The Expressive Power of Binary Submodular Functions

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Problem

Which submodular polynomials can be expressed by (or decomposed into) quadratic submodular polynomials?

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Not all.

Motivation

Expressible by quadratic submodular polynomials
= minimisation via (s,t)-Min-Cut.

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pseudo-Boolean: domain $\{0, 1\}$, coefficients in $\mathbb{R}$

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Submodularity

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Definition

(Quadratic) p is submodular $\Leftrightarrow$ $a_{ij} \le 0$ for all $1 \le i < j \le n$.

Submodularity, cont'd

$$p(x_1, \ldots, p_n) = a_0 + \sum_{i=1}^{n} a_i x_i + \sum_{1 \le i < j \le n} a_{ij} x_i x_j + \sum_{1 \le i < j < k \le n} a_{ijk} x_i x_j x_k + \ldots + \sum_{|I|=k} a_I x_I$$

Submodularity, cont'd

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Definition

p is submodular $\Leftrightarrow$ $\delta_{ij}(\boldsymbol{x}) \leq 0$, where $\delta_{1,2}(\boldsymbol{x}) = p(1,1,\boldsymbol{x}) - p(1,0,\boldsymbol{x}) - p(0,1,\boldsymbol{x}) + p(0,0,\boldsymbol{x})$ is the second-order derivate of p.

Submodularity, cont’d

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  $\psi(A \cup B) + \psi(A \cap B) \leq \psi(A) + \psi(B)$ for all $A, B \subseteq V$
- examples: cut fns, matroid rank fns, entropy fns

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Expressibility

Definition

$$p \in \langle L \rangle \;\Leftrightarrow\; p(\boldsymbol{x}) = \min_{\boldsymbol{z}} \sum_i p_i(\boldsymbol{x}, \boldsymbol{z}) + c, \text{ where } p_i \in L, c \in \mathbb{R}.$$

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Another example

$$p(x_1, x_2, x_3, x_4) = x_1x_2x_3x_4 - x_1x_2 - x_1x_3x_4 - x_2x_3x_4$$

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new: 2 extra variables, 1 is provably not enough

Yet another example

$$p(x_1, x_2, x_3, x_4) = -x_1x_2x_3x_4 + x_1x_3x_4 + x_2x_3x_4 \\ - x_1x_3 - x_1x_4 - x_2x_3 - x_2x_4 - x_3x_4$$

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new: not expressible!

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- minimisation of quadratic submodular polynomials
  = (s, t)-Min-Cut [Hammer '65]

Reduction to (s, t)-Min-Cut

If

$$p(x_1, \ldots, x_n) = a_0 + \sum_{i=1}^{n} a_i x_i + \sum_{1 \leq i < j \leq n} a_{ij} x_i x_j,$$

where $a_{ij} \leq 0$ for all $1 \leq i < j \leq n$.

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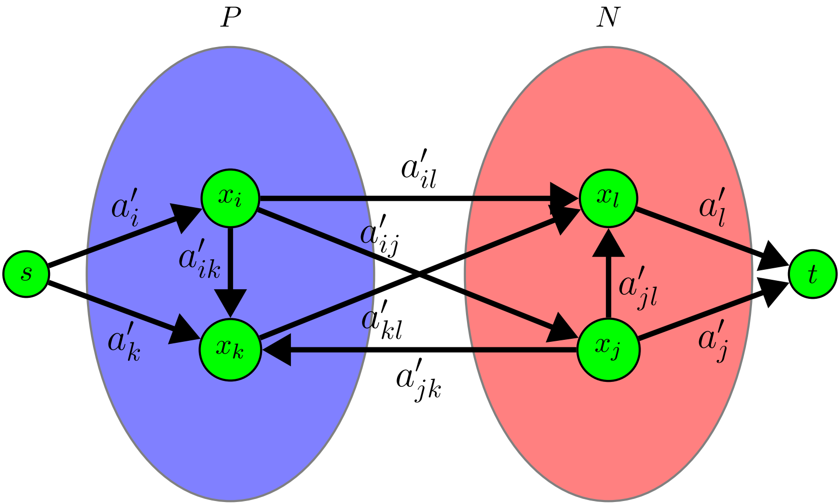
where $a_{ij} \le 0$ for all $1 \le i < j \le n$.
Then,

$$p = a'_0 + \sum_{i\in P} a'_i x_i + \sum_{j\in N} a'_j (1-x_j) + \sum_{1\le i<j\le n} a'_{ij} (1-x_i) x_j$$

s.t. $a'_i, a'_j, a'_{ij} \ge 0$.

Reduction to (s, t)-Min-Cut, cont'd



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- here we assume submodularity (CSP, CV)

Result: Expressibility of fans

Theorem [Ž., Cohen, Jeavons '08]

All FANS are expressible by quadratic submodular polynomials.

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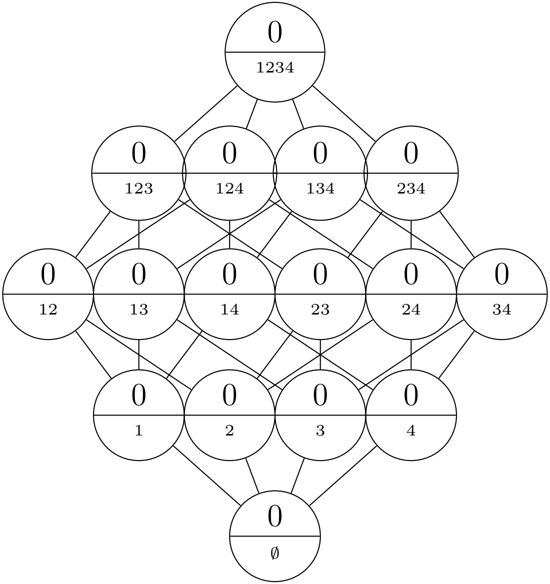
(generalise previous expressibility results)
(arity k fan expressible with $O(k)$ extra variables)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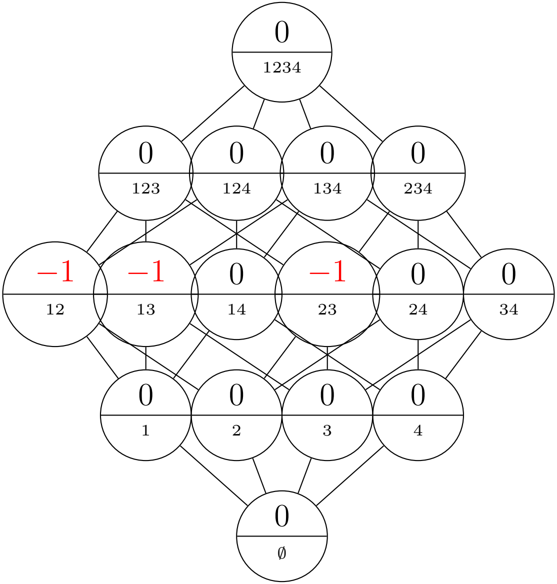
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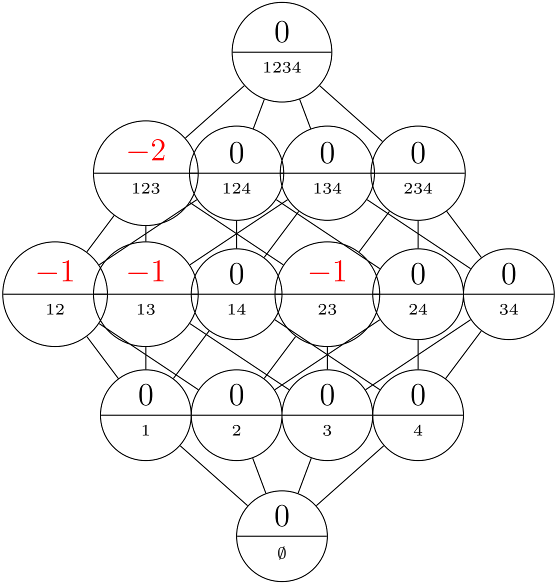
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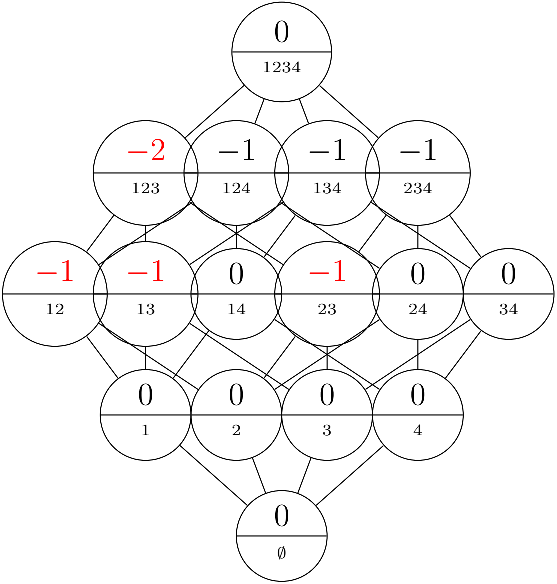
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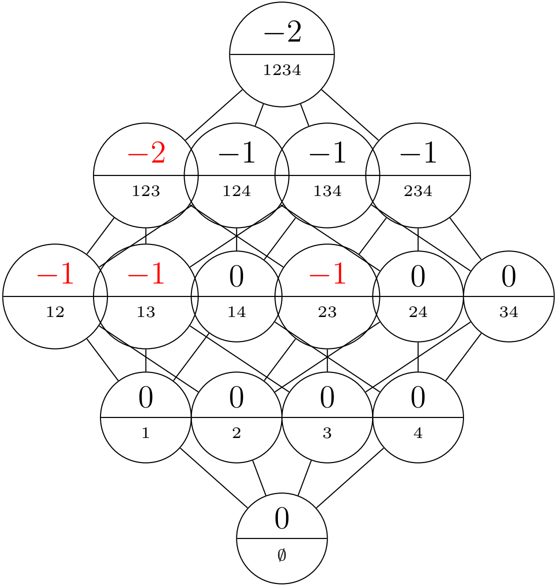
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More on expressibility

Theorem [Cooper, Cohen, Jeavons '06]

1. $p \in \langle L \rangle \;\Leftrightarrow\; \mathsf{fPol}(L) \subseteq \mathsf{fPol}(\{p\})$
2. $p \in \langle L \rangle \;\Rightarrow\; p \in \langle L \rangle$ with at most 2^{2^k} extra variables

Result: Non-expressibility

Theorem [Ž., Cohen, Jeavons '08]

Let $p \in L_{sub,4}$. Then the following are equivalent:

1. $p \in \langle L_{sub,2} \rangle$
2. $p \in \mathrm{Cone}(L_{fans,4})$
3. $\mathcal{F}_{sep} \in \mathsf{fPol}(\{p\})$
4. $\forall \{i,j\},\{k,l\} \subseteq [n]$ distinct : $a_{ij} + a_{kl} + a_{ijk} + a_{ijl} \leq 0$.

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Proof: $(2) \Rightarrow (1)$ (Fans expressible), $(1) \Rightarrow (3)$ (**Characterisation of fPol**), $(3) \Rightarrow (2)$ (The same polyhedra), $(3) \Leftrightarrow (4)$

Results, cont'd

Theorem [Ž., Cohen, Jeavons '08]

- Canonical examples qin.
- Testing whether $p \in L_{sub,4} \cap \langle L_{sub,2} \rangle$ is co-NP-complete.
- Not all extreme rays are Fans (Promislow & Young).
- Optimal number of variables.

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- KEY: fPol$(L_{sub,2})$ are conservative Hamming distance non-increasing weighted mappings.

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- VCSP instance $\Rightarrow p = \sum_i p_i$

Conclusions

- not all submodular polynomials are expressible over $L_{sub,2}$
- most submodular submodular polynomials are not
- extreme rays vs. expressibility

Q&A

Thank you!

Many thanks to the organisers!

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