On the Chvatál-Complexity of Binary Knapsack Problems

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where $\mathbf{A}$ is an $m \times n$ matrix, $\mathbf{b}$ and $\mathbf{x}$ are vectors of m and n dimensions, respectively.

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2. if the Chvátal cuts added to the set of inequalities (1) and in this way a new the polyhedral set is defined, and the whole procedure is repeated, then after finite many iterations the polyhedral set becomes equal to the integer hull.

Definition.

The number of iterations is called Chvátal rank.

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where $a_1, a_2, \ldots, a_n$ and b are positive integers. Furthermore

$$a_1 \leq a_2 \leq \cdots \leq a_n. \tag{4}$$

2.1 Indexing of Constraints

index	Right-Hand Side		Left-Hand Side
0	$a_1x_1 + a_2x_2 + \cdots + a_nx_n$	$\leq$	b
1	x_1	$\leq$	1
2	x_2	$\leq$	1
	$\vdots$		
n	x_n	$\leq$	1
$n+1$	$-x_1$	$\leq$	0
$n+2$	$-x_2$	$\leq$	0
	$\vdots$		
$2n$	$-x_n$	$\leq$	0

Using the same index set the multipliers of the inequalities of this original constraint set are denoted by $\lambda_0, \ldots, \lambda_{2n}$.

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$\lambda_0 = 1/a_1$, $\lambda_1 = \lambda_2 = 0$ implies $x_1 \leq 0$.

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The case of $n = 3$.

The maximal elements of the feasible solutions belong to one of the cases of the table below:

case	maximal vectors	inequalities of the feasible set
1	$(0,0,0)$	$y_i \leq 0$
2	$(1,0,0)$	$y_2 \leq 0,\ y_3 \leq 0$
3	$(1,0,0),\ (0,1,0)$	$y_1 + y_2 \leq 1,\ y_3 \leq 0$
4	$(1,1,0)$	$y_3 \leq 0$
5	$(1,0,0),\ (0,1,0),\ (0,0,1)$	$y_1 + y_2 + y_3 \leq 1$
6	$(0,0,1),\ (1,1,0)$	$y_1 + y_3 \leq 1,\ y_2 + y_3 \leq 1$
7	$(1,1,0),\ (1,0,1)$	$y_2 + y_3 \leq 1$
8	$(1,1,0),\ (1,0,1),\ (0,1,1)$	$y_1 + y_2 + y_3 \leq 2$
9	$(1,1,1)$	empty

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the multipliers are:

$$\lambda_0 = \frac{1}{a_3}, \lambda_1 = 0, \lambda_2 = 0, \lambda_3 = 0,$$
$$\lambda_4 = \frac{a_1}{a_3}, \lambda_5 = \frac{a_2}{a_3}, \lambda_6 = 0.$$

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$\lambda_0 = \frac{1}{b}, \lambda_1 = 1 - \frac{a_1}{b}, \lambda_2 = 0, \lambda_3 = 1 - \frac{a_3}{b}$,
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$$\lambda_4=0, \lambda_5=0, \lambda_6=\frac{a_3}{a_2}-1.$$

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The set of maximal feasible solutions:
$(1, 1, 1, 0), (0, 0, 1, 1), (0, 1, 0, 1), (1, 0, 0, 1)$

The hyperplane $y_1 + y_2 + y_3 + 2y_4 = 3$

contains all of these maximal feasible points.

4 A Counterexample in Dimension 4

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contains all of these maximal feasible points.

Therefore
$$y_1 + y_2 + y_3 + 2y_4 \leq 3$$

is a valid cut of the integer hull.

4.1 Linear constraints for the generation of the cut

$$
\begin{aligned}
12\lambda_0 + \lambda_1 - \lambda_5 &= 1\\
12\lambda_0 + \lambda_2 - \lambda_6 &= 1\\
14\lambda_0 + \lambda_3 - \lambda_7 &= 1\\
30\lambda_0 + \lambda_4 - \lambda_8 &= 2\\
53\lambda_0 + \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 &< 4
\end{aligned}
$$

4.2 LP formulation

$$\min\ 53\lambda_0 + \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4$$

$$\begin{aligned}
12\lambda_0 + \lambda_1 - \lambda_5 &= 1 \\
12\lambda_0 + \lambda_2 - \lambda_6 &= 1 \\
14\lambda_0 + \lambda_3 - \lambda_7 &= 1 \\
30\lambda_0 + \lambda_4 - \lambda_8 &= 2 \\
\lambda_0, \ldots \lambda_8 &\geq 0
\end{aligned}$$

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In general there are 27 different sets of maximal feasible solutions in dimension 4 if inequality (4) is satisfied.

10	(1,0,0,0), (0,1,0,0), (0,0,1,0), (0,0,0,1)	$y_1 + y_2 + y_3 + y_4 \leq 1$
11	(0,0,1,0), (0,0,0,1), (1,1,0,0)	$y_1 + y_3 + y_4 \leq 1,\ y_2 + y_3 + y_4 \leq 1$
12	(0,0,0,1), (1,1,0,0), (1,0,1,0)	$y_1 + y_2 + y_3 + 2y_4 \leq 2,\ y_2 + y_3 + y_4 \leq 1$
13	(0,0,0,1), (1,1,0,0), (1,0,1,0), (0,1,1,0)	$y_1 + y_2 + y_3 + 2y_4 \leq 2$
14	(0,0,0,1), (1,1,1,0)	$y_1 + y_4 \leq 1,\ y_2 + y_4 \leq 1,\ y_3 + y_4 \leq 1$
15	(1,1,0,0), (1,0,1,0), (1,0,0,1)	$y_2 + y_3 + y_4 \leq 1$
16	(1,1,0,0), (1,0,1,0), (0,1,1,0), (1,0,0,1)	$y_1 + y_2 + y_3 + y_4 \leq 2,\ y_2 + y_4 \leq 1,\ y_3 + y_4 \leq 1$
17	(1,0,0,1), (1,1,1,0)	$y_2 + y_4 \leq 1,\ y_3 + y_4 \leq 1$
18	(1,1,0,0), (1,0,1,0), (0,1,1,0), (1,0,0,1), (0,1,0,1)	$y_1 + y_2 + y_3 + y_4 \leq 2,\ y_3 + y_4 \leq 1$
19	(1,0,0,1), (0,1,0,1), (1,1,1,0)	$y_1 + y_2 + y_4 \leq 2,\ y_3 + y_4 \leq 1$
20	(1,1,1,0), (1,1,0,1)	$y_3 + y_4 \leq 1$
21	(1,1,0,0), (1,0,1,0), (0,1,1,0) and	$y_1 + y_2 + y_3 + y_4 \leq 2$
	(1,0,0,1), (0,1,0,1), (0,0,1,1)	
22	(1,0,0,1), (0,1,0,1), (0,0,1,1), (1,1,1,0)	$y_1 + y_2 + y_3 + 2y_4 \leq 3$
23	(0,0,1,1), (1,1,0,1)	$y_1 + y_3 + y_4 \leq 2,\ y_2 + y_3 + y_4 \leq 2$
24	(0,1,0,1), (1,1,1,0), (1,0,1,1)	$y_1 + y_2 + y_4 \leq 2,\ y_2 + y_3 + y_4 \leq 2$
25	(1,1,1,0), (1,1,0,1), (1,0,1,1)	$y_2 + y_3 + y_4 \leq 2$
26	(1,1,1,0), (1,1,0,1), (1,0,1,1), (0,1,1,1)	$y_1 + y_2 + y_3 + y_4 \leq 3$
27	(1,1,1,1)	$0 \leq y_i \leq 1$ ß $= 1, 2, 3, 4$

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$y_1 + y_2 + y_3 + 2y_4 \leq 3.$

All other cases have Chvátal rank 1.

Theorem

The Chvátal rank of case 22 is higher than 1 if and only if

$$\begin{aligned} a_1 + a_2 + a_3 &\le b \\ a_3 + a_4 &\le b \\ a_1 + a_2 + a_4 &> b \\ a_3 &< \frac{a_4}{2} \\ a_1 + a_2 + a_3 + \frac{a_4}{2} &\le b \end{aligned}$$

5 The [Dahl-Foldnes 2003] Knapsack Problem Polytope

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$$x_1 + \cdots + x_{m_1} + p x_{m_1+1} + \cdots + p x_{m_1+m_2} \leq b. \tag{5}$$

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Let $T = \{m_1 + 1, \ldots, m_1 + m_2\}$ and $S \subseteq \{1, 2, \ldots, m_1\}$ and $s = |S|$, and $1 \leq q < p$.

Then

$$h(s, q) =_{df} \max\{x(S) + qx(T) : x \in F\}$$

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Then

$$h(s,q) =_{df} \max\{x(S) + qx(T) : x \in F\}$$

$$= \begin{cases} b & \text{if } s \geq b \\ \max\{s + q\lfloor \frac{b-s}{p} \rfloor, \\ b - (p-q)\lceil \frac{b-s}{p} \rceil\} & \text{if } b > s \end{cases}$$

Among the last m_2 variables at most $l_{\max} = \min\left\{m_2, \left\lfloor \frac{b}{p} \right\rfloor\right\}$ can have value 1 in any feasible solution.

Theorem *[Dahl-Foldnes 2003]*

(A) The integer hull of the knapsack problem is described by the following system of inequalities:

- *(5),*
- $x(T) \leq l_{\max}$,
- $x(S) + qx(T) \leq h(s,q), \forall S : \emptyset \neq S \subseteq \{1, 2, \ldots, m_1\}$ *and* $\forall q : 1 \leq q < p$,
- $0 \leq x_i \leq 1, i \in \{1, 2, \ldots, m_1 + m_2\}$.

(B) The inequality $x(S) + qx(T) \leq h(s, q)$ *defines a facet of the integer hull if and only if (*$s > q$ *or* $s = q = 1$*) and* $s \in \{q+b-p, q+b-2p, \ldots, q+b-pl_{\max}\}$.

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The above theorem implies that for each $q > 1$ the only value of s giving a facet of the integer hull is

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Hence it follows that $h(s, q) = s$.

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Hence it follows that $h(s, q) = s$.

When has a facet with parameters satisfying (6) a Chvátal rank 1?

The LP model of the best cut of this type in the first Chvátal iteration:

$$
\begin{aligned}
\min\ b\lambda_0 + \lambda_1 + \lambda_2 + \ldots + \lambda_{m_1+1} & & \\
\lambda_0 + \lambda_1 - \lambda_{m_1+2} &= 1 \\
\cdots & \\
\lambda_0 + \lambda_s - \lambda_{m_1+s+1} &= 1 \\
\lambda_0 + \lambda_{s+1} - \lambda_{m_1+s+2} &= 0 \\
\cdots & \\
\lambda_0 + \lambda_{m_1} - \lambda_{2m_1+1} &= 0 \\
p\lambda_0 + \lambda_{m_1+1} - \lambda_{2m_1+2} &= q \\
\lambda_0, \ldots \lambda_{2m_1+2} &\geq 0.
\end{aligned}
$$

The Path of the Simplex Method

The variables $\lambda_1, \ldots \lambda_{m_1+1}$ form a feasible basis.

	λ_0	λ_1		λ_s				λ_{m_1+1}	λ_{m_1+2}						λ_{2m_1+2}	RHS
λ_1	1	1							-1							1
$\vdots$			$\ddots$							$\ddots$						
λ_s	1			1							-1					1
λ_{s+1}	1				1							-1				0
$\vdots$						$\ddots$							$\ddots$			
λ_{m_1}	1						1							-1		0
λ_{m_1+1}	p							1							-1	q
OBF	$b - m_1 - p$	0	$\cdots$	0	0	$\cdots$	0	0	1	$\cdots$	1	1	$\cdots$	1	1	$-q - s$

Case $b \geq m_1 + p$.

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The Chvátal rank is 0. The simplex tableau is optimal.

Case $m_1 + p > b$ and $m_1 > s$.

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After the interchange $\lambda_0 \Leftrightarrow \lambda_{s+1}$ the simplex tableau is this:

	λ_0	λ_1 λ_s	λ_{s+1}		λ_{m_1+1}	λ_{m_1+2}			λ_{2m_1+2}	RHS
λ_1	0	1	-1			-1	1			1
$\vdots$		$\ddots$				$\ddots$				
λ_s	0	1	-1			-1	1			1
λ_0	1		1				-1			0
λ_{s+2}	0		-1	1			1	-1		0
$\vdots$				$\ddots$				$\ddots$		
λ_{m_1}	0		-1	1			1	-1		0
λ_{m_1+1}	0		$-p$		1		p		-1	q
OBF	0	$0 \cdots 0$	$-b + m_1 + p$	$0 \cdots 0$	0	$1 \cdots 1$	$b + 1 - m_1 - p$	$1 \cdots 1$	1	$-q - s$

The current basis is optimal if and only if

$m_1 = s, \; q = 1.$

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It implies that $b + 1 = m_1 + p$. Then this case is a generalization of case 26 in Dimension 4. Hence the only inequality what must be generated to obtain the integer hull is

$$\sum_{j=1}^{m_1+1} x_j \leq m_1.$$

It can be generated by the following weights:

$$\lambda_0 = \frac{1}{p},\ \lambda_1 = \cdots = \lambda_{m_1} = \frac{p-1}{p},$$
$$\lambda_{m_1+1} = \cdots = \lambda_{2m_1+2} = 0.$$

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The case of non-optimality.

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The case of non-optimality.

The sequence of entering variables is

$\lambda_{m_1+s+2},\ \lambda_{m_1+s+3},\ \ldots,\ \lambda_{2m_1},\ \lambda_{2m_1+1}.$

At this moment the simplex tableau is as follows:

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	λ_0	λ_1		λ_s				λ_{m_1+1}							λ_{2m_1+2}	RHS
λ_1	0	1						$-\frac{1}{p}$	-1						$\frac{1}{p}$	$1-\frac{q}{p}$
$\vdots$			$\ddots$							$\ddots$						
λ_s	0			1				$-\frac{1}{p}$			-1				$\frac{1}{p}$	$1-\frac{q}{p}$
λ_0	1							$\frac{1}{p}$							$-\frac{1}{p}$	$\frac{q}{p}$
λ_{m_1+s+2}	0				-1			$\frac{1}{p}$				1			$-\frac{1}{p}$	$\frac{q}{p}$
$\vdots$						$\ddots$							$\ddots$			
λ_{2m_1}	0						-1	$\frac{1}{p}$						1	$-\frac{1}{p}$	$\frac{q}{p}$
OBF	0	0	$\cdots$	0	1	$\cdots$	1	$\frac{q}{p}$	1	$\cdots$	1	0	$\cdots$	0	$1-\frac{q}{p}$	$\frac{q^2}{p}-q-s$

At this moment the simplex tableau is as follows:

	λ_0	λ_1		λ_s				λ_{m_1+1}							λ_{2m_1+2}	RHS
λ_1	0	1						$-\frac{1}{p}$	-1						$\frac{1}{p}$	$1-\frac{q}{p}$
$\vdots$			$\ddots$							$\ddots$						
λ_s	0			1				$-\frac{1}{p}$			-1				$\frac{1}{p}$	$1-\frac{q}{p}$
λ_0	1							$\frac{1}{p}$							$-\frac{1}{p}$	$\frac{q}{p}$
λ_{m_1+s+2}	0				-1			$\frac{1}{p}$				1			$-\frac{1}{p}$	$\frac{q}{p}$
$\vdots$						$\ddots$							$\ddots$			
λ_{2m_1}	0						-1	$\frac{1}{p}$						1	$-\frac{1}{p}$	$\frac{q}{p}$
OBF	0	0	$\cdots$	0	1	$\cdots$	1	$\frac{q}{p}$	1	$\cdots$	1	0	$\cdots$	0	$1-\frac{q}{p}$	$\frac{q^2}{p}-q-s$

This is the optimal simplex tableau.

The optimal objective function value is

$$-\frac{q^2}{p} + q + s.$$

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Thus the Chvátal rank of the facet defining cut is 1 if and only if

$$-\frac{q^2}{p} + q + s < h(s, q) + 1 = s + 1.$$

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Thus the Chvátal rank of the facet defining cut is 1 if and only if

$$-\frac{q^2}{p} + q + s < h(s, q) + 1 = s + 1.$$

This is equivalent to the inequality

$$q^2 - pq + p > 0. \tag{7}$$

If $m_1 + p > b$ and $s = m_1$ then we obtain the same inequality.

If $m_1 + p > b$ and $s = m_1$ then we obtain the same inequality.

Lemma

Let m_1, p, and b be positive integers such that $m_1 + p > b + 1$. Then the Chvátal rank of the integer hull of the set

$$\{\mathbf{x} \in \mathbb{R}^{m_1+1} \mid x_1 + \cdots + x_{m_1} + p x_{m_1+1} \leq b; \\ 0 \leq x_i \leq 1,\ i = 1, \ldots, m_1\} \tag{8}$$

is 1 if and only if no positive integer q with $q < p$ exists such that (7) is violated.

Theorem

Let m_1, p, and b be positive integers such that $m_1 + p > b + 1$ and $p \geq 4$. Then the Chvátal rank of the integer hull of the set (8) is at least 2.

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The main content of the theorem is that although the set defined in (8) has one of the simplest definitions among the sets of binary vectors, its Chvátal rank is still large.

An upper bound of the Chvátal rank

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Iterative step:

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Case $q = i + 1$.

We got a facet defining inequality if $s = b - p + i + 1$. For the sake of simplicity assume that $S = \{1, \ldots, s\}$.

The inequality for $(s, q) =$

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$$+v \cdot (\text{the inequality for } (b-i, p-i) +$$

$$+ \sum_{j=s+1}^{b-i} (-x_j \leq 0)),$$

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where

$$u = \frac{p-2i-1}{(s-1)(p-2i)-i}, v = \frac{s-i-1}{(s-1)(p-2i)-i}.$$

Case $q = p - i - 1$, $s = b - i - 1$.

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$$+(-x_{b-i} \leq 0))$$

where

$$u = \frac{1}{\binom{b-i-2}{b-p+i-1}\left(p - i - \frac{b-i-1}{b-p+i} i\right)}, v = 1 - \frac{1}{p - i - \frac{b-i-1}{b-p+i} i}.$$

Initial step.

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The inequality of $q = 1$ and $s = b - p + 1$ is

$$x_1 + \cdots + x_{m_1} + (p-1)x_{m_1+1} \leq b - 1. \quad (9)$$

Initial step.

The inequality of $q = 1$ and $s = b - p + 1$ is

$$x_1 + \cdots + x_{m_1} + (p-1)x_{m_1+1} \le b - 1. \quad (9)$$

(9) can be generated by the following multipliers:

$$\lambda_0 = \frac{p-1}{p}, \ \lambda_1 = \lambda_2 = \cdots = \lambda_{m_1} = \frac{1}{p},$$
$$\lambda_{m_1+1} = \cdots = \lambda_{2m_1+2} = 0.$$

These results can be summarize in the following statement.

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Theorem

Let m_1, p, and b be positive integers such that $m_1 + p > b + 1$ and $p \geq 4$. Then the Chvátal rank of the integer hull of the set (8) is at most

$$\left\lfloor \frac{p}{2} \right\rfloor .$$

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