PERFORMANCE ANALYSIS ON A SHOESTRING

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SOME ALTERNATE TITLES

- Crossing a non-Jackson network, with or without a map
- Latency in Markov-routed networks with node and link delays
- Path-additive functionals on networks with Markovian routing
- Markovian routing approximations to address-based routing

OVERVIEW

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- Problem summary and basics
- Review of the traffic equation
- Path-additive functionals
- Node-to-node values
  - Cross-network values
- Three models of address-based routing
- Conclusion

PROBLEM NUTSHELLS

- Networks deliver services by transporting certain objects from point to point
  - Physical objects
    - Packages
    - Oil, gas, electricity
  - Notional objects
    - Packets
- Service quality includes metrics on how long it takes to complete a trip across the network
  - Cross-network delay

PROBLEM NUTSHELLS

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- Computing cross-network delay in a product-form network is easy
  - Everything is independent
- Many factors can cause the standard models to be unsuitable
  - Packet size distribution not exponential
  - Delays on links
- How can we analyze cross-network delays absent the standard framework?

BASICS
NETWORK

- A network is a graph $G = (\mathcal{N}, \mathcal{L})$
  - $\mathcal{N}$ = set of nodes, $|\mathcal{N}| = n$
  - $\mathcal{L}$ = set of links $\subset \mathcal{N} \times \mathcal{N}$
  - $\mathcal{O} \subset \mathcal{N}$ = set of originating nodes
  - $\mathcal{D} \subset \mathcal{N}$ = set of destination nodes
  - For $i \in \mathcal{O}$ and $j \in \mathcal{D}$, u_{ij} = exogenous demand = traffic to be carried from i to j
    - + Barrels per hour
    - + Terabits per second
    - + etc.

BASICS
ROUTING

- Let X_n represent the node at which a customer is located at the n^{th} step in its routing process
- $r_{ij} = P\{X_{n+1} = j \mid X_n = i, X_{n-1}, X_{n-2}, \ldots, X_1\}$
- Routing is homogeneous
  - Doesn't depend on n
- Routing is Markovian
  - No dependence on $X_{n-1}, X_{n-2}, \ldots, X_1$
- $r_{ij} = P\{X_{n+1} = j \mid X_n = i\}$ is routing matrix
  - Usually $r_{ii} = 0$

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TRAFFIC EQUATION

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- Network in equilibrium
  - Rate of flow into each node equals the rate of flow out of the node
- λ_i = total arrival rate into node i
- $u_i^* = u_{i1} + \cdots + u_{in}$ = total arrival rate of exogenous demand into node i
- Then $\lambda_i = u_i^* + \sum_{j=1}^{n} \lambda_j r_{ji}, \quad i = 1, \ldots, n$
- If R is convergent, then $\lambda = u^*(I - R^T)^{-1}$

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PATHS

- A *path* in G is a sequence of alternating nodes and (single) links joining them, beginning and ending with a node
  - Set of all paths in G is $\mathcal{P}(G)$
- A single node is a legitimate path
- So is a pair of nodes connected by a single link

PATH-ADDITIVE FUNCTIONALS

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- $a : \mathcal{P}(G) \to \mathbf{R}$ is *path-additive* if for every $\pi \in \mathcal{P}(G)$,

$$a(\pi) = \sum_{m=1}^{k-1} \left[a(v_m) + a(v_m, v_{m+1}) \right] + a(v_k)$$

CUSTOMER SAMPLE PATHS

- Customer ω travels a path $X_0(\omega)$ (= i), $X_1(\omega)$, …, $X_k(\omega)$ (= j) called $\pi(\omega)$
- If a is path-additive, we define $A_{ij}(\omega)$ by

$$A_{ij}(\omega) = a(i, j) I\{X_0 = i, X_1 = j\} +$$

$$+ \sum_{k=1}^{\infty} a(\pi(\omega)) I\{X_0 = i, X_1 = \nu_1, \ldots, X_k = \nu_k, X_{k+1} = j\}$$

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NODE-TO-NODE VALUES

- Define $s_{km} = [a(k) + a(k, m)]I\{(k, m) \in \mathcal{L}\}$
  - Assume these are random variables with distributions F_{km}
    - Could be degenerate
  - Independent of the routing process
- Let $\bar{s}_{km} = \mathrm{E}s_{km}$ and S be the matrix of the $\bar{s}_{km}$
- Let $\bar{A}$ be the matrix of $\mathrm{E}a_{ij} - \mathrm{E}a(j)$

NODE-TO-NODE VALUES

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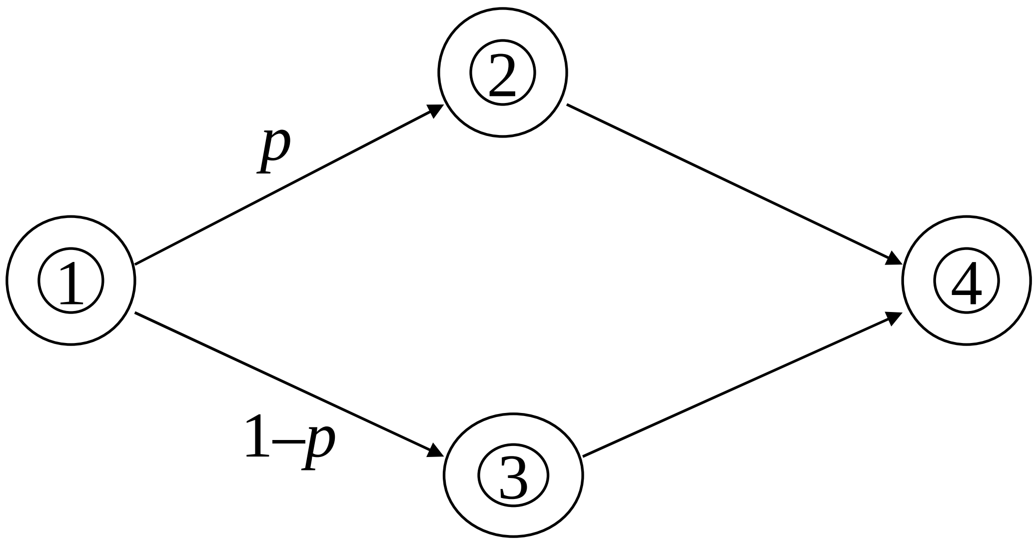
- Theorem: In an open network, $\bar{A} = (I - R)^{-1} \cdot S \# R \cdot (I - R)^{-1}$
  - Single-link values independent of routing process
- Example: let $s_{km} = 1$ for every $(k, m) \in \mathcal{L}$
  - Then A_{ij} is the (random) number of links traveled from i to j
  - The expected number of links traveled from i to j is (the (i, j) entry in) $R(I - R)^{-2}$

NODE-TO-NODE VALUES

- Example:



EXAMPLE

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- $R = \begin{pmatrix} 0 & p & 1-p & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

- $R(I - R)^{-2} = \begin{pmatrix} 0 & p & 1-p & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

EXAMPLE

- Single-link delays d_{km}
- $(I - R)^{-1} \cdot S \# R \cdot (I - R)^{-1} =$

$$\begin{pmatrix} 0 & d_{12}p & d_{13}(1-p) & p\left(d_{12}+d_{24}\right)+(1-p)\left(d_{13}+d_{34}\right) \\ 0 & 0 & 0 & d_{24} \\ 0 & 0 & 0 & d_{34} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

NODE-TO-NODE VALUES

- Let $w_{ij} = Es_{ij}^2$ and $W = (w_{ij})$

NODE-TO-NODE VALUES

- Theorem: In an open network in which the s_{km} are uncorrelated, $E(A_{ij} - \mathrm{a}(j))^2$ is given by the (i, j) entry of $(I - R)^{-1}\left[W\#R + 2S\#R\cdot(I - R)^{-1}\cdot S\#R\right](I - R)^{-1}$
- Not a problem if single-element delays are deterministic
  - The delay on a single link counts the processing delay at the link's head node

CROSS-NETWORK DELAY

- Expected value is *latency*
- Standard deviation is *jitter*
- Packet loss probability is $P\{\text{delay} = \infty\}$

DISTRIBUTIONS

- Let F and G be $n \times n$ matrices of CDFs
- Define matrix convolution of F and G by

$$\left(F * G\right)_{ij}(x) = \sum_{k=1}^{n} F_{ik} * G_{kj}(x)$$

- Repeated convolutions denoted by superscripts $*2$, $*3$, etc.

DISTRIBUTIONS

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- Theorem: Let s_{km} be nonnegative and mutually stochastically independent and let $M(x)$ denote the matrix $\sum_{m=1}^{\infty}(R\#F)^{*m}(x)$.

  Then, in an open network, $P\{A_{ij} - a(j) \le x\} = M_{ij}(x)$ and M satisfies the matrix-integral equation

$$M = R\#F + (R\#F)*M.$$

EXAMPLE
A JACKSON NETWORK

- Consider a network of K nodes in series
- $O = \{1\}$, $\mathcal{D} = \{K\}$

$$R_K = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & 0 \\ 0 & 0 & 0 & 0 & \vdots & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

EXAMPLE
A JACKSON NETWORK

$$\left(I-R_K\right)^{-1}=\begin{pmatrix}1&1&1&1&\cdots&1\\0&1&1&1&\cdots&1\\0&0&1&1&\cdots&1\\\vdots&\vdots&\vdots&\vdots&\vdots&\vdots\\0&0&0&0&\cdots&1\\0&0&0&0&\cdots&1\end{pmatrix}\text{ and }\left(I-R_K^T\right)^{-1}=\begin{pmatrix}1&0&0&0&\cdots&0\\1&1&0&0&\cdots&0\\1&1&1&0&\cdots&0\\\vdots&\vdots&\vdots&\vdots&\vdots&\vdots\\1&1&1&1&\cdots&0\\1&1&1&1&\cdots&1\end{pmatrix}$$

EXAMPLE
A JACKSON NETWORK

- The traffic equation gives $\lambda_k = \sum_{m=1}^{k} u_k \,, \quad k = 1, \ldots, K.$
- Jackson's theorem: if $\lambda_k < \mu_k$ for all k, then nodes are independent M/M/1 queues with $a(k) = (\mu_k - \lambda_k)^{-1}$
- Put $s_{jk} = (\mu_k - \lambda_k)^{-1}\delta_{j,k-1}$ for $j, k = 1, \ldots, K$

EXAMPLE
A JACKSON NETWORK

- Mean inter-node delays are

$$\begin{pmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 1 & 1 & \cdots & 1 \\ 0 & 0 & 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & 0 & \cdots & 1 \end{pmatrix} \begin{pmatrix} 0 & a_1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & a_2 & 0 & \cdots & 0 \\ 0 & 0 & 0 & a_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & a_{K-1} \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 1 & 1 & \cdots & 1 \\ 0 & 0 & 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$$

EXAMPLE
A JACKSON NETWORK

- Simplifies to

$$\begin{pmatrix} 0 & a_1 & a_1+a_2 & a_1+a_2+a_3 & \cdots & a_1+\cdots+a_{K-1} \\ 0 & 0 & a_2 & a_2+a_3 & \cdots & a_2+\cdots+a_{K-1} \\ 0 & 0 & 0 & a_3 & \cdots & a_3+\cdots+a_{K-1} \\ 0 & 0 & 0 & 0 & \cdots & a_4+\cdots+a_{K-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

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ADDRESS-BASED ROUTING

- Every arriving customer has a stated destination
- Routers (nodes) contains lookup tables that determine the next node given
  - The destination address and
  - The current state of congestion in the network
- Incompatible with the Markovian routing model

ADDRESS-BASED ROUTING

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- In address-based routing,
  - A customer leaves the network immediately upon reaching destination
  - A customer visits a given node at most once
  - Every origin has different routing tables
- Can we mimic any of these (approximately) in a *Markovian* routing model?

CUSTOMER REPEATS NO NODES

- Let $t_{ij}^{(m)}$ be the probability that a customer reaches j from i in m steps with no repeats
- Let $T_{ij} = t_{ij}^{(1)} + t_{ij}^{(2)} + \cdots + t_{ij}^{(n)}$
- Let T be the matrix of the T_{ij}
- Let B be the matrix whose (i, j) element is $r_{ij} - \sum_{k \neq i, j} r_{ik} t_{kj}^{(n)}$

CUSTOMER REPEATS NO NODES

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- Theorem: $T = (I - R)^{-1} B$
- Note $t_{ij}^{(n)} = \sum_{\sigma \in S_n} r_{\sigma_1 \sigma_2} r_{\sigma_2 \sigma_3} \cdots r_{\sigma_{n-1} \sigma_n}$
- Let $M_{ij}^{(m)}$ be the probability that a customer starting from i reaches j for the first time in m steps
- Of interest: $|M_{ij}^{(m)} - t_{ij}^{(m)}|$ and other measures of how far M is from T

INDIVIDUAL ROUTING MATRICES

- There is a separate routing matrix R_i for each node i as an originating node
- Given $X_0 = i$, the Markov chain is acyclic
- Requires treating each originating node separately

CONCLUSION

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- Generalized cross-network delay
  - Markovian routing
  - Link delays
  - Non-Poisson arrivals
  - Non-exponential service times
- Path-additive functionals
- Markovian approximations for address-based routing