Comparison of proof techniques in game-theoretic probability and measure-theoretic probability

Akimichi Takemura, Univ. of Tokyo

March 31, 2008

Outline: A.Takemura

0. **Background and our contributions**
1. **Setup of various games and notions**
2. **Examples of the first part of Borel-Cantelli**
3. **Kolmogorov's 0-1 law**
4. **Martingale convergence theorem for non-negative martingales**

Purpose of the talk: show that game-theoretic proofs are much more intuitive!

0. Background and our contributions

- My own background:
  - U.Tokyo, undergrad, Master
  - Stanford Ph.D (1982, statistics)
  - U.Tokyo since 1984
  - Main field: classical multivariate statistics
- Our group on game-theoretic probability: Kei Takeuchi, Masayuki Kumon, me and I gently push some students.

- **The BOOK: Shafer and Vovk (2001).** *Probability and Finance: It's Only a Game!.*
  - **I knew Glenn at Stanford "Theory of evidence"**
  - **Takeuchi got interested in 2002**
  - **I got interested in 2003**
- **Takeuchi wrote a book in Japanese in 2004.**
- **Japanese translation of The BOOK in 2006.**
- **By now my group wrote 7 papers (5 published).**

1. **Simple strategy for strong law of large numbers (bounded case)**
2. **Exposition of pricing formulas**
3. **SLLN for unbounded variables**
4. **Bayesian strategy in game-theoretic probability**
5. **Consideration of contrarian strategies**
6. **Application of Bayesian strategy to continuous-time game.**
7. **Multistep Bayesian strategies**

- **So at least in Japan, there are some disciples.**

1. Setup of various games and notions

Setup of various games:[a]

- **Complete information game between two players**
  - **Skeptic (statistician, investor) bets on some outcome.**
  - **Reality (nature, market) decides the outcome.;**
- **Skeptic → Reality → S → R → .**
  They play in turn.

---

[a] Games with explicit prices of tickets, but for simplicity without Forecaster

- **One round is (Skeptic's turn, Reality's turn) in this order**

- $n = 1, 2, \ldots$ **denote rounds.**

- **Skeptic's initial capital:** $\mathcal{K}_0 = 1$

- **At each round, Skeptic first announces how much he bets** $M_n \in \mathbb{R}$**.** M_n **can be any real number and can be arbitrarily small. Negative** M_n **allowed (selling).**

- **He has to pay some predetermined price p_n per unit bet (ticket) at round n.**
- **After knowing M_n, Reality chooses the outcome $x_n \in X \subset \mathbb{R}$.**
- **We consider various move spaces X of Reality**
- **Payoff to Skeptic:** $M_n x_n - M_n p_n = M_n(x_n - p_n)$
- **Skeptic's capital changes as**

$$\mathcal{K}_n = \mathcal{K}_{n-1} + M_n(x_n - p_n).$$

In summary:

```
$\mathcal{K}_0 = 1$
FOR $n = 1, 2, \dots$
    Skeptic announces $M_n \in \mathbb{R}$.
    Reality announces $x_n \in X$.
    $\mathcal{K}_n := \mathcal{K}_{n-1} + M_n(x_n - p_n)$.
END FOR
```

Fair coin game

- $X = \{-1, 1\}$ **and** $p_n \equiv 0$

  or equivalently

- $X = \{0, 1\}$ **and** $p_n \equiv 1/2$**.**

We take the second parameterization

- **Reality can choose the sign of** $x_n - 1/2 = \pm 1/2$ **as the opposite of the sign of** M_n**. Therefore Reality can always decrease Skeptic's capital.**

- **Skeptic can bet**

$$M_1 = \frac{1}{2},\ \ M_2 = \frac{1}{4},\ \ M_3 = \frac{1}{8}, \ldots$$

and avoid bankruptcy.

- **No-win situation for Skeptic?**
- **But then Reality is “forced” to observe SLLN!**

Theorem There exists a Skeptic's strategy $\mathcal{P}$. (he can announce this strategy even before the start of the game.) If Skeptic uses $\mathcal{P}$, then he is never bankrupt and furthermore whenever Reality violates

$$\lim_{n\to\infty} \frac{1}{n}(x_1 + \cdots + x_n) = \frac{1}{2},$$

then

$$\lim_{n\to\infty} \mathcal{K}_n = \infty.$$

Returning to general setup:

- **Path:** $\xi = x_1 x_2 \dots$ is an infinite sequence of Reality's moves
- **Sample space:** $\Xi = \{\xi\} = X^\infty$, the set of all paths
- **Event:** $E \subset \Xi$
- **Partial path:** $\xi^n = x_1 x_2 \dots x_n$
- **Finite event:** $E \subset X^n$

In game-theoretic probability we do not introduce a σ-field on Ξ. [However do events $E \subset \Xi$ have to be approximated by finite events?]

- **Skeptic's strategy $\mathcal{P}$:**

$$\mathcal{P}: \ \xi^{n-1} = x_1 x_2 \dots x_{n-1} \mapsto M_n$$

- **Capital process for $\mathcal{P}$:**

$$\mathcal{K}_n^{\mathcal{P}}(\xi) = \mathcal{K}_n^{\mathcal{P}}(\xi^n) = \mathcal{K}_0 + \sum_{i=1}^{n} M_i(\xi^{i-1})(x_i - p_i)$$

- **Collateral duty: $\mathcal{P}$ satisfies the collateral duty for Skeptic with the initial capital $\mathcal{K}_0 = \delta > 0$ if**

$$\mathcal{K}_n^{\mathcal{P}}(\xi) \geq 0, \qquad \forall \xi, \forall n.$$

- **Weak forcing of an event** $\mathcal{P}$ **weakly forces an event** $E \subset \Xi$ **if** $\mathcal{P}$ **satisfies the collateral duty with some** $\delta > 0$ **and**

$$\limsup_n \mathcal{K}_n^{\mathcal{P}}(\xi) = \infty, \quad \forall \xi \notin E.$$

- **Forcing of an event** $\mathcal{P}$ **forces an event** $E \subset \Xi$ **if** "$\limsup_n$" **is replaced by** "$\lim_n$" **above.**

- **"Skeptic can (weakly) force** E**": if Skeptic can construct a strategy** $\mathcal{P}$ **as above.**
  We also say "E **happens almost surely".**

- **Upper probability $\bar{P}(E)$ of an event E:** **Let I_E denote a ticket which payes 1 dollar if E occurs. The upper probability $\bar{P}(E)$ of E is the price of the ticket I_E.**
  - **Definition**

    $$\bar{P}(E) = \inf\{\mathcal{K}_0^{\mathcal{P}} \mid \exists \mathcal{P} \text{ s.t. } \mathcal{K}_n^{\mathcal{P}}(\xi) \geq I_E(\xi),\ \forall \xi \in \Xi\},$$

  - **If we start with the initial capital $\delta > \bar{P}(E)$, then we can superreplicate the ticket I_E. So that the value of the ticket I_E is at most δ.**

2. Examples of the first part of Borel-Cantelli

Example 1 $X = \{0, 1\}$ **(coin-tossing) and** $\sum_{n=1}^{\infty} p_n < \infty$ **:**

```
$\mathcal{K}_0 = 1$
FOR $n = 1, 2, \dots$
    Skeptic announces $M_n \in \mathbb{R}$.
    Reality announces $x_n \in \{0, 1\}$.
    $\mathcal{K}_n := \mathcal{K}_{n-1} + M_n(x_n - p_n)$.
END FOR
```

- **Let E be the event that $x_n = 1$ for only finite n.**
- **Skeptic can force E.**

Proof. Let $C = \sum_n p_n < \infty$. Starting with the initial capital $\delta = 1$, consider the strategy $M_n \equiv 1/C$. The capital process is

$$\begin{aligned} \mathcal{K}_n^{\mathcal{P}}(\xi) &= 1 + \frac{1}{C}\sum_{i=1}^{n}(x_i - p_i) = 1 - \frac{1}{C}\sum_{i=1}^{n} p_i + \frac{1}{C}\sum_{i=1}^{n} x_i \\ &\geq \frac{1}{C}\sum_{i=1}^{n} x_i . \end{aligned}$$

If $x_n = 1$ for infinitely many x_n, then $\lim_n \mathcal{K}_n^{\mathcal{P}}(\xi) = \infty$. $\square$

Example 2

- $X = [0, \infty)$ **and the price** $p_n = \nu$ **is a constant:**

```
$\mathcal{K}_0 = 1$
FOR $n = 1, 2, \dots$
    Skeptic announces $M_n \in \mathbb{R}$.
    Reality announces $x_n \geq 0$.
    $\mathcal{K}_n := \mathcal{K}_{n-1} + M_n(x_n - \nu)$.
END FOR
```

- **Let E_n be the event $x_n \geq n^{1+\epsilon}$, $\epsilon > 0$.**
- **Skeptic can force the event**

$$E = \{E_n \textbf{ only for finite } n\}$$

Proof. We combine Markov inequality with Borel-Cantelli argument. Let $C = \sum_{n=1}^{\infty} 1/n^{1+\epsilon} < \infty$. Consider the strategy $M_n = 1/(C\nu n^{1+\epsilon})$. Starting with

$\delta = 1$, the capital process is

$$\begin{aligned}\mathcal{K}_n^{\mathcal{P}}(\xi) &= 1 + \sum_{i=1}^{n} \frac{1}{C\nu i^{1+\epsilon}}(x_i - \nu) \\ &= 1 - \sum_{i=1}^{n} \frac{1}{Ci^{1+\epsilon}} + \frac{1}{C\nu} \sum_{i=1}^{n} \frac{x_i}{i^{1+\epsilon}} \\ &\geq \frac{1}{C\nu} \sum_{i=1}^{n} \frac{x_i}{i^{1+\epsilon}}.\end{aligned}$$

If $x_n \geq n^{1+\epsilon}$ for infinitely many x_n, then $\lim_n \mathcal{K}_n^{\mathcal{P}}(\xi) = \infty$. $\square$

General game-theoretic statement:

Let p_n be the price for the event E_n. If $\sum \bar{P}(E_n) < \infty$ then E_n happens only for finite n almost surely.

Suppose that for each event E_n there is a unit ticket I_{E_n} which pays you 1 dollar when E_n happens. Assume that the sum of the prices for all the tickets is finite $\sum_n p_n < \infty$. Then you can buy all the tickets with a finite amount of money. Now if E_n happens for infinitely many n, then you become infinitely rich!

Such a simple argument!

Review of the measure-theoretic proof

- $\limsup_n E_n = \cap_{n=1}^{\infty} \cup_{m=n}^{\infty} E_m$. **(O.K.)**
- **Since** $\{D_n = \cup_{m=n}^{\infty} E_m\}$ **is a decreasing sequence of events**

$$P(\limsup_n E_n) = \lim_n P(\cup_{m=n}^{\infty} E_m)$$

$\Leftarrow$ **uses the continuity of probability measure (why do we need continuity?)**

- $P(\cup_{m=n}^{\infty} E_m) \leq \sum_{m=n}^{\infty} P(E_m) \to 0 \quad (n \to \infty)$ **(O.K.)**

Kolmogorov's 0-1 law

- $E \subset \Xi$ **is a tail event if**

$$x_1 \dots x_N x_{N+1} \dots \in E \;\Leftrightarrow\; \forall N \;\; * \dots * x_N x_{N+1} \dots \in E.$$

- **Suppose that** $\bar{P}(E) < 1$**. Actually we have to define** $\bar{P}(E)$ **carefully because** E **is a subset of the set of infinite sequences** X^∞**.**

- **Define** $\bar{P}(E) < 1$ **as follows. There exist** $\delta < 1$ **and a strategy** $\mathcal{P}$ **satisfying the collateral duty with initial** δ **such that**

$$\liminf_n \mathcal{K}_n^{\mathcal{P}}(\xi) \geq 1 \quad \forall \xi \in E.$$

- **In words, if E happens then starting with $\delta < 1$ you can wait and there is a time point n such that $\mathcal{K}_n^{\mathcal{P}}(\xi) \geq 1 - \epsilon$, where ϵ is arbitrary small.**
- **Multiplying everything by $1/\delta$, $\bar{P}(E) < 1$ means the following: There exists $\epsilon > 0$ such that starting with $\delta = 1$, there will be a time point where your capital is at least $1 + \epsilon$.**

Now we have the following game-theoretic 0-1 law.

$\boxed{\textbf{Let } E \textbf{ be a tail event. If } \bar{P}(E) < 1 \textbf{ then } \bar{P}(E) = 0\textbf{.}}$

Proof. (In words). Suppose that E happens. You start with the initial capital of $\delta = 1$. Wait until your capital becomes $1 + \epsilon$. Then save ϵ. Now start all over again. Because E is a tail event, your situation is "the same" as the beginning of the game. Therefore there will be a time point that you get another ϵ. This repeats infinite number of times and your capital becomes infinite. □

($\bar{P}(E) = 0$ if you can get arbitrarily rich when E happens.)

This material on Kolmogorov's 0-1 law is being written up in a paper

> **"The martingales behind the zero-one laws" by Akimichi Takemura, Vladimir Vovk, and Glenn Shafer.**

Measure-theoretic statement: Suppose that $X_1, X_2, \ldots$ are independent random variables. If E is a tail event, then $P(E) = 0$ or 1.

Proof.

- Approximate E by $E_n \in \sigma(X_1, \ldots, X_n)$.
- Because E is a tail event, E is independent of E_n and

$$P(E \cap E_n) = P(E) \times P(E_n)$$

- Taking the limit we have $P(E) = P(E)^2$. Then $P(E) = 0$ or 1.

□

Unfortunately this proof is so artificial.

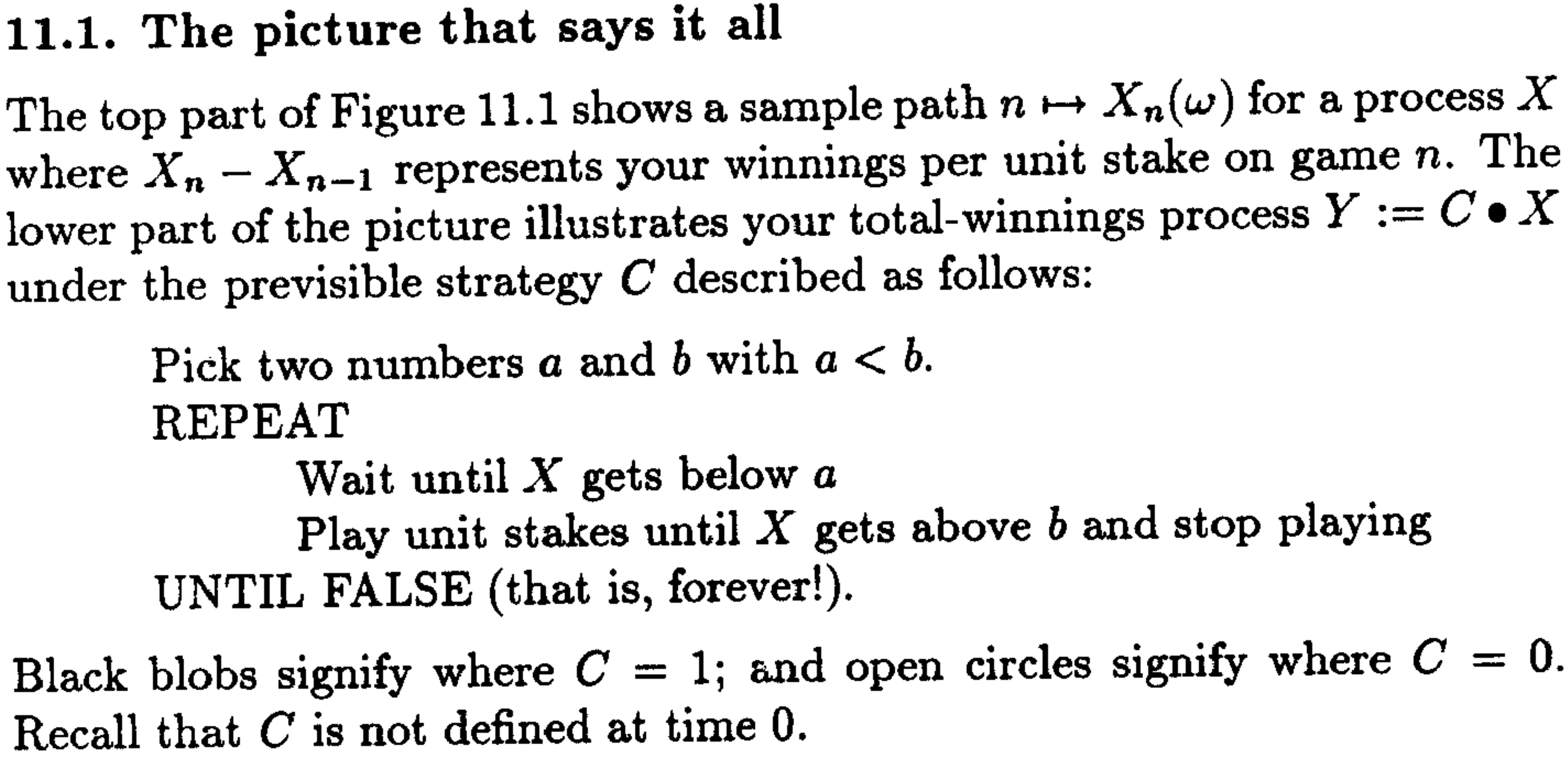
Martingale convergence theorem for non-negative martingales

- If there exists a Skeptic's strategy $\mathcal{P}$ satisfying the collateral duty with initial δ, then its capital process $\mathcal{K}_n^{\mathcal{P}}$ is called a game-theoretic non-negative martingale.

Then we have the following statement (Lemma 4.5 of Shafer and Vovk (2005)).

A non-negative martingale $\mathcal{K}_n^{\mathcal{P}}$ converges to a non-negative finite value almost surely.

From Williams book ("Probability with Martingales"):



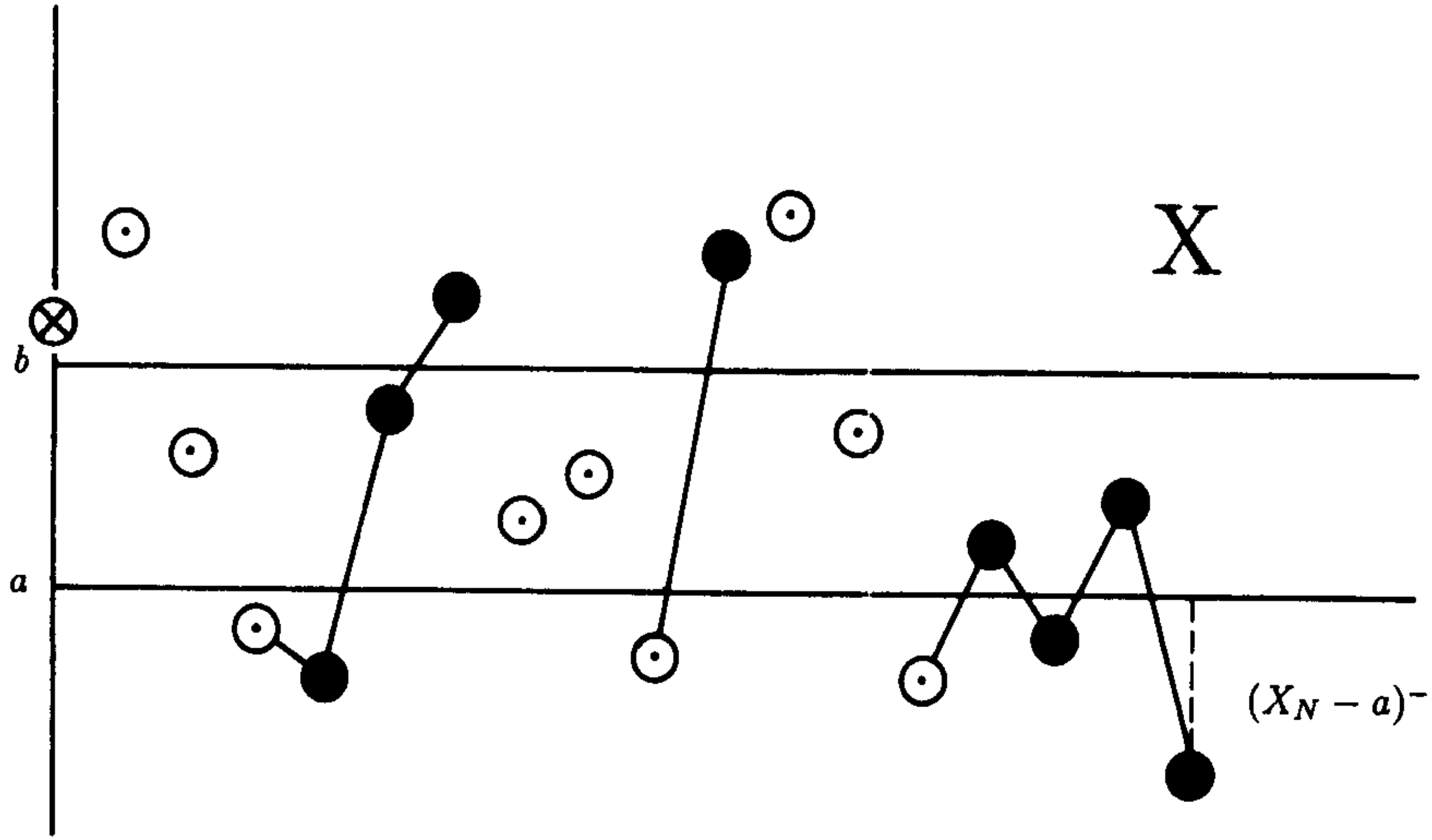
11.1. The picture that says it all

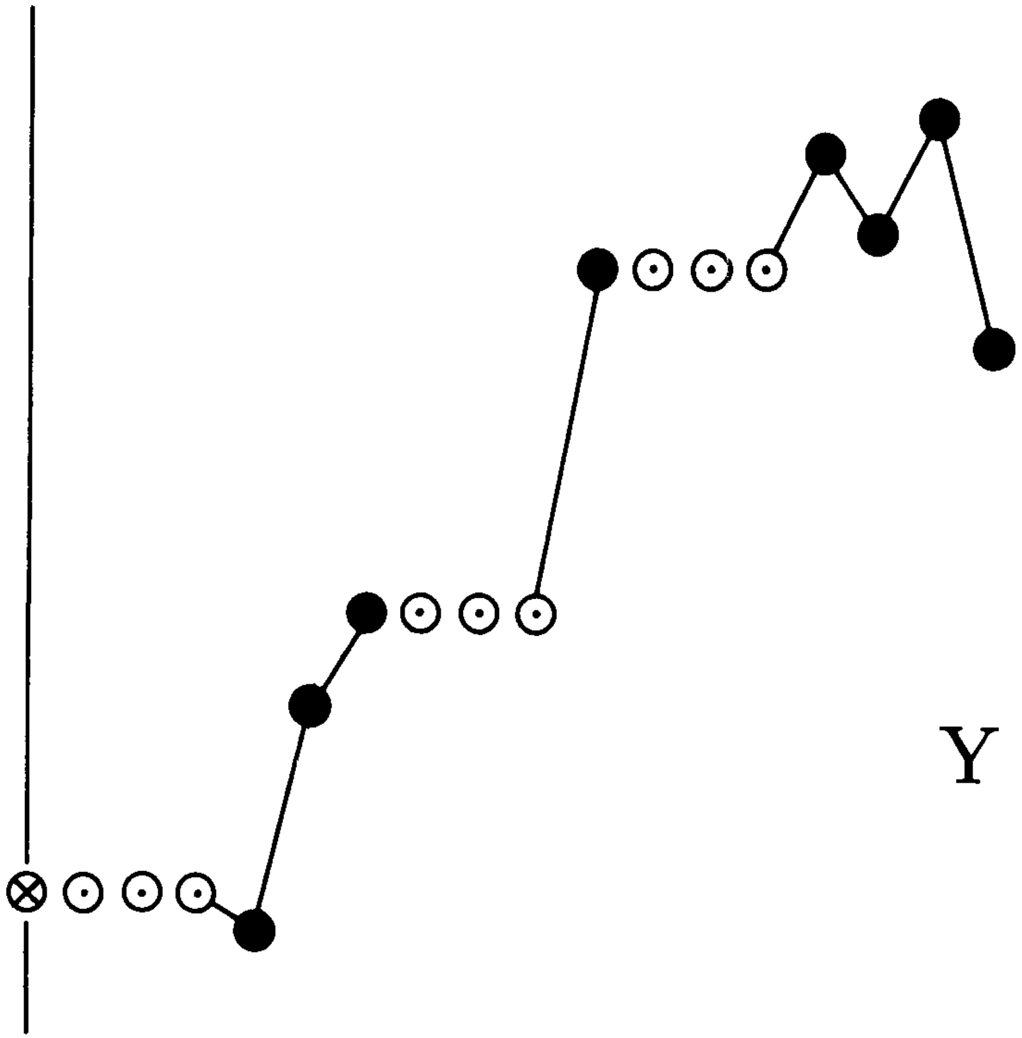
The top part of Figure 11.1 shows a sample path $n \mapsto X_n(\omega)$ for a process X where $X_n - X_{n-1}$ represents your winnings per unit stake on game n. The lower part of the picture illustrates your total-winnings process $Y := C \bullet X$ under the previsible strategy C described as follows:

```
Pick two numbers $a$ and $b$ with $a < b$.
REPEAT
    Wait until $X$ gets below $a$
    Play unit stakes until $X$ gets above $b$ and stop playing
UNTIL FALSE (that is, forever!).
```

Black blobs signify where $C = 1$; and open circles signify where $C = 0$. Recall that C is not defined at time 0.

Figure 11.1





Y

Proof.

- Let E denote the set of paths such that $\mathcal{K}_n^{\mathcal{P}}$ converges to a finite value. We need to construct a strategy $\mathcal{Q}$ such that $\limsup_n \mathcal{K}_n^{\mathcal{Q}} = \infty$ for each $\xi \notin E$.
- Use $\mathcal{P}$ itself with the initial capital of $\delta = 1/2$. If $\limsup_n \mathcal{K}_n^{\mathcal{P}} = \infty$, we do not need any other strategy.
- Divide the remaining initial capital $1/2$ as

$$\frac{1}{2} = \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots$$

- Enumerate pairs of positive rational numbers (a_i, b_i), $0 < a_i < b_i$, $i = 1, 2, \dots$. Note that the pairs of rational numbers are countable.

- Assign the initial capital $1/2^{i+1}$ to the account and strategy $\mathcal{Q}^{(i)}$ based on the Doob's upcrossing lemma for (a_i, b_i).

- The strategy associated $\mathcal{Q}^{(i)}$ is watching the capital process $\mathcal{K}_n^{\mathcal{P}}$. If $\mathcal{K}_n^{\mathcal{P}}$ comes below a_i, then $\mathcal{Q}^{(i)}$ tells us to start betting as $\mathcal{P}$ until $\mathcal{K}_n^{\mathcal{P}}$ exceeds b_i.

- Form the convex combination

$$\mathcal{Q} = \frac{1}{2}\mathcal{P} + \sum_{i=1}^{\infty} \frac{1}{2^{i+1}} \mathcal{Q}^{(i)}$$

- Then for $\xi \notin E$, $\limsup_n \mathcal{K}_n^{\mathcal{Q}}(\xi) = \infty$.

$\square$

**Now we look at measure-theoretic proof of the same statement:
"A non-negative martingale converges to a non-negative finite value almost surely".**

I take a look at Chapter 11 of Williams (1991).

Proof.

- Let X be a martingale. Let $U_N[a,b]$ be the number of upcrossings of $[a,b]$ by time N. Then

$$(b-a)E(U_N[a,b]) \leq E[(X_N-a)^-], \tag{1}$$

where $x^- = \max(-x,0)$. Proof of this is very instructive but very counterintuitive and hard to explain to students.

- Let X be a non-negative martingale, then

$$P(U_\infty[a,b]=\infty)=0. \tag{2}$$

This follows easily from (1)

- Use (2) for enumeration of pairs $\{(a_i, b_i)\}$ of positive rational numbers. Countable sum of 0 probability is 0. This proves the theorem.

$\square$

- **Again the game-theoretic proof is much more intuitive. You can explain it by words and pictures.**
- **On the other hand, for measure-theoretic proof, you definitely need monotone convergence theorem and other machinery of measure theory.**
- **You need to say "almost surely" so many times in measure-theoretic proof.**

Summary and Discussion:

- **I presented some examples, where game-theoretic arguments are much easier.**
- **Game-theoretic statement is "pathwise"**
- **What is the role of measure theory? Why do you need measurability? For some statements, we can use outer measure (like Borel-Cantelli).**