Advanced Operations Management: Solutions to Homework 1

Fall, 2000

Problem 1

Solve unconstrained problem:

$$\begin{array}{ll} \max & f(x) = 100x - 10x^2 - 150 \\ \text{s.t.} & x \in \mathbb{R} \end{array}$$

$$\begin{aligned} f'(x) &= 100 - 20x = 0 \\ f''(x) &= -20 < 0 \end{aligned}$$

imply that $x = 5$ is the only local maximum. Thus it is an optimal solution and corresponding sales are $f(5) = 100$.

Problem 2

Total cost per pound is

$$f(x) = \frac{TC(x)}{x} = \frac{4}{3}x^3 - 3x^2 + 2x + 5.$$

Solve

$$\begin{array}{ll} \min & f(x) = \frac{4}{3}x^3 - 3x^2 + 2x + 5 \\ \text{s.t.} & x \geq \frac{3}{10} \end{array}$$

Compute derivatives:

$$\begin{aligned} f'(x) &= 4x^2 - 6x + 2 \\ f''(x) &= 8x - 6 \end{aligned}$$

Equation

$$f'(x) = 4x^2 - 6x + 2 = 0$$

has solutions $x = 1$ and $x = \frac{1}{2}$.

$$f''(1) = 2 > 0 \Rightarrow \min,\ f''(\frac{1}{2}) = -2 \Rightarrow \max$$

Two candidates for the minimum are $x = \frac{3}{10}$ (case III) and $x = 1$ (case I). Compare:

$$f(1) = 5.333,\ f(\frac{3}{10}) = 5.366 > f(1)$$

Thus $x = 1$ is an optimal solution.

Problem 3

(a) $20 + 5x = 40$ implies $x = 4$.

(b) Average travel time:

$$f(x) = (20 + 5x)x + 40(10 - x) = 5x^2 - 20x + 400$$

Solve

$$\begin{array}{ll} \min & f(x) = 5x^2 - 20x + 400 \\ \text{s.t.} & 0 \le x \le 10 \end{array}$$

$$f'(x) = 10x - 20 = 0$$

when $x = 2$ and

$$f''(x) = 10 > 0$$

implies that $x = 2$ is a local minimum.

$$f(2) = 380, \ f(0) = 400 > 380, \ f(10) = 700 > 380$$

Thus, $x = 2$ is an optimal solution. (Also, can check that $f'(0) > 0$ and $f'(10) < 0$.)

Problem 4

(a) Revenues:

$$R(x_1, x_2) = p_1 x_1 + p_2 x_2 = \frac{1}{2}(32 - x_1)x_1 + (22 - x_2)x_2$$

Profit:

$$\begin{aligned} P(x_1, x_2) &= R(x_1, x_2) - TC(x_1, x_2) \\ &= -x_1^2 - 2x_1x_2 - 2x_2^2 + 16x_1 + 22x_2 - 73 \end{aligned}$$

(b) Solve

$$\begin{array}{ll} \max & P(x_1, x_2) = -x_1^2 - 2x_1x_2 - 2x_2^2 + 16x_1 + 22x_2 - 73 \\ \text{s.t.} & x_1, x_2 \in \mathbb{R} \end{array}$$

$$\begin{aligned} \frac{\partial P}{\partial x_1} &= 16 - 2x_1 - 2x_2 = 0 \\ \frac{\partial P}{\partial x_2} &= 22 - 2x_1 - 4x_2 = 0 \end{aligned}$$

give a stationary point $x_1 = 5$, $x_2 = 3$.

Second partial derivatives:

$$\frac{\partial^2 P}{\partial {x_1}^2} = -2, \ \frac{\partial^2 P}{\partial {x_2}^2} = -4, \ \frac{\partial^2 P}{\partial x_1 \partial x_2} = -2$$

give a Hessian:

$$H = \left\| \begin{array}{cc} -2 & -2 \\ -2 & -4 \end{array} \right\|, \ H_1 = -2 < 0, \ H_2 = 8 - 4 = 4 > 0.$$

Thus, $x_1 = 5$, $x_2 = 3$ is the only local maximum. Corresponding prices are $p_1 = 13.5$ and $p_2 = 19$.

Problem 5

Solve

$$\begin{array}{ll} \max & E(x_1, x_2) \\ \text{s.t.} & x_1, x_2 \in \mathbb{R} \end{array}$$

$$\begin{aligned} \frac{\partial E}{\partial x_1} &= -6x_1 + 2x_2 + 30 = 0 \\ \frac{\partial E}{\partial x_2} &= 2x_1 - 12x_2 + 24 = 0 \end{aligned}$$

gives $x_1 = 6$, $x_2 = 3$. Second partial derivatives:

$$\frac{\partial^2 E}{\partial {x_1}^2} = -6, \ \frac{\partial^2 E}{\partial {x_2}^2} = -12, \ \frac{\partial^2 E}{\partial x_1 \partial x_2} = 2$$

give a Hessian:

$$H = \left\| \begin{array}{cc} -6 & 2 \\ 2 & -12 \end{array} \right\|, \ H_1 = -6 < 0, \ H_2 = 72 - 4 = 68 > 0.$$

Thus $x_1 = 6$, $x_2 = 3$ is the only local maximum with $E(6, 3) = 40$.

Problem 6

Express profit as a function of prices:

$$\begin{aligned} P(p_1, p_2) &= p_1 q_1 + p_2 q_2 - 25 q_1 - 72 q_2 \\ &= (p_1 - 25)(60 - 3p_1 + p_2) + (p_2 - 72)(80 - 2p_1 + p_2) \end{aligned}$$

Solve

$$\begin{array}{ll} \max & P(x_1, x_2) = (p_1 - 25)(60 - 3p_1 + p_2) + (p_2 - 72)(80 - 2p_2 + p_1) \\ \text{s.t.} & x_1, x_2 \in \mathbb{R} \end{array}$$

$$\begin{aligned} \frac{\partial P}{\partial x_1} &= 63 - 6p_1 + 2p_2 = 0 \\ \frac{\partial P}{\partial x_2} &= 199 + 2p_1 - 4p_2 = 0 \end{aligned}$$

give a stationary point $p_1 = 32.5$, $p_2 = 66$.

Second partial derivatives:

$$\frac{\partial^2 P}{\partial {p_1}^2} = -6, \ \frac{\partial^2 P}{\partial {p_2}^2} = -4, \ \frac{\partial^2 P}{\partial p_1 \partial p_2} = 2$$

give a Hessian:

$$H = \left\| \begin{array}{cc} -6 & 2 \\ 2 & -4 \end{array} \right\|, \ H_1 = -6 < 0, \ H_2 = 24 - 4 = 20 > 0.$$

Thus, $p_1 = 32.5$, $p_2 = 66$ is the only local maximum. However, one of the corresponding product quantities is negative due to unrealistically selected numerical constants of this problem: $q_1 = 28.5$, $q_2 = -19.5$