Cube Partitions and Nonrepeating Decision Trees

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Outline

1. Introduction
2. DNF with many prime implicants
3. Cube partitions
    - Neighboring partitions for NUD–k-term DNF
    - General Splitting Problem for Cube Partitions
4. Open Problems

Main Result in a Nutshell

- Exact characterization of the $2^k - 1$ prime implicants of those k-term DNF having $2^k - 1$ prime implicants, which has been known to be the maximum possible since late 1970's.
- Relates to a particular type of decision tree.

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- Relates to a particular type of decision tree.
- Next: One reason it's important—at least to me. Then on to a few definitions and main talk.

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- But now, "Bitte ein Bit!", (Slogan of Bitburger Brauerie in 1951 and again today), Volume 4 is being released in Fascicles, and in Pre-Fascicle 0B, Section 7.1.1, "Boolean Basics" Problem 32 is about this, and solution to 32(b) cites our ECCC Report!

Basic Definitions

- n-dimensional hypercube: $\{0,1\}^n$
- Cube or term: $0 * 1$ or $\bar{x} \wedge z$
- Union of k cubes or k-term DNF: $T_1 \vee \ldots \vee T_k$
- Implicant of $A \subseteq \{0,1\}^n$: cube contained in A
- Prime implicant of $A \subseteq \{0,1\}^n$: maximal cube contained in A

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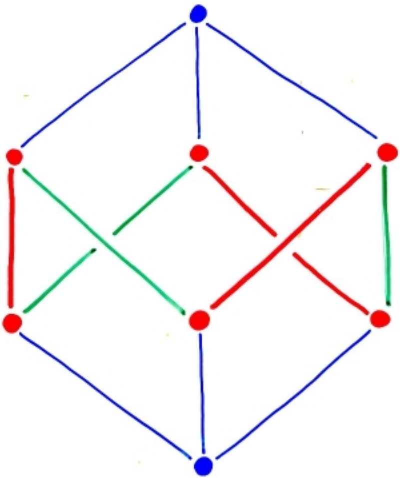
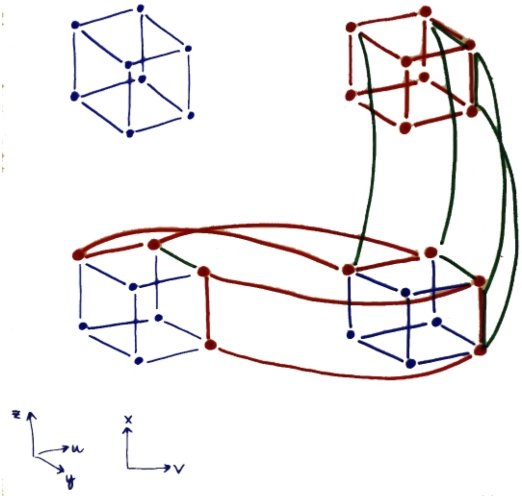
- Chandra, Markowsky (1978)
- Laborde (1980)
- A. A. Levin (1981)
- McMullen, Shearer (1986)

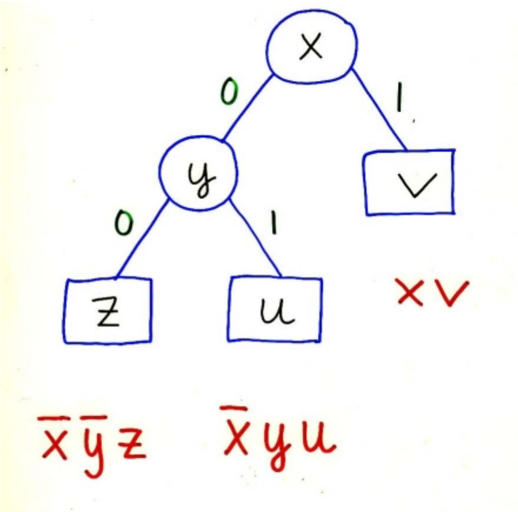
Example: A 3-term DNF with 6 Prime Implicants



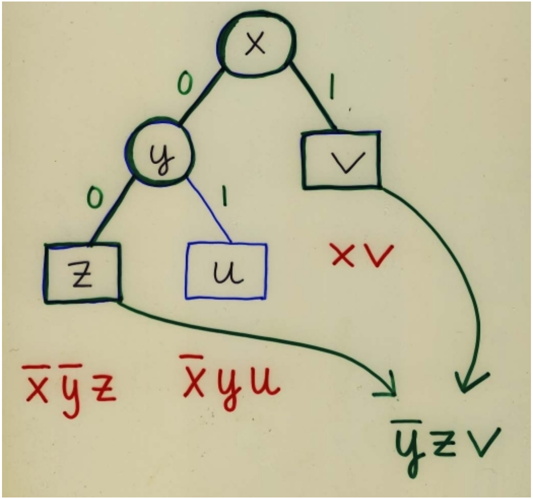
A 3-term DNF with 7 Prime Implicants: $xv \vee u\bar{x}y \vee \bar{x}\bar{y}z$



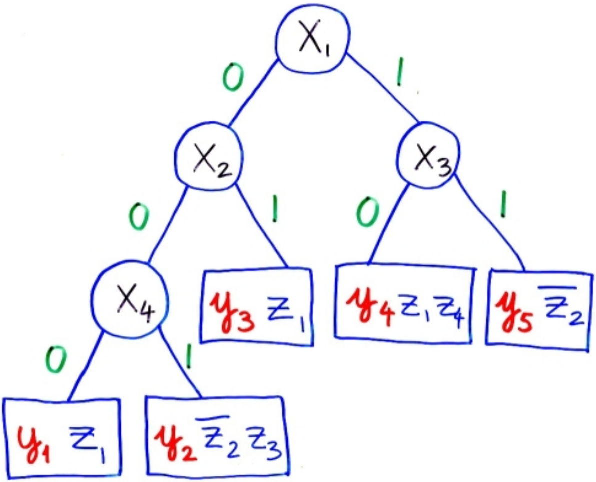
A 3-term DNF with 7 Prime Implicants: Another View



How to Find Prime Implicants



Nonrepeating Unate-leaf Decision Tree (NUD)



k-term DNF with the Largest Number of Prime Implicants

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Theorem

A k-term DNF has $2^k - 1$ prime implicants iff it is a NUD.

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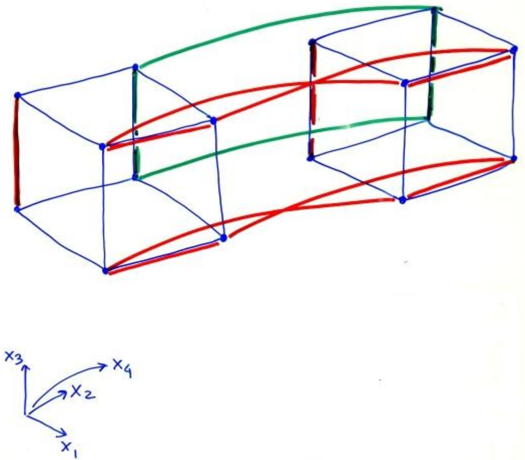
Definitions: Distance of Disjoint Subcubes of $\{0,1\}^n$

- Distance of two cubes: number of conflicting coordinates
- $dist(0*10,\ 110*) = dist(\bar{x}_1 x_3 \bar{x}_4,\ x_1 x_2 \bar{x}_3) = 2$
- Partition of the hypercube into cubes is distance-k if any two of its cubes have distance at most k
- Distance 1: neighboring

Neighboring Cube Partition



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- Kullmann (2000), using matroid theory
- Related results in satisfiablity theory—Aharoni, Linial (1986), Davydov, Davydova, Kleine-Büning (1998)
- Ours: elementary combinatorial proof, simplified by Sgall

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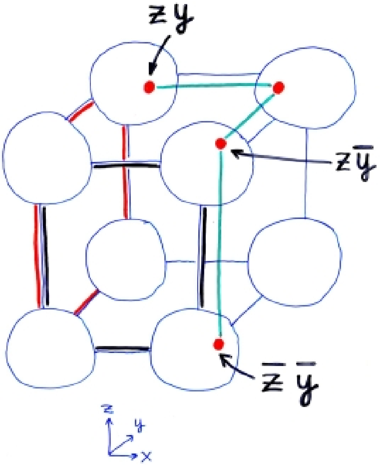
- For contradiction, assume there is some term t in the partition not containing variable z, and let a be a vector covered by t.

Proof of the Splitting Lemma: Induction Step



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- Define:

  Influence of variable x_i on $\mathcal{T}$:

  $$v_i^{\mathcal{T}} = \sum \{2^{-|T_j|} : x_i \in T_j \text{ or } \bar{x}_i \in T_j\} \ ,$$

  where $|T|$ = number vectors satisfying T.

  $$\alpha_n = \min_{\mathcal{T}} \max_i \ v_i^{\mathcal{T}}$$

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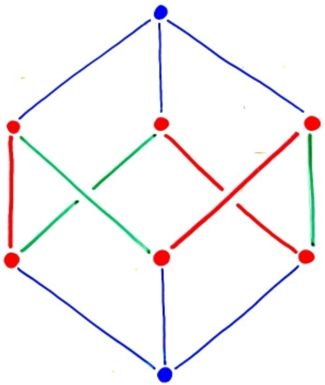
Example for $\alpha_n^3 < 1$: Partition into 5 Subcubes



Bounds on Splittability

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Theorem (Szörényi)

$$\alpha_n^2 = 1$$

Some Open Problems

- How many shortest prime implicants can a k-term DNF have?
  Example: k-term DNF

$$x_1\bar{x}_2 \vee x_2\bar{x}_3 \vee \cdots \vee x_{k-1}\bar{x}_k \vee x_k\bar{x}_1$$

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- Bounds for α_n and α_n^d for $d \geq 3$
- How many prime implicants can any n-variable Boolean function have?—between $\Omega\left(\frac{3^n}{n}\right)$ and $O\left(\frac{3^n}{\sqrt{n}}\right)$

Summary

- Characterized the $2^k - 1$ prime implicants of k-term DNF using all the nonempty subsets of a k-leaf NUD.
- Using Splitting Lemma.
- Bit about general partitions and how they respect splits.
- Paper has some related results on partitions of complete graphs into complete bipartite graphs.