mathbb{R}^n$ called the *normal manifold*

The normal manifold of a pentagon



Summary of the Formulation

- Start with the given function F and set C; fix $w_0 = (x_0, x_0^*) \in \operatorname{gph} N_C$ with $F(x_0, x_0^*) = 0$
- Construct the Lipschitz homeomorphisms L and M
- Construct $E = (F \circ M) : Z \to \mathbb{R}^n$, which is the map we will analyze
- As M will usually be nonsmooth, so will be E, even if F is smooth. This is the price we pay for dealing with nasty graphs

Well-behaved equations

- In ordinary analysis, a nice function f from an open subset of $\mathbb{R}^n$ into $\mathbb{R}^n$ could be a (local) homeomorphism: f and f^{-1} are both (locally) single-valued and continuous
- Even better, it could be a Lipschitz homeomorphism: f and f^{-1} each obey a Lipschitz condition
- With C^1 functions from $\mathbb{R}^n$ to $\mathbb{R}^n$, the inverse function theorem says we have a local Lipschitz homeomorphism at x_0 when the derivative $df(x_0)$ is nonsingular
- This theorem is the foundation for local analysis of C^1 functions, with innumerable applications

The situation with variational problems

- In nonlinear programming, complementarity, or other equilibrium problems we typically *do not* have anything like this, even with very nice problems (e.g., the cloud, or the projection problem in $\mathbb{R}^1$)

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- This is a strong argument for looking at variational problems in this way, rather than in the traditional way

Nonsingularity for a piecewise affine function

- When is a piecewise affine function from a normal manifold $\mathcal{N}_C$ for a polyhedral convex $C \subset \mathbb{R}^n$ to $\mathbb{R}^n$ a Lipschitz homeomorphism?
- On each n-cell of the manifold, the function has an affine representative; the linear part of that affine function has a determinant
- f is a Lipschitz homeomorphism *if and only if* those determinants all have the same nonzero sign (so that f is *coherently oriented*)
- This extends the classical case, in which there is just one n-cell ($\mathbb{R}^n$)

What about nonlinear problems?

- Just as in the classical case, a problem with a C^1 function is a local Lipschitz homeomorphism if and only if the linearized problem has that property
- We linearize $f(x) + x^*$ by passing to the problem $f(x_0) + df(x_0)(x - x_0) + x^*$
- The proof of this nonlinear result is a little harder than the proof for the classical case
- But the proof of the coherent orientation test for piecewise affine problems is *very much* harder than that for the classical (linear) case

This gives us a good set of tools

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  - Do convergence analysis for algorithms,
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- In the final section we'll look at some possible applications to stochastic problems

A stochastic variational problem

- Suppose we have a vector-valued stochastic process $\{f_n(\omega, x) \in \mathbb{R}^m \mid n = 1, 2, \ldots\}$ with the following properties
- For all $n \geq 1$ and $x \in \mathbb{R}^k$, the random variables $f_n(\omega, x)$ are defined on a common probability space $(\Omega, \mathcal{F}, P)$, and for almost all ω the $f_n(\omega, \cdot)$ converge pointwise to a deterministic function $f(\cdot)$
- We look for a point x_0 such that the function f satisfies $0 \in f(x) + N_C(x)$, where C is polyhedral convex
- Motivation: the f_n are estimates obtained by simulation

One approach to solution (Gürkan *et al.*, 1999)

- Fix a large n and a sample point ω
- Solve the deterministic variational inequality with $f(\cdot) = f_n(\omega, \cdot)$
- Take the solution $x_n(\omega)$ as an estimate of x_0
- One can give conditions ensuring that with probability 1, when n is sufficiently large the $x_n(\omega)$ exist and are close to x_0
- This approach has been applied to energy market problems (interruptions in natural gas supply), as well as option pricing and network design, among other areas
- Early versions of some of the results already discussed provided the justification for that analysis

Other examples

- One can use the theory described here in constructing confidence regions for variational problems (Demir, 2000)
- A slightly more comprehensive form provides tools for analyzing the behavior of robust statistical estimators
- There are many other possibilities

The point: in the hands of people more expert than I am, these tools could extend our ability to analyze stochastic problems where variational behavior is a key aspect.