

PERFORMANCE ANALYSIS ON A SHOESTRING

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SOME ALTERNATE TITLES

- ❖ Crossing a non-Jackson network, with or without a map
- ❖ Latency in Markov-routed networks with node and link delays
- ❖ Path-additive functionals on networks with Markovian routing
- ❖ Markovian routing approximations to address-based routing

OVERVIEW

- ❖ Problem summary and basics
- ❖ Review of the traffic equation
- ❖ Path-additive functionals
- ❖ Node-to-node values
 - Cross-network values
- ❖ Three models of address-based routing
- ❖ Conclusion

PROBLEM NUTSHELLS

- ❖ Networks deliver services by transporting certain objects from point to point
 - Physical objects
 - + Packages
 - + Oil, gas, electricity
 - Notional objects
 - + Packets

- ❖ Service quality includes metrics on how long it takes to complete a trip across the network
 - Cross-network delay

PROBLEM NUTSHELLS

- ❖ Computing cross-network delay in a product-form network is easy
 - Everything is independent
- ❖ Many factors can cause the standard models to be unsuitable
 - Packet size distribution not exponential
 - Delays on links
- ❖ How can we analyze cross-network delays absent the standard framework?

BASICS NETWORK

- ❖ A network is a graph $G = (\mathcal{N}, \mathcal{L})$
 - \mathcal{N} = set of nodes, $|\mathcal{N}| = n$
 - \mathcal{L} = set of links $\subset \mathcal{N} \times \mathcal{N}$
 - $O \subset \mathcal{N}$ = set of originating nodes
 - $\mathcal{D} \subset \mathcal{N}$ = set of destination nodes
 - For $i \in O$ and $j \in \mathcal{D}$, u_{ij} = exogenous demand = traffic to be carried from i to j
 - + Barrels per hour
 - + Terabits per second
 - + etc.

BASICS ROUTING

- ❖ Let X_n represent the node at which a customer is located at the n^{th} step in its routing process
- ❖ $r_{ij} = P\{X_{n+1} = j \mid X_n = i, X_{n-1}, X_{n-2}, \dots, X_1\}$
- ❖ Routing is homogeneous
 - Doesn't depend on n
- ❖ Routing is Markovian
 - No dependence on $X_{n-1}, X_{n-2}, \dots, X_1$
- ❖ $r_{ij} = P\{X_{n+1} = j \mid X_n = i\}$ is routing matrix
 - Usually $r_{ii} = 0$

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TRAFFIC EQUATION

❖ Network in equilibrium

- Rate of flow into each node equals the rate of flow out of the node

❖ $\lambda_i =$ total arrival rate into node i

❖ $u_i^* = u_{i1} + \dots + u_{in} =$ total arrival rate of exogenous demand into node i

❖ Then $\lambda_i = u_i^* + \sum_{j=1}^n \lambda_j r_{ji}$, $i = 1, \dots, n$

❖ If R is convergent, then $\lambda = u^* (I - R^T)^{-1}$

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PATHS

- ❖ A *path* in G is a sequence of alternating nodes and (single) links joining them, beginning and ending with a node
 - Set of all paths in G is $\mathcal{P}(G)$
- ❖ A single node is a legitimate path
- ❖ So is a pair of nodes connected by a single link

PATH-ADDITIVE FUNCTIONALS

❖ $a : \mathcal{P}(G) \rightarrow \mathbf{R}$ is *path-additive* if for every $\pi \in \mathcal{P}(G)$,

$$a(\pi) = \sum_{m=1}^{k-1} [a(v_m) + a(v_m, v_{m+1})] + a(v_k)$$

CUSTOMER SAMPLE PATHS

- ❖ Customer ω travels a path $X_0(\omega) (= i)$, $X_1(\omega), \dots, X_k(\omega) (= j)$ called $\pi(\omega)$
- ❖ If a is path-additive, we define $A_{ij}(\omega)$ by

$$A_{ij}(\omega) = a(i, j)I \{ X_0 = i, X_1 = j \} + \\ + \sum_{k=1}^{\infty} a(\pi(\omega))I \{ X_0 = i, X_1 = v_1, \dots, X_k = v_k, X_{k+1} = j \}$$

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NODE-TO-NODE VALUES

- ❖ Define $s_{km} = [a(k) + a(k, m)]I\{(k, m) \in \mathcal{L}\}$
 - Assume these are random variables with distributions F_{km}
 - + Could be degenerate
 - Independent of the routing process
- ❖ Let $\bar{s}_{km} = \mathbb{E}s_{km}$ and S be the matrix of the \bar{s}_{km}
- ❖ Let \bar{A} be the matrix of $\mathbb{E}a_{ij} - \mathbb{E}a(j)$

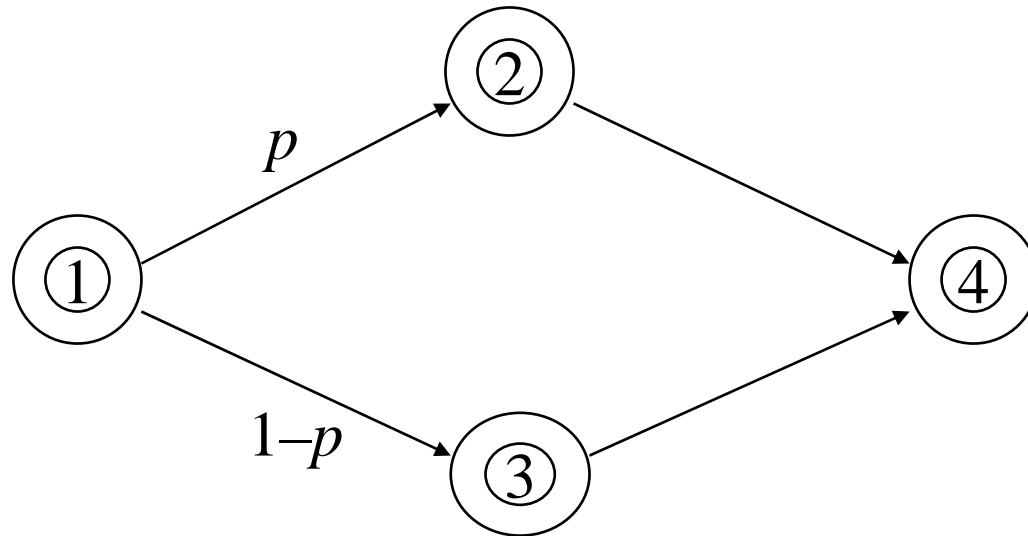
NODE-TO-NODE VALUES

- ❖ Theorem: In an open network,
 $\bar{A} = (I - R)^{-1} \cdot S \# R \cdot (I - R)^{-1}$
 - Single-link values independent of routing process

- ❖ Example: let $s_{km} = 1$ for every $(k, m) \in \mathcal{L}$
 - Then A_{ij} is the (random) number of links traveled from i to j
 - The expected number of links traveled from i to j is (the (i, j) entry in) $R(I - R)^{-2}$

NODE-TO-NODE VALUES

❖ Example:



EXAMPLE

$$\diamond R = \begin{pmatrix} 0 & p & 1-p & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\diamond R(I - R)^{-2} = \begin{pmatrix} 0 & p & 1-p & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$


EXAMPLE

❖ Single-link delays d_{km}

❖ $(I - R)^{-1} \cdot S \# R \cdot (I - R)^{-1} =$

$$\begin{pmatrix} 0 & d_{12}p & d_{13}(1-p) & p(d_{12} + d_{24}) + (1-p)(d_{13} + d_{34}) \\ 0 & 0 & 0 & d_{24} \\ 0 & 0 & 0 & d_{34} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

NODE-TO-NODE VALUES



❖ Let $w_{ij} = E s_{ij}^2$ and $W = (w_{ij})$

NODE-TO-NODE VALUES

- ❖ Theorem: In an open network in which the s_{km} are uncorrelated, $E(A_{ij} - a(j))^2$ is given by the (i, j) entry of $(I - R)^{-1} [W \# R + 2S \# R \cdot (I - R)^{-1} \cdot S \# R] (I - R)^{-1}$
- ❖ Not a problem if single-element delays are deterministic
 - The delay on a single link counts the processing delay at the link's head node

CROSS-NETWORK DELAY

- ❖ Expected value is *latency*
- ❖ Standard deviation is *jitter*
- ❖ Packet loss probability is $P\{\text{delay} = \infty\}$

DISTRIBUTIONS

- ❖ Let F and G be $n \times n$ matrices of CDFs
- ❖ Define matrix convolution of F and G by

$$(F * G)_{ij}(x) = \sum_{k=1}^n F_{ik} * G_{kj}(x)$$

- ❖ Repeated convolutions denoted by superscripts $*2$, $*3$, etc.

DISTRIBUTIONS

❖ Theorem: Let s_{km} be nonnegative and mutually stochastically independent and let $M(x)$ denote the matrix $\sum_{m=1}^{\infty} (R \# F)^{*m}(x)$.

Then, in an open network,

$P\{A_{ij} - a(j) \leq x\} = M_{ij}(x)$ and M satisfies the matrix-integral equation

$$M = R \# F + (R \# F) * M.$$

EXAMPLE

A JACKSON NETWORK

❖ Consider a network of K nodes in series

❖ $O = \{1\}$, $D = \{K\}$

$$R_K = \begin{pmatrix} 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \dots & 0 \\ 0 & 0 & 0 & 0 & \vdots & 1 \\ 0 & 0 & 0 & 0 & \dots & 0 \end{pmatrix}$$

EXAMPLE

A JACKSON NETWORK

$$(I - R_K)^{-1} = \begin{pmatrix} 1 & 1 & 1 & 1 & \dots & 1 \\ 0 & 1 & 1 & 1 & \dots & 1 \\ 0 & 0 & 1 & 1 & \dots & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & 0 & \dots & 1 \end{pmatrix} \quad \text{and} \quad (I - R_K^T)^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots & 0 \\ 1 & 1 & 0 & 0 & \dots & 0 \\ 1 & 1 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & 1 & \dots & 0 \\ 1 & 1 & 1 & 1 & \dots & 1 \end{pmatrix}$$

EXAMPLE

A JACKSON NETWORK

- ❖ The traffic equation gives $\lambda_k = \sum_{m=1}^k u_m$, $k = 1, \dots, K$.
- ❖ Jackson's theorem: if $\lambda_k < \mu_k$ for all k , then nodes are independent M/M/1 queues with $a(k) = (\mu_k - \lambda_k)^{-1}$
- ❖ Put $s_{jk} = (\mu_k - \lambda_k)^{-1} \delta_{j,k-1}$ for $j, k = 1, \dots, K$

EXAMPLE

A JACKSON NETWORK

❖ Mean inter-node delays are

$$\begin{pmatrix} 1 & 1 & 1 & 1 & \dots & 1 \\ 0 & 1 & 1 & 1 & \dots & 1 \\ 0 & 0 & 1 & 1 & \dots & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & 0 & \dots & 1 \end{pmatrix}
 \begin{pmatrix} 0 & a_1 & 0 & 0 & \dots & 0 \\ 0 & 0 & a_2 & 0 & \dots & 0 \\ 0 & 0 & 0 & a_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & a_{K-1} \\ 0 & 0 & 0 & 0 & \dots & 0 \end{pmatrix}
 \begin{pmatrix} 1 & 1 & 1 & 1 & \dots & 1 \\ 0 & 1 & 1 & 1 & \dots & 1 \\ 0 & 0 & 1 & 1 & \dots & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & 0 & \dots & 1 \end{pmatrix}$$

EXAMPLE

A JACKSON NETWORK

❖ Simplifies to

$$\begin{pmatrix} 0 & a_1 & a_1 + a_2 & a_1 + a_2 + a_3 & \cdots & a_1 + \cdots + a_{K-1} \\ 0 & 0 & a_2 & a_2 + a_3 & \cdots & a_2 + \cdots + a_{K-1} \\ 0 & 0 & 0 & a_3 & \cdots & a_3 + \cdots + a_{K-1} \\ 0 & 0 & 0 & 0 & \cdots & a_4 + \cdots + a_{K-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

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ADDRESS-BASED ROUTING

- ❖ Every arriving customer has a stated destination
- ❖ Routers (nodes) contains lookup tables that determine the next node given
 - The destination address and
 - The current state of congestion in the network
- ❖ Incompatible with the Markovian routing model

ADDRESS-BASED ROUTING

- ❖ In address-based routing,
 - A customer leaves the network immediately upon reaching destination
 - A customer visits a given node at most once
 - Every origin has different routing tables

- ❖ Can we mimic any of these (approximately) in a Markovian routing model?

CUSTOMER REPEATS NO NODES

- ❖ Let $t_{ij}^{(m)}$ be the probability that a customer reaches j from i in m steps with no repeats
- ❖ Let $T_{ij} = t_{ij}^{(1)} + t_{ij}^{(2)} + \dots + t_{ij}^{(n)}$
- ❖ Let T be the matrix of the T_{ij}
- ❖ Let B be the matrix whose (i, j) element is $r_{ij} - \sum_{k \neq i, j} r_{ik} t_{kj}^{(n)}$

CUSTOMER REPEATS NO NODES

- ❖ Theorem: $T = (I - R)^{-1} B$
- ❖ Note $t_{ij}^{(n)} = \sum_{\sigma \in \mathcal{S}_n} r_{\sigma_1 \sigma_2} r_{\sigma_2 \sigma_3} \cdots r_{\sigma_{n-1} \sigma_n}$
- ❖ Let $M_{ij}^{(m)}$ be the probability that a customer starting from i reaches j for the first time in m steps
- ❖ Of interest: $|M_{ij}^{(m)} - t_{ij}^{(m)}|$ and other measures of how far M is from T

INDIVIDUAL ROUTING MATRICES

- ❖ There is a separate routing matrix R_i for each node i as an originating node
- ❖ Given $X_0 = i$, the Markov chain is acyclic
- ❖ Requires treating each originating node separately

CONCLUSION

- ❖ Generalized cross-network delay
 - Markovian routing
 - Link delays
 - Non-Poisson arrivals
 - Non-exponential service times

- ❖ Path-additive functionals

- ❖ Markovian approximations for address-based routing