Answers and questions in selected topics of probabilistic programming

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*International Colloquium on Stochastic Modeling and Optimization*
*Dedicated to the 80th birthday of Professor András Prékopa*

RUTCOR, Rutgers University

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The Book of Answers

René Henrion

80th birthday of Professor András Prékopa
Theorem (Prékopa 1971)

If a random vector $\xi$ has a log-concave density, then its law is log-concave.

What makes the beauty of this Theorem?

- The statement is very simple.
- The result is powerful.

Insertion

The consequences are manifold
- Key for convexity theory in probabilistic programming.
- Impact on Numerics
- Stability
- Structure

80th birthday of Professor András Prékopa
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Stability

Optimization problem: \( \min \{ f(x) \mid x \in C, \mathbb{P}(\xi \leq Ax) \geq p \} \)

Distribution of \( \xi \) rarely known \( \implies \) Approximation by some \( \eta \) \( \implies \) Stability?

Solution set mapping: \( \Psi(\eta) := \arg\min \{ f(x) \mid x \in C, \mathbb{P}(\eta \leq Ax) \geq p \} \)
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**Theorem (R.H./W.Römisch 2004)**

- \( f \) convex, \( C \) convex, closed, \( \xi \) has log-concave distribution function
- \( \Psi(\xi) \) nonempty and bounded
- \( \exists x \in C : \ \mathbb{P}(\xi \leq Ax) > p \) (Slater point)

Then, \( \Psi \) is upper semicontinuous at \( \xi \):

\[ \Psi(\eta) \subseteq \Psi(\xi) + \varepsilon B \quad \text{for} \quad d_K(\mathbb{P} \circ \eta^{-1}, \mathbb{P} \circ \xi^{-1}) < \delta \]
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\]

If in addition

- \( f \) convex-quadratic, \( C \) polyhedron,
- \( \xi \) has strongly log-concave distribution function,

then \( \Psi \) is locally Hausdorff-Hölder continuous at \( \xi \):

\[
d_{\text{Haus}}(\Psi(\eta), \Psi(\xi)) \leq \sqrt{d_K(P \circ \eta^{-1}, P \circ \xi^{-1})} \quad \text{(locally around} \, \xi)\]
When is a distribution function $F_\xi$ strongly log-concave?

$$\log F_\xi(\lambda x + (1 - \lambda)y) \geq \lambda \log F_\xi(x) + (1 - \lambda) \log F_\xi(y) + \kappa \lambda(1 - \lambda)\|x - y\|^2$$
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**Proposition**

$\xi_i$ independent, $F_{\xi_i}$ strongly log-concave $\implies F_\xi$ strongly log-concave.

**Example**

- The multivariate normal distribution function with independent components is strongly log-concave on bounded convex sets.
- The uniform distribution on multivariate intervals $[a, b]$ is strongly log-concave on int $[a, b]$. 
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**Question 1:** Density strongly log-concave $\implies$ Distribution function strongly log-concave?

**Question 1'**: Multivariate normal distribution function strongly log-concave?
Failure of rotation invariance of strong log-concavity

uniform distribution on unit square and log

$45^\circ$ rotation of unit square
Lipschitz continuity of distribution functions of quasi-concave probability measures

**Definition**

A probability measure $\mathbb{P}$ on $\mathbb{R}^n$ is *quasi-concave* if

$$\mathbb{P}(\lambda A + (1 - \lambda)B) \geq \min\{\mathbb{P}(A), \mathbb{P}(B)\}$$

for all $\lambda \in [0, 1]$, $A, B \in \mathcal{B}(\mathbb{R}^n)$.

Log-concavity (and $\alpha$-concavity) implies quasi-concavity.

**Theorem (R.H./W.Römisch 2005)**

Let $\xi$ have a quasi-concave law $\mathbb{P}$ and denote its distribution function by $F_\xi$.

- $F_\xi$ is Lipschitz $\iff$ $F_\xi$ is continuous
- $\iff$ $\text{supp } \mathbb{P} \notin \text{canonic hyperplane} \iff \text{Var } \xi_i \neq 0 \ \forall i$
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Corollary
If the $s$-dimensional random vector has a density $f_\xi$ such that $f_\xi^{-1/s}$ is convex, then its distribution function $F_\xi$ is Lipschitz continuous.
Lipschitz continuity of singular normal distributions

\[ \Sigma = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \Sigma = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad \Sigma = \begin{pmatrix} 1 \\ -1 \\ -1 & 1 \end{pmatrix} \]

\[ \text{supp } P \]

Fig. 1 Distribution functions of 2-dimensional singular normal distributions with covariance matrix having rank one (see text)

2 Lipschitz continuity of quasi-concave distributions

We start this section by introducing the class of quasi-concave probability measures (see Prékopa 1995). By \( P(\mathbb{R}^s) \) we denote the set of probability measures on \( \mathbb{R}^s \).

Definition 2.1 A probability measure \( \mu \in P(\mathbb{R}^s) \) is called quasi-concave whenever

\[ \mu(\lambda A + (1-\lambda)B) \geq \min \{ \mu(A), \mu(B) \} \]

holds true for all convex and Borel measurable subsets \( A, B \subseteq \mathbb{R}^s \) and all \( \lambda \in [0,1] \) such that \( \lambda A + (1-\lambda)B \) is Borel measurable.

It is well known that a large class of prominent multivariate distributions shares the property of being quasi-concave. Among those are the multivariate normal distribution (nondegenerate or singular), the Dirichlet-, Pareto-, Gamma-, Log-normal distributions (possibly with a restricted range of parameters) as well as uniform distributions over compact, convex subsets of \( \mathbb{R}^s \) (see Prékopa 1995; Borell 1975). Consequently, all future statements in this section apply in particular to singular normal distributions.

For the proof of our Lipschitz criterion, we shall make use of the following three propositions:

Proposition 2.1 A quasiconcave measure \( \mu \in P(\mathbb{R}) \) has either a density or coincides with some Dirac measure, i.e., \( \mu = \delta_x \) for some \( x \in \mathbb{R} \).

Proof Follows immediately from Theorem 3.2 in Borell (1975). \( \square \)

Proposition 2.2 If for all marginal distributions \( \mu_i \) of \( \mu \in P(\mathbb{R}^s) \) there exist bounded densities on \( \mathbb{R} \), then the distribution function \( F_\mu \) of \( \mu \) is Lipschitz continuous.

Proof See Proposition 3.8 in Römisch and Schultz (1993). \( \square \)

Proposition 2.3 If \( \mu \in P(\mathbb{R}) \) is a quasiconcave measure with density \( f_\mu \), then \( f_\mu \) is bounded.

Proof According to Theorem 3.2 in Borell (1975), the possibly extended-valued function \( 1/f_\mu \) is convex and the support of \( \mu \) is a convex subset of \( \mathbb{R} \). Assuming that \( f_\mu \) is unbounded, there exists a sequence \( \{x_n\} \subseteq \mathbb{R} \) such that \( f_\mu(x_n) \geq n \). If \( \{x_n\} \) is unbounded, then, without loss of generality, it is increasing, hence \( [x_1,\infty) \subseteq \text{supp } \mu \) and \( \{1/f_\mu(x_n)\} \) is decreasing. Since \( 1/f_\mu \) is convex, it follows that \( 1/f_\mu \) is decreasing on \( [x_1,\infty) \). Therefore,
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Linear probabilistic constraints with Gaussian coefficient matrix: structural properties

Feasible Set: \( M_p := \{ x \in \mathbb{R}^n \mid \mathbb{P}(\Xi x \leq a) \geq p \} \)

Denoting by \( \xi_i \) the rows of \( \Xi \), we assume that \( \xi_i \sim \mathcal{N}(\mu_i, \Sigma_i) \quad \forall i \).

Theorem (R.H. 2007)

- \( M_p \) is compact if \( p > \min_i \{ \Phi(\|\mu_i\|\Sigma_i^{-1}) \} \)
  \((\Phi = \text{one-dimensional standard normal distribution function})\)

- \( M_p \) is empty if \( p \geq \min_{a_i < 0} \{ \Phi(\|\mu_i\|\Sigma_i^{-1}) \} \)

- If \( a \geq 0 \) then \( M_p \) is a nonempty and star-shaped (= connected) set.
Linear probabilistic constraints with Gaussian coefficient matrix: convextity

\[ M_p := \{ x \in \mathbb{R}^n \mid \mathbb{P}(\Xi x \leq a) \geq p \} \quad \text{vec} \, \Xi \sim \mathcal{N}(\mu, \Sigma) \quad (\xi_i = \text{rows of } \Xi) \]

Theorem (Prékopa 1974)

*If all cross covariance matrices \( \text{Cov}(\xi_i, \xi_j) \) are proportional, then \( M_p \) is convex for \( p \geq 0.5 \). The same holds true, if the \( \xi_i \) refer to the columns of \( \Sigma \).*
Linear probabilistic constraints with Gaussian coefficient matrix: **convexity**

\[ M_p := \{ x \in \mathbb{R}^n \mid P(\Xi x \leq a) \geq p \} \quad \text{vec } \Xi \sim \mathcal{N}(\mu, \Sigma) \quad (\xi_i = \text{rows of } \Xi) \]

**Theorem (Prékopa 1974)**

*If all cross covariance matrices Cov($\xi_i, \xi_j$) are proportional, then $M_p$ is convex for $p \geq 0.5$. The same holds true, if the $\xi_i$ refer to the columns of $\Sigma$.***

**Theorem (R.H./C. Strugarek 2008)**

*If all $\xi_i$ are pairwise independent (not the components of $\xi_i$!), then $M_p$ is convex for $p \geq p^* := \Phi(\max\{\sqrt{3}, \tau\})$ with*

\[
\tau := \max_i \lambda_{\max}^{(i)} \left[ \lambda_{\min}^{(i)} \right]^{-3/2} \| \mu_i \|
\]

\[
\lambda_{\max}^{(i)}, \lambda_{\min}^{(i)} := \text{largest and smallest eigenvalue of } \Sigma_i.
\]

*Question: Does it hold true that $M_p$ is convex for $p \geq 0.5$ (or: $p \geq \bar{p}$) for any multivariate normal distribution of the elements of $\Xi$?*
Linear probabilistic constraints with Gaussian coefficient matrix: **convexity**

\[ M_p := \{ x \in \mathbb{R}^n \mid \mathbb{P}(\Xi x \leq a) \geq p \} \quad \text{vec } \Xi \sim \mathcal{N}(\mu, \Sigma) \quad (\xi_i = \text{rows of } \Xi) \]

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\( \lambda_{\text{max}}, \lambda_{\text{min}} \): = largest and smallest eigenvalue of \( \Sigma_i \).*

**Question:** Does it hold true that \( M_p \) is convex for \( p \geq 0.5 \) (or: \( p \geq \bar{p} \)) for any multivariate normal distribution of the elements of \( \Xi \)?
Linear probabilistic constraints with a random coefficient row vector

\[ M^\alpha_p := \{ x \in \mathbb{R}^n \mid \mathbb{P}(\langle \xi, x \rangle \leq \alpha) \geq p \} \]

Theorem (R.H. 2007)

Let \( \xi \) have a density. Then, there exists some \( d \) such that

\[ M^\alpha_p = \{ d \} + \left( M^{\neg\alpha}_{1-p} \right)^c \quad \forall \alpha \neq 0 \forall p \in (0, 1) \]

Result has an impact on properties which are not affected by translation and closure (e.g., convexity, boundedness, nontriviality).
Linear probabilistic constraints with Gaussian coefficient matrix: **Gradients**

\[ M_p := \{ x \in \mathbb{R}^n \mid P(\Xi x \leq a) \geq p \} \]

\[ \text{vec} \, \Xi \sim \mathcal{N}(\mu, \Sigma) \quad (\xi_i = \text{rows of } \Xi) \]
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\[ \mathbb{P}(\Xi x \leq a) \geq p \quad = \quad \Phi^{0,\Sigma(x)}(\alpha(x)) \]

\[ (\Sigma(x))_{i,j} := \langle x, \text{Cov} (\xi_i, \xi_j) x \rangle \]

\[ \alpha_i(x) := a_i - \langle \mu_i, x \rangle \]

\[ \implies \text{ for fixed } x: \text{ value of a multivariate normal distribution function} \]
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Question:

Gradients?

Sensitivities of normal distribution functions w.r.t. correlations?
80th birthday of Professor András Prékopa
Boldog éveket,
jó egészséget
és még sok új tételt
kivánok önnek!
Uniform Distribution on the Disk

Formula for the distribution function:

\[ F_\xi(x, y) = \begin{cases} 
\varphi(x, y) & \text{if } x \leq 0, y \leq 0 \\
2\varphi(0, y) - \varphi(-x, y) & \text{if } y \leq 0, x > 0 \\
2\varphi(x, 0) - \varphi(x, -y) & \text{if } y > 0, x \leq 0 \\
1 - 2\varphi(0, -y) - 2\varphi(-x, 0) + \varphi(-x, -y) & \text{if } y > 0, x > 0 
\end{cases} \]

where

\[ \varphi(x, y) := \begin{cases} 
\frac{1}{2\pi} \left( x\sqrt{1 - x^2} + y\sqrt{1 - y^2} + 2yx + \arcsin x - \arcsin \left( -\sqrt{1 - y^2} \right) \right) & \text{if } x^2 + y^2 \leq 1 \\
0 & \text{else} 
\end{cases} \]

Check log-concavity!

Density is a constant \( \implies \) log-concave.
Probabilistic constraints in hydro power management

Numerical experiments for Electricité de France (A. Möller, WIAS Berlin)
Profit maximization in hydro power production, reservoir with random inflow and prices, probabilistic filling level constraints.
Optimal release policy with weekly decisions over one year (n=52):

Optimal release policy with daily decisions over half a year (n=182):

René Henrion 80th birthday of Professor András Prékopa