Comparison of proof techniques in game-theoretic probability and measure-theoretic probability

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Outline: A. Takemura

0. Background and our contributions
1. Setup of various games and notions
2. Examples of the first part of Borel-Cantelli
3. Kolmogorov’s 0-1 law
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Purpose of the talk: show that game-theoretic proofs are much more intuitive!
0. Background and our contributions

- My own background:
  - U.Tokyo, undergrad, Master
  - Stanford Ph.D (1982, statistics)
  - U.Tokyo since 1984
  - Main field: classical multivariate statistics

- Our group on game-theoretic probability:
  Kei Takeuchi, Masayuki Kumon, me and I gently push some students.
• The BOOK: Shafer and Vovk (2001). *Probability and Finance: It’s Only a Game!.*
  – I knew Glenn at Stanford “Theory of evidence”
  – Takeuchi got interested in 2002
  – I got interested in 2003
• Japanese translation of The BOOK in 2006.
• By now my group wrote 7 papers (5 published).
1. Simple strategy for strong law of large numbers (bounded case)
2. Exposition of pricing formulas
3. SLLN for unbounded variables
4. Bayesian strategy in game-theoretic probability
5. Consideration of contrarian strategies
6. Application of Bayesian strategy to continuous-time game.
7. Multistep Bayesian strategies

• So at least in Japan, there are some disciples.
1. Setup of various games and notions

Setup of various games:

- Complete information game between two players
  - **Skeptic** (statistician, investor) bets on some outcome.
  - **Reality** (nature, market) decides the outcome.

- **Skeptic → Reality → S → R →**. They play in turn.

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Games with explicit prices of tickets, but for simplicity without Forecaster.
• One round is (Skeptic’s turn, Reality’s turn) in this order

• $n = 1, 2, \ldots$ denote rounds.

• **Skeptic’s initial capital:** $K_0 = 1$

• At each round, Skeptic first announces how much he bets $M_n \in \mathbb{R}$. $M_n$ can be any real number and can be arbitrarily small. Negative $M_n$ allowed (selling).
• He has to pay some predetermined price \( p_n \) per unit bet (ticket) at round \( n \).

• After knowing \( M_n \), Reality chooses the outcome \( x_n \in X \subset \mathbb{R} \).

• We consider various move spaces \( X \) of Reality.

• Payoff to \textbf{Skeptic}: \( M_n x_n - M_n p_n = M_n (x_n - p_n) \)

• Skeptic’s capital changes as

\[
\mathcal{K}_n = \mathcal{K}_{n-1} + M_n (x_n - p_n).
\]
In summary:

\[ K_0 = 1 \]

**FOR** \( n = 1, 2, \ldots \)

- **Skeptic** announces \( M_n \in \mathbb{R} \).
- **Reality** announces \( x_n \in X \).

\[ K_n := K_{n-1} + M_n(x_n - p_n). \]

**END FOR**
Fair coin game

- $X = \{-1, 1\}$ and $p_n \equiv 0$

  or equivalently

- $X = \{0, 1\}$ and $p_n \equiv 1/2$.

We take the second parameterization

- Reality can choose the sign of $x_n - 1/2 = \pm 1/2$ as the opposite of the sign of $M_n$. Therefore Reality can always decrease Skeptic’s capital.
• **Skeptic** can bet

\[ M_1 = \frac{1}{2}, \quad M_2 = \frac{1}{4}, \quad M_3 = \frac{1}{8}, \ldots \]

and avoid bankruptcy.

• No-win situation for **Skeptic**?

• But then **Reality** is “forced” to observe SLLN!
Theorem  There exists a Skeptic’s strategy $\mathcal{P}$. (he can announce this strategy even before the start of the game.) If Skeptic uses $\mathcal{P}$, then he is never bankrupt and furthermore whenever Reality violates

$$\lim_{n \to \infty} \frac{1}{n} (x_1 + \cdots + x_n) = \frac{1}{2},$$

then

$$\lim_{n \to \infty} K_n = \infty.$$
Returning to general setup:

- **Path:** $\xi = x_1 x_2 \ldots$ is an infinite sequence of Reality’s moves
- **Sample space:** $\Xi = \{\xi\} = X^\infty$, the set of all paths
- **Event:** $E \subset \Xi$
- **Partial path:** $\xi^n = x_1 x_2 \ldots x_n$
- **Finite event:** $E \subset X^n$

In game-theoretic probability we do not introduce a $\sigma$-field on $\Xi$. [However do events $E \subset \Xi$ have to be approximated by finite events?]
• **Skeptic’s strategy** \( \mathcal{P} \):

\[
\mathcal{P} : \quad \xi^{n-1} = x_1 x_2 \ldots x_{n-1} \mapsto M_n
\]

• **Capital process for** \( \mathcal{P} \):

\[
K^\mathcal{P}_n(\xi) = K^\mathcal{P}_n(\xi^n) = K_0 + \sum_{i=1}^{n} M_i(\xi^{i-1})(x_i - p_i)
\]

• **Collateral duty**: \( \mathcal{P} \) satisfies the collateral duty for Skeptic with the initial capital \( K_0 = \delta > 0 \) if

\[
K^\mathcal{P}_n(\xi) \geq 0, \quad \forall \xi, \forall n.
\]
- **Weak forcing of an event** \( \mathcal{P} \) weakly forces an event \( E \subset \Xi \) if \( \mathcal{P} \) satisfies the collateral duty with some \( \delta > 0 \) and
\[
\limsup_{n} K_{n}^{\mathcal{P}}(\xi) = \infty, \quad \forall \xi \notin E.
\]

- **Forcing of an event** \( \mathcal{P} \) forces an event \( E \subset \Xi \) if “\( \limsup_{n} \)” is replaced by “\( \lim_{n} \)” above.

- **“Skeptic can (weakly) force \( E \)”**: if Skeptic can construct a strategy \( \mathcal{P} \) as above. We also say “\( E \) happens almost surely”.
• **Upper probability** $\bar{P}(E)$ of an event $E$: Let $I_E$ denote a ticket which pays 1 dollar if $E$ occurs. The upper probability $\bar{P}(E)$ of $E$ is the price of the ticket $I_E$.

  - **Definition**
    
    $$\bar{P}(E) = \inf \{ K^0 \mid \exists \mathcal{P} \text{ s.t. } K^P_n(\xi) \geq I_E(\xi), \forall \xi \in \Xi \}$$

  - If we start with the initial capital $\delta > \bar{P}(E)$, then we can superreplicate the ticket $I_E$. So that the value of the ticket $I_E$ is at most $\delta$. 


2. Examples of the first part of Borel-Cantelli

Example 1 $X = \{0, 1\}$ (coin-tossing) and $\sum_{n=1}^{\infty} p_n < \infty$:

$K_0 = 1$

FOR $n = 1, 2, \ldots$

Skeptic announces $M_n \in \mathbb{R}$.

Reality announces $x_n \in \{0, 1\}$.

$K_n := K_{n-1} + M_n(x_n - p_n)$.

END FOR
• Let $E$ be the event that $x_n = 1$ for only finite $n$.

• Skeptic can force $E$.

Proof. Let $C = \sum_n p_n < \infty$. Starting with the initial capital $\delta = 1$, consider the strategy $M_n \equiv 1/C$. The capital process is

$$K_n^P(\xi) = 1 + \frac{1}{C} \sum_{i=1}^n (x_i - p_i) = 1 - \frac{1}{C} \sum_{i=1}^n p_i + \frac{1}{C} \sum_{i=1}^n x_i$$

$$\geq \frac{1}{C} \sum_{i=1}^n x_i.$$  

If $x_n = 1$ for infinitely many $x_n$, then $\lim_n K_n^P(\xi) = \infty$.

□
Example 2

\• \(X = [0, \infty)\) and the price \(p_n = \nu\) is a constant:

\[ \mathcal{K}_0 = 1 \]

\textbf{FOR} \(n = 1, 2, \ldots\)

\begin{align*}
\textbf{Skeptic} & \text{ announces } M_n \in \mathbb{R}. \\
\textbf{Reality} & \text{ announces } x_n \geq 0. \\
\mathcal{K}_n & := \mathcal{K}_{n-1} + M_n(x_n - \nu).
\end{align*}

\textbf{END FOR}
• Let $E_n$ be the event $x_n \geq n^{1+\epsilon}$, $\epsilon > 0$.

• Skeptic can force the event

$$E = \{E_n \text{ only for finite } n\}$$

Proof. We combine Markov inequality with Borel-Cantelli argument. Let $C = \sum_{n=1}^{\infty} 1/n^{1+\epsilon} < \infty$. Consider the strategy $M_n = 1/(C\nu n^{1+\epsilon})$. Starting with
δ = 1, the capital process is

\[ \mathcal{K}_n^P(\xi) = 1 + \sum_{i=1}^{n} \frac{1}{C' \nu i^{1+\epsilon}} (x_i - \nu) \]

\[ = 1 - \sum_{i=1}^{n} \frac{1}{C' \nu i^{1+\epsilon}} + \frac{1}{C'} \sum_{i=1}^{n} \frac{x_i}{i^{1+\epsilon}} \]

\[ \geq \frac{1}{C'} \sum_{i=1}^{n} \frac{x_i}{i^{1+\epsilon}}. \]

If \( x_n \geq n^{1+\epsilon} \) for infinitely many \( x_n \), then
\[ \lim_n \mathcal{K}_n^P(\xi) = \infty. \]
General game-theoretic statement:

Let $p_n$ be the price for the event $E_n$. If 
\[ \sum \tilde{P}(E_n) < \infty \] 
then $E_n$ happens only for finite $n$ almost surely.

Suppose that for each event $E_n$ there is a unit ticket $I_{E_n}$ which pays you 1 dollar when $E_n$ happens. Assume that the sum of the prices for all the tickets is finite 
\[ \sum_n p_n < \infty \] 
Then you can buy all the tickets with a finite amount of money. Now if $E_n$ happens for infinitely many $n$, then you become infinitely rich!

Such a simple argument!
Review of the measure-theoretic proof

- \( \limsup_n E_n = \bigcap_{n=1}^\infty \bigcup_{m=n}^\infty E_m. \) (O.K.)

- Since \( \{D_n = \bigcup_{m=n}^\infty E_m\} \) is a decreasing sequence of events

\[
P(\limsup_n E_n) = \lim_{n \to \infty} P(\bigcup_{m=n}^\infty E_m)
\]

\( \Leftarrow \) uses the continuity of probability measure
(why do we need continuity?)

- \( P(\bigcup_{m=n}^\infty E_m) \leq \sum_{m=n}^\infty P(E_m) \to 0 \ (n \to \infty) \) (O.K.)
Kolmogorov’s 0-1 law

- $E \subset \Xi$ is a tail event if
  \[ x_1 \ldots x_Nx_{N+1} \ldots \in E \iff \forall N \ast \cdots \ast x_Nx_{N+1} \cdots \in E. \]

- Suppose that $\bar{P}(E) < 1$. Actually we have to define $\bar{P}(E)$ carefully because $E$ is a subset of the set of infinite sequences $X^\infty$.

- Define $\bar{P}(E) < 1$ as follows. There exist $\delta < 1$ and a strategy $\mathcal{P}$ satisfying the collateral duty with initial $\delta$ such that
  \[ \liminf_n \mathcal{K}_n^\mathcal{P}(\xi) \geq 1 \quad \forall \xi \in E. \]
• In words, if $E$ happens then starting with $\delta < 1$ you can wait and there is a time point $n$ such that $\mathcal{K}_n^P(\xi) \geq 1 - \epsilon$, where $\epsilon$ is arbitrary small.

• Multiplying everything by $1/\delta$, $\bar{P}(E) < 1$ means the following: There exists $\epsilon > 0$ such that starting with $\delta = 1$, there will be a time point where your capital is at least $1 + \epsilon$. 
Now we have the following game-theoretic 0-1 law.

Let $E$ be a tail event. If $\bar{P}(E) < 1$ then $\bar{P}(E) = 0$.

Proof. (In words). Suppose that $E$ happens. You start with the initial capital of $\delta = 1$. Wait until your capital becomes $1 + \epsilon$. Then save $\epsilon$. Now start all over again. Because $E$ is a tail event, your situation is “the same” as the beginning of the game. Therefore there will be a time point that you get another $\epsilon$. This repeats infinite number of times and your capital becomes infinite. □

$(\bar{P}(E) = 0$ if you can get arbitrarily rich when $E$ happens.)
This material on Kolmogorov’s 0-1 law is being written up in a paper

“The martingales behind the zero-one laws” by Akimichi Takemura, Vladimir Vovk, and Glenn Shafer.
Measure-theoretic statement: Suppose that $X_1, X_2, \ldots$ are independent random variables. If $E$ is a tail event, then $P(E) = 0$ or 1.

Proof. • Approximate $E$ by $E_n \in \sigma(X_1, \ldots, X_n)$.

• Because $E$ is a tail event, $E$ is independent of $E_n$ and

$$P(E \cap E_n) = P(E) \times P(E_n)$$

• Taking the limit we have $P(E) = P(E)^2$. Then $P(E) = 0$ or 1.

Unfortunately this proof is so artificial.
Martingale convergence theorem for non-negative martingales

- If there exists a Skeptic’s strategy $\mathcal{P}$ satisfying the collateral duty with initial $\delta$, then its capital process $\mathcal{K}_n^\mathcal{P}$ is called a game-theoretic non-negative martingale.

Then we have the following statement (Lemma 4.5 of Shafer and Vovk (2005)).

A non-negative martingale $\mathcal{K}_n^\mathcal{P}$ converges to a non-negative finite value almost surely.
From Williams book ("Probability with Martingales"):  

11.1. The picture that says it all

The top part of Figure 11.1 shows a sample path $n \mapsto X_n(\omega)$ for a process $X$ where $X_n - X_{n-1}$ represents your winnings per unit stake on game $n$. The lower part of the picture illustrates your total-winnings process $Y := C \cdot X$ under the previsible strategy $C$ described as follows:

- Pick two numbers $a$ and $b$ with $a < b$.
- **REPEAT**
  - Wait until $X$ gets below $a$
  - Play unit stakes until $X$ gets above $b$ and stop playing
- **UNTIL FALSE** (that is, forever!).

Black blobs signify where $C = 1$; and open circles signify where $C = 0$. Recall that $C$ is not defined at time 0.
Proof. • Let $E$ denote the set of paths such that $\mathcal{K}_n^P$ converges to a finite value. We need to construct a strategy $Q$ such that $\limsup_n \mathcal{K}_n^Q = \infty$ for each $\xi \notin E$.

• Use $P$ itself with the initial capital of $\delta = 1/2$. If $\limsup_n \mathcal{K}_n^P = \infty$, we do not need any other strategy.

• Divide the remaining initial capital $1/2$ as

\[
\frac{1}{2} = \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots
\]

• Enumerate pairs of positive rational numbers $(a_i, b_i)$, $0 < a_i < b_i$, $i = 1, 2, \ldots$. Note that the pairs of rational numbers are countable.
• Assign the initial capital $1/2^{i+1}$ to the account and strategy $Q^{(i)}$ based on the Doob’s upcrossing lemma for $(a_i, b_i)$.

• The strategy associated $Q^{(i)}$ is watching the capital process $K_n^P$. If $K_n^P$ comes below $a_i$, then $Q^{(i)}$ tells us to start betting as $P$ until $K_n^P$ exceeds $b_i$.

• Form the convex combination

$$Q = \frac{1}{2} P + \sum_{i=1}^{\infty} \frac{1}{2^{i+1}} Q^{(i)}$$

• Then for $\xi \notin E$, $\limsup_n K_n^Q(\xi) = \infty$. 

□
Now we look at measure-theoretic proof of the same statement:
“A non-negative martingale converges to a non-negative finite value almost surely”.

Proof. • Let $X$ be a martingale. Let $U_N[a, b]$ be the number of upcrossings of $[a, b]$ by time $N$. Then

$$(b - a)E(U_N[a, b]) \leq E[(X_N - a)^-],$$

(1)

where $x^- = \max(-x, 0)$. Proof of this is very instructive but very counterintuitive and hard to explain to students.

• Let $X$ be a non-negative martingale, then

$$P(U_\infty[a, b] = \infty) = 0.$$  

(2)

This follows easily from (1)
• Use (2) for enumeration of pairs \{(a_i, b_i)\} of positive rational numbers. Countable sum of 0 probability is 0. This proves the theorem.

• Again the game-theoretic proof is much more intuitive. You can explain it by words and pictures.

• On the other hand, for measure-theoretic proof, you definitely need monotone convergence theorem and other machinery of measure theory.

• You need to say “almost surely” so many times in measure-theoretic proof.
Summary and Discussion:

- I presented some examples, where game-theoretic arguments are much easier.
- Game-theoretic statement is “pathwise”
- What is the role of measure theory? Why do you need measurability? For some statements, we can use outer measure (like Borel-Cantelli).