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A TCHEBYCHEFF DEA MODEL

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Abstract. In this study a new Data Envelopment Analysis (DEA) model is proposed that minimizes the L_∞ (or Tchebycheff) distance of each Decision Making Unit (DMU) to the efficient frontier. This model is obtained by using the same constraints of the additive model but with a different objective function. This function is based on the L_∞ metric and, contrary to the additive model, relies on the minimization of the distance of the actual operation point to the observed extreme frontier. This last issue is important since a better efficiency evaluation can be achieved with this model with respect to the additive model. The conditions in which the normal (instead of the augmented) Tchebycheff distance could be used to guarantee the attainment of efficient solutions are shown. Simple graphical examples (in 2D and 3D) are used to illustrate the proposed DEA model.

Keywords: Data Envelopment Analysis (DEA); L_∞ metric.

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1 Introduction

In the Data Envelopment Analysis (DEA) literature four models have been considered as the basic DEA models (see Charnes et al., 1994, for a presentation and comparative analysis of these models):

1) CCR model – This model was presented in the DEA seminal work of Charnes, Cooper and Rhodes (1978). The CCR model is based on the radial minimization (maximization) of all inputs (outputs) and assumes an environment of Constant Returns to Scale (CRS);
2) BCC model – The Banker, Charnes and Cooper (1984) model is the Variable Returns to Scale (VRS) version of the CCR model;
3) Additive model – The additive model was originated in the work of Charnes et al. (1985). This model maximizes the L1 distance of the analyzed Decision Making Unit (DMU) to the observed efficient frontier (or envelopment surface) and assumes VRS;
4) Multiplicative model – This model was first presented in the work of Charnes et al. (1982, 1983) and results from the application of the other basic models to the logarithms of the original data values.

In this study a new DEA model is proposed that minimizes the L_∞ (or Tchebycheff) distance of the analyzed DMU (k) to the envelopment surface. This model is obtained by using the same constraints of the additive model but a different objective function is considered. This function is based on the L_∞ metric and, contrary to the additive model, it relies on the minimization of the distance of the actual DMU point of operation to the observed extreme frontier. This last issue is important since a better efficiency evaluation can be achieved with this model with respect to the additive model (see also Briec, 1999).

In section 2 a simple DEA example with 1 input and 1 output is presented, where the additive and Tchebycheff projections are shown. In section 3 the Tchebycheff DEA model formulation is presented. The circumstances in which it is necessary to use the augmented Tchebycheff distance, instead of the normal Tchebycheff distance, are explained in section 4. In section 5 the Tchebycheff model is applied to an illustrative example with 2 inputs and 1 output. Finally some conclusions are drawn and future directions of possible work are outlined.

2 An illustrative example with 1 input and 1 output

The example in this section is based on the example of Charnes et al. (1994). This example illustrates a case of 7 DMUs with 1 input and 1 output (see figure 1).

DMUs	X	Y
P1	2	2
P2	3	5
P3	6	7
P4	9	8
P5	5	3
P6	4	1
P7	10	7

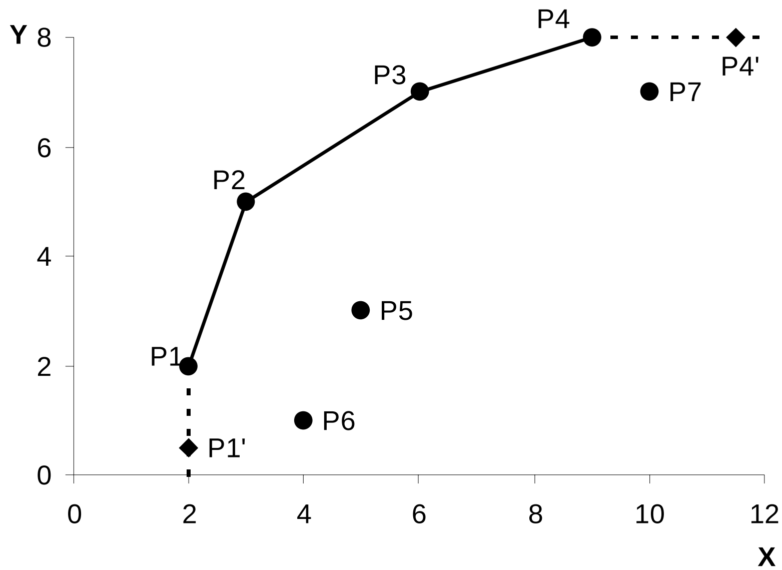

Figure 1: Illustrative example.

In figure 1 it is possible to recognize that P_1, P_2, P_3 and P_4 units are operating better than the others, in an environment of VRS. This is because a DMU is (Pareto) efficient if there is no other DMU, or a nonnegative convex linear combination of m inputs and s outputs of some DMUs, that improves one factor (output or input) without worsening at least one of the other factors.

DEA sets an envelopment surface in the space of dimension $m+s$, which is defined by the factors' observed values of the efficient DMUs (or obtained by a nonnegative convex linear combination of some efficient DMUs). The DMUs that do not belong to this envelopment surface (or efficient frontier), and are part of its interior, are operating inefficiently.

The efficient frontier (figure 1) is given by the lines $\overline{P_1P_2}$, $\overline{P_2P_3}$ and $\overline{P_3P_4}$. The inefficient DMUs (P_5, P_6 and P_7) are enveloped inside this surface.

Supposing that $P_1^{'}$ is an extra DMU, then $P_1^{'}$ is a weakly[1] efficient unit, but it is clearly inefficient since P_1 produces a higher level of output with the same input consumption. Similar comments could be stated about $P_4^{'}$.

Usually, besides classifying each DMU as efficient/inefficient, a DEA model returns an efficient projection point on the efficient frontier for each inefficient DMU.

[1] One solution is (Pareto) weakly efficient, if and only if there is no efficient solution that is strictly better in all performance measures.

The additive model returns a projected point based on the maximum L1 distance between the analyzed DMU's point of operation and the efficient frontier. For example (see figure 2), DMU P_6 has its maximum L1 distance[2] to the envelopment surface in P_2 and the minimum in A (if weakly efficient points are considered) or P_1 (if only efficient points are considered). Dashed lines in figure 2 are the L1-metric isodistance contours, that is the loci of points of the same distance away from P_6.

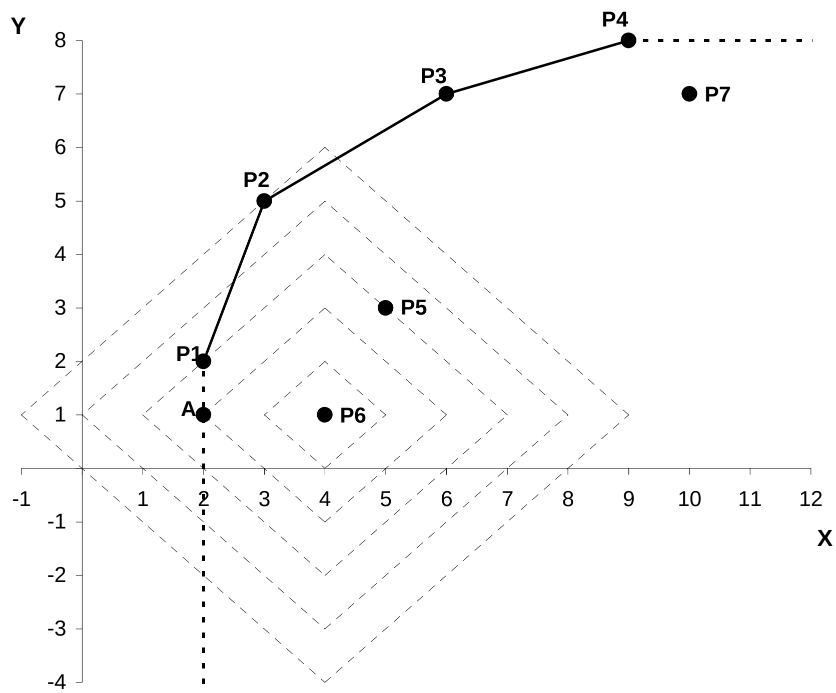

Figure 2: Example of L1 metric.

[2] For a Lp metric, the distance between two points, $a,b \in \Re^n$, is given by

$$\|a-b\|_p = \sqrt[p]{\sum_{i=1}^{n} |a_i - b_i|^p} \qquad p \in \{1,2,3,...\} \cup \{\infty\}$$

Within the family of Lp metrics, L1 and L∞ metrics are of particular interest in the field of MultiObjective Linear Programming. This is because they are the only Lp metrics that result in linear scalar problems, when minimizing a distance of the frontier to a reference point

$$\|a-b\|_1 = \sum_{i=1}^{n} |a_i - b_i|$$

$$\|a-b\|_\infty = \max_{i=1,\dots,n} (|a_i - b_i|)$$

L1 metric is a distance function in which every factors' difference (both inputs and outputs) is taken into consideration in the direct proportion of its magnitude (thus leading to the solution with maximum aggregate achievement; see figure 2).

The L∞ metric is a distance function where the absolute value of the largest coordinates' difference between two points absolutely dominates. Therefore, if this metric is used in DEA, only the largest factors' difference is taken into account (thus leading to the most balanced solution between achievements of different factors). In figure 3 it is possible to see how point B that minimizes the L∞ distance between P_6 and the efficient frontier is obtained. Dashed lines in figure 3 are the L∞-metric distance isocontours.

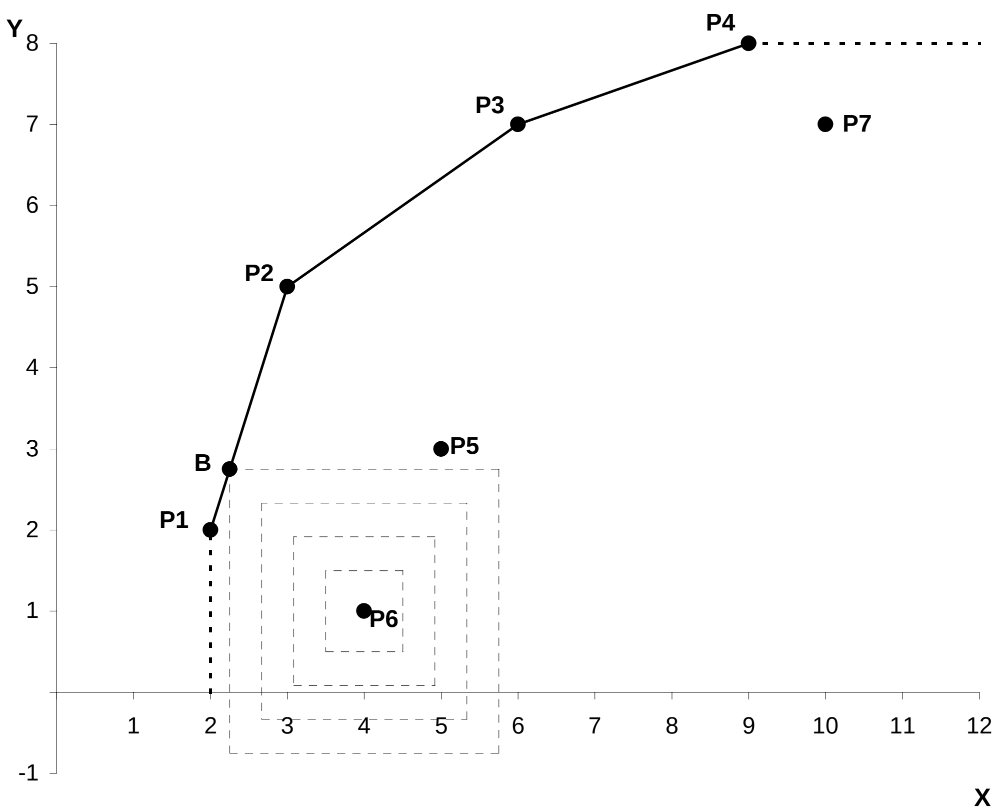

Figure 3: Example of L∞ metric.

3 Tchebycheff DEA model formulation

Before presenting the Tchebycheff DEA model for a DMU k, it is shown how its actual point of operation, its projected point and the decision space in the DEA context are represented:

- Let x_{ik} represent input i of DMU k and y_{rk} represent output r of DMU k. Then its actual point of operation is given by

$$(X,Y)=(x_{1k},...,x_{mk},y_{1k},...,y_{sk})$$

If $\lambda_{jk}\geq 0,\ j=1,...,n \quad \wedge \quad \sum_{j=1}^{n}\lambda_{jk}=1$, then the projected point of DMU k is given by

$$(\hat{X},\hat{Y})=\left(\sum_{j=1}^{n}\lambda_{jk}x_{1j},...,\sum_{j=1}^{n}\lambda_{jk}x_{mj},\sum_{j=1}^{n}\lambda_{jk}y_{1j},...,\sum_{j=1}^{n}\lambda_{jk}y_{sj}\right) \tag{1}$$

- In general, DEA models assume that every input of a DMU k projected point should be lower than or equal to its actual value and every output of a DMU k projected point should be higher than or equal to its actual value:

$$\sum_{j=1}^{n}\lambda_{jk}x_{ij}\leq x_{ik}, \qquad i=1,...,m$$

$$\sum_{j=1}^{n}\lambda_{jk}y_{rj}\geq y_{rk}, \qquad r=1,...,s$$

By having into consideration the last assumptions and $s_{ik}^{-}=x_{ik}-\sum_{j=1}^{n}\lambda_{jk}x_{jk}, i=1,...,m$, $s_{rk}^{+}=\sum_{j=1}^{n}\lambda_{jk}y_{jk}-y_{rk},\ \ r=1,...,s$, the L1 distance between DMU k actual point of operation and its projected point is given by

$$\begin{aligned}
&\left\|(X,Y)-(\hat{X},\hat{Y})\right\|_1\\
&=\left|x_{1k}-\sum_{j=1}^{n}\lambda_{jk}x_{1j}\right|+...+\left|x_{mk}-\sum_{j=1}^{n}\lambda_{jk}x_{mj}\right|+\\
&+\left|y_{1k}-\sum_{j=1}^{n}\lambda_{jk}y_{1j}\right|+...+\left|y_{sk}-\sum_{j=1}^{n}\lambda_{jk}y_{sj}\right|\\
&=x_{1k}-\sum_{j=1}^{n}\lambda_{jk}x_{1j}+...+x_{mk}-\sum_{j=1}^{n}\lambda_{jk}x_{mj}+\\
&+\sum_{j=1}^{n}\lambda_{jk}y_{1j}-y_{1k}+...+\sum_{j=1}^{n}\lambda_{jk}y_{sj}-y_{sk}\\
&=s_{1k}^{-}+...+s_{mk}^{-}+s_{1k}^{+}+...+s_{sk}^{+}\\
&=\sum_{i=1}^{m}s_{ik}^{-}+\sum_{r=1}^{s}s_{rk}^{+}
\end{aligned}$$

The corresponding L∞ metric is given by

$$\left\|(X,Y)-\left(\hat{X},\hat{Y}\right)\right\|_\infty = \max\left|\begin{array}{l}\left|x_{1k}-\sum_{j=1}^{n}\lambda_{jk}x_{1j}\right|,\ldots,\left|x_{mk}-\sum_{j=1}^{n}\lambda_{jk}x_{mj}\right|, \\ \left|y_{1k}-\sum_{j=1}^{n}\lambda_{jk}y_{1j}\right|,\ldots,\left|y_{sk}-\sum_{j=1}^{n}\lambda_{jk}y_{sj}\right|\end{array}\right| = \max\left(s_{1k}^{-},\ldots,s_{mk}^{-},s_{1k}^{+},\ldots,s_{sk}^{+}\right)$$

The following linear programming formulation is the DEA additive model (ADD) applied to DMU k:

$$\max_{\lambda_k, s_k^+, s_k^-} \sum_{i=1}^{m} s_{ik}^{-} + \sum_{r=1}^{s} s_{rk}^{+} \qquad \text{(ADD)}$$

subject to:

$$\begin{aligned} s_{ik}^{-} &= x_{ik} - \sum_{j=1}^{n}\lambda_{jk}x_{ij}, && i=1,\ldots,m \\ s_{rk}^{+} &= \sum_{j=1}^{n}\lambda_{jk}y_{rj} - y_{rk}, && r=1,\ldots,s \\ \sum_{j=1}^{n}\lambda_{jk} &= 1 && \\ s_{ik}^{-} &\geq 0, && i=1,\ldots,m \\ s_{rk}^{+} &\geq 0, && r=1,\ldots,s \\ \lambda_{jk} &\geq 0, && j=1,..,n \end{aligned} \qquad (2)$$

The additive model uses a maximization and not a minimization objective function because in the later case the solution is always $s_{ik}^{-}=0, i=1,\ldots,m$, $s_{rk}^{+}=0,\ r=1,\ldots,s$. In that case, the DMU k projected point coincides with its actual operation point. This happens because the reference point (or actual point of operation) belongs to the decision space.

The minimization of a distance function based on the L∞ metric is interesting because it captures the attitude of minimizing the worst case and it does not imply a heavier computational burden (regarding the L1 metric) because the min-max problem can be transformed into a linear programming problem.

Assuming that each factor projection is a function ($\bar{f}_k$) to be maximized in the case of an output and a function to be minimized in the case of an input, then the following model[3] is a

[3] The inputs minimization was transformed in maximization through their symmetrical values.

Multiple Objective Linear Programming (MOLP) problem that reflects the efficiency of DMU k:

$$\max_{\lambda_k, s_k^+, s_k^-} \bar{f}_k$$

$$\bar{f}_k \equiv \left| \begin{array}{ll} \mathrm{f}_k(i) = -\sum_{j=1}^{n} \lambda_{jk} x_{ij}, & i = 1,\ldots,m \\ \mathrm{f}_k(r+m) = \sum_{j=1}^{n} \lambda_{jk} y_{rj}, & r = 1,\ldots,s \end{array} \right. \qquad \text{(M1)}$$

subject to (2)

One of the so-called scalarizing processes generally used in MOLP to compute efficient solutions consists in minimizing a distance to a reference point (which is generally taken as the ideal solution, that is the point in the objective function space that would optimize all the objective functions simultaneously and which is not feasible whenever the functions are conflicting). The problem to be solved by using the L∞ metric is

$$\min_{\lambda_k, s_k^+, s_k^-} \left\| \bar{f}_k^A - \bar{f}_k \right\|_\infty \iff \min_{\lambda_k, s_k^+, s_k^-} \max_{p=1,\ldots,m+s} \left| f_k^A(p) - f_k(p) \right| \qquad \text{(M2)}$$

subject to (2) subject to (2)

Since $\bar{f}_k^A$ is the reference point or the actual DMU k operation point:

$$\bar{f}_k^A - \bar{f}_k$$

$$= \left| \begin{array}{ll} f_k^A(i) - f_k(i) = -x_{ik} - \; -\sum_{j=1}^{n} \lambda_{jk} x_{ij} \;, & i = 1,\ldots,m \\ f_k^A(r+m) - f_k(r+m) = y_{rk} - \sum_{j=1}^{n} \lambda_{jk} y_{rj}, & r = 1,\ldots,s \end{array} \right.$$

$$= \left| \begin{array}{ll} f_k^A(i) - f_k(i) = -x_{ik} + \sum_{j=1}^{n} \lambda_{jk} x_{ij} = -s_{ik}^-, & i = 1,\ldots,m \\ f_k^A(r+m) - f_k(r+m) = -s_{rk}^+, & r = 1,\ldots,s \end{array} \right.$$

$$= \left(-s_{1k}^-, \ldots, -s_{mk}^-, -s_{1k}^+, \ldots, -s_{sk}^+ \right)$$

The minimax objective function in M2 can be formulated as:

$$\min_{V,\lambda_k,s_k^+,s_k^-} V \qquad \text{(M3)}$$

subject to:

$$V \geq f_k^A(p) - f_k(p),\ p = 1,\dots,m+s$$

(

V free

V is a free variable because the reference point is inside the decision space. However, solutions given by model M3 can be weakly efficient (Steuer, 1986). To avoid these points, it is possible to use the Tchebycheff augmented distance (D) instead of the ordinary Tchebycheff distance[4] (V):

$$D = V + \varepsilon \sum_{p=1}^{m+s} \left[f_k^A(p) - f_k(p) \right]$$

$$= V - \varepsilon \ \sum_{i=1}^{m} s_{ik}^- + \sum_{r=1}^{s} s_{rk}^+ \ |$$

By making $V = -U_k$ and using D, the following formulation for M3 is obtained:

$$\max_{U_k,\lambda_k,s_k^+,s_k^-} U_k + \varepsilon \ \sum_{i=1}^{m} s_{ik}^- + \sum_{r=1}^{s} s_{rk}^+ \ | \qquad \text{(M4)}$$

subject to:

$$U_k \leq s_{ik}^-, \quad i = 1,\dots,m$$

$$U_k \leq s_{rk}^+, \quad r = 1,\dots,s$$

(2)

U_k free

The solution to M4 is always nonnegative: cf. the constraints of M4, in particularly (2) ($s_{ik}^- \geq 0, i = 1,\dots,m$ and $s_{rk}^+ \geq 0,\ r = 1,\dots,s$). It is then possible to assume that $U \geq 0$ and the following Tchebycheff DEA model (TCH) formulation is obtained:

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[4] ε is a small positive quantity. Usually ε=10E-6.

$$\max_{U_k,\lambda_k,s_k^+,s_k^-} U_k + \varepsilon \left(\sum_{i=1}^{m} s_{ik}^- + \sum_{r=1}^{s} s_{rk}^+ \right) \qquad \text{(TCH)}$$

subject to:

$$\begin{aligned}
& U_k \le s_{ik}^-, && i=1,\dots,m \\
& U_k \le s_{rk}^+, && r=1,\dots,s \\
& s_{ik}^- = x_{ik} - \sum_{j=1}^{n} \lambda_{jk} x_{ij}, && i=1,\dots,m \\
& s_{rk}^+ = \sum_{j=1}^{n} \lambda_{jk} y_{rj} - y_{rk}, && r=1,\dots,s \\
& \sum_{j=1}^{n} \lambda_{jk} = 1 && \\
& s_{ik}^- \ge 0, && i=1,\dots,m \\
& s_{rk}^+ \ge 0, && r=1,\dots,s \\
& \lambda_{jk} \ge 0, && j=1,..,n \\
& U_k \ge 0 &&
\end{aligned}$$

Theorem 1:

DMU k is operating efficiently if and only if the optimal solution to TCH model applied to DMU k is 0.

Proof: The projected point of an efficient DMU k is given by its actual operation point, which is given by $\lambda_{kk} = 1 \wedge \lambda_{jk} = 0$, $j = 1,\dots,n,\ j \ne k$. Substituting these values into the constraints of the TCH model null slack values ($s_{ik}^- = 0, i = 1,\dots,m$ and $s_{rk}^+ = 0,\ r = 1,\dots,s$) are obtained. By substituting these slack values into the other constraints then $U_k \le 0$. This implies that $U_k^* = 0$, and since all components of the objective function are zero then its value is zero. ■

4 Weakly efficient points

In the previous section, it was shown that if it is necessary to attain efficient solutions in the Tchebycheff model, then the augmented Tchebycheff distance should be used. In this section, the conditions in which the normal Tchebycheff distance could be used to guarantee the attainment of efficient solutions are shown.

The TCH model without the augmented portion in the objective function leads to a simplified version (M5) of that model because the variables $s_{ik}^-, i = 1,\dots,m$ and $s_{rk}^+,\ r = 1,\dots,s$ could be removed. This is possible because their non-negativity is guaranteed by the constraints of the TCH model: $0 \le U_k \le s_{ik}^-, i = 1,\dots,m$ and $0 \le U_k \le s_{rk}^+, r = 1,\dots,s$.

$$\max_{U_k,\lambda_k} U_k \tag{M5}$$

subject to:

$$U_k \le x_{ik} - \sum_{j=1}^{n} \lambda_{jk} x_{ij}, \quad i = 1,\ldots,m$$

$$U_k \le \sum_{j=1}^{n} \lambda_{jk} y_{rj} - y_{rk}, \quad r = 1,\ldots,s$$

$$\sum_{j=1}^{n} \lambda_{jk} = 1$$

$$\lambda_{jk} \ge 0, \qquad j = 1,..,n$$

$$U_k \ge 0$$

The following results will be illustrated at the end of this section.

<u>Theorem 2</u>:

If a DMU k exists that has one best factor value ($x_{ik} = \min_{j=1,\ldots,n}(x_{ij}), i = 1 \vee \ldots \vee i = m$ or $y_{rk} = \max_{j=1,\ldots,n}(y_{rj}),\ r = 1 \vee \ldots \vee r = s$) then the optimal objective function of M5 is 0 ($U_k^* = 0$).

<u>Proof</u>: Without loss of generality for the output case, let us suppose that DMU k has an input i verifying $x_{ik} = \min_{j=1,\ldots,n}(x_{ij})$. By combining convexity and input i constraints, the following result is obtained:

$$\left|\begin{array}{l} U_k \le x_{ik} - \sum_{j=1}^{n} \lambda_{jk} x_{ij} \\ \sum_{j=1}^{n} \lambda_{jk} = 1 \\ \lambda_{jk} \ge 0,\ j = 1,..,n \\ U_k \ge 0 \end{array}\right. \Leftrightarrow \left|\begin{array}{l} U_k \le x_{ik} - \sum_{j=1}^{n} \lambda_{jk} x_{ij} \\ \lambda_{kk} = 1 - \sum_{\substack{j=1 \\ j\ne k}}^{n} \lambda_{jk} \\ \lambda_{jk} \ge 0,\ j = 1,..,n \\ U_k \ge 0 \end{array}\right. \Leftrightarrow \left|\begin{array}{l} U_k \le x_{ik} - \lambda_{kk} x_{ik} - \sum_{\substack{j=1 \\ j\ne k}}^{n} \lambda_{jk} x_{ij} \\ \lambda_{kk} = 1 - \sum_{\substack{j=1 \\ j\ne k}}^{n} \lambda_{jk} \\ \lambda_{jk} \ge 0,\ j = 1,..,n \\ U_k \ge 0 \end{array}\right.$$

$$\left|\begin{array}{l} U_k \le x_{ik} - x_{ik} + x_{ik} \sum_{\substack{j=1 \\ j\ne k}}^{n} \lambda_{jk} - \sum_{\substack{j=1 \\ j\ne k}}^{n} \lambda_{jk} x_{ij} \\ \lambda_{jk} \ge 0,\ j = 1,..,n \wedge U \ge 0 \end{array}\right. \Leftrightarrow \left|\begin{array}{l} U_k \le \sum_{\substack{j=1 \\ j\ne k}}^{n} \lambda_{jk} (x_{ik} - x_{ij}) \\ \lambda_{jk} \ge 0,\ j = 1,..,n \wedge U_k \ge 0 \end{array}\right. \tag{3}$$

$$x_{ik} \le x_{ij} \wedge \lambda_{jk} \ge 0,\ j = 1,..,n \Rightarrow$$
$$\lambda_{jk} x_{ik} \le \lambda_{jk} x_{ij},\ j = 1,..,n \Rightarrow \sum_{\substack{j=1 \\ j \ne k}}^{n} \lambda_{jk} \left(x_{ik} - x_{ij} \right) \le 0 \tag{4}$$

By combining (3) and (4), it is easy to verify that $0 \le U_k \le 0$. Therefore, the only possible solution is $U_k^* = 0$. ■

Theorem 3:

If a DMU k exists that has one best factor value ($x_{ik} = \min_{j=1,...,n} \left(x_{ij} \right), i = 1 \vee ... \vee i = m$ or $y_{rk} = \max_{j=1,...,n} \left(y_{rj} \right),\quad r = 1 \vee ... \vee r = s$) then one possible optimal solution for M5 is $\lambda_{kk}^* = 1 \wedge \lambda_{jk}^* = 0,\ j = 1,..,n,\ j \ne k$.

Proof: Without loss of generality for the output case, let us suppose that DMU k has an input i verifying $x_{ik} = \min_{j=1,...,n} \left(x_{ij} \right)$. By using theorem 2 and its proof, it can be concluded that constraint i is binding. Therefore, by using (3) the following equation is obtained:

$$\sum_{\substack{j=1 \\ j \ne k}}^{n} \lambda_{jk} \left(x_{ik} - x_{ij} \right) = 0 \tag{5}$$

$\lambda_{jk}^* = 0,\ j = 1,..,n,\ j \ne k$ is a possible solution to (5). Therefore $\lambda_{kk}^* = 1 - \sum_{i=1,\ j \ne k}^{n} \lambda_{jk}^* = 1$. Moreover, this solution is unique if and only if $x_{ik} < x_{ij},\ j = 1,...,n,\ j \ne k$. This occurs because equation (5) is a sum of products and the second product factor is always negative ($x_{ik} - x_{ij} < 0,\ j = 1,...,n,\ j \ne k$), implying that the first product component shall always be zero ($\lambda_{jk}^* = 0,\ j = 1,..,n,\ j \ne k$). ■

Theorem 4:

If a DMU k exists that has a factor (i) value that has a value difference δ to a DMU p ($\delta = x_{ik} - x_{ip} \wedge \delta \ge 0$, $i = 1 \vee ... \vee i = m$ or $\delta = y_{rp} - y_{rk} \wedge \delta \ge 0$, $r = 1 \vee ... \vee r = s$) and the global optimum of M5 is δ, then one possible optimal solution for M5 is $\lambda_{pk}^* = 1 \wedge \lambda_{jk}^* = 0,\ j = 1,..,n,\ j \ne p$.

Proof: Without loss of generality for the output case, it is possible to conclude (3) for the case of input i. If $U_k^* = \delta$ then the i constraint is binding and the following equation is obtained:

$$\sum_{\substack{j=1 \\ j \ne k}}^{n} \lambda_{jk} \left(x_{ik} - x_{ij} \right) = \delta \Leftrightarrow \lambda_{pk} \delta + \sum_{\substack{j=1 \\ j \ne k,p}}^{n} \lambda_{jk} \left(x_{ik} - x_{ij} \right) = \delta \tag{6}$$

By analyzing (6) it is clear that $\lambda^*_{pk} = 1 \wedge \lambda^*_{jk} = 0$, $j = 1,..,n, j \neq p$ is a solution to (6). Consequently, it is only necessary to show that the other constraints are satisfied. This is done for the case of output r. Let us consider the corresponding constraint:

$$\begin{aligned} &\sum_{\substack{j=1 \\ j\neq k}}^{n} \lambda_{jk}\left(y_{rj} - y_{rk}\right) \geq \delta \\ &\Leftrightarrow \lambda_{pk}\left(y_{rp} - y_{rk}\right) + \sum_{\substack{j=1 \\ j\neq k,p}}^{n} \lambda_{jk}\left(y_{rj} - y_{rk}\right) \geq \delta \end{aligned} \tag{7}$$

$\lambda^*_{pk} = 1 \wedge \lambda^*_{jk} = 0, j = 1,..,n, j \neq p$ is a possible solution to (7) because the optimal solution is $U^*_k = \delta$, implying that the minimum deviation of output r between DMUs p and k should be greater or equal to δ ($y_{rp} - y_{rk} \geq \delta$), according to the L∞ metric definition. ■

Theorem 5:
M5 can have optimal weak efficient solutions, if and only if there exists at least one weakly efficient DMU.
Proof: This result is a consequence of previous theorems 2, 3 and 4. ■

The results of theorems 2, 3, 4 and 5, are illustrated by using the example of figure 1, to which 3 more DMUs (P_8, P_9 and P_{10}; see figure 4) have been added.

DMUs	X	Y
P1	2	2
P2	3	5
P3	6	7
P4	9	8
P5	5	3
P6	4	1
P7	10	7
P8	2	1
P9	2.25	0.5
P10	10	7.5

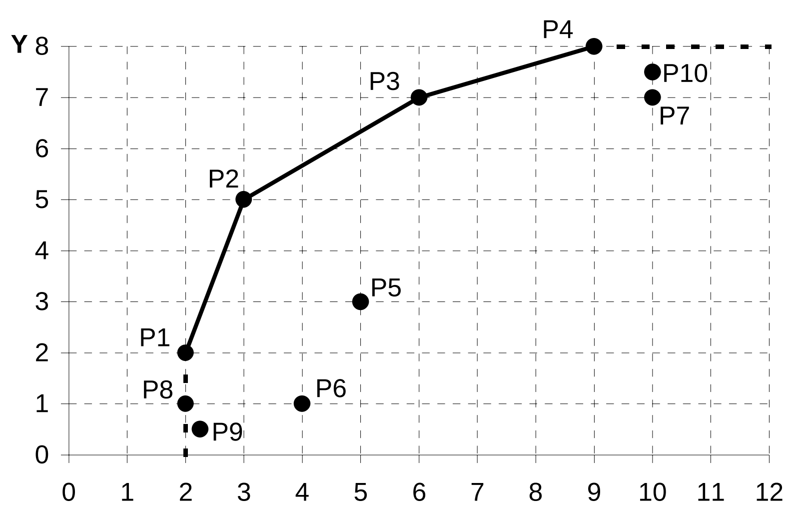

Figure 4: Illustrative example of weak efficient projections.

DMU 8 is a weakly efficient DMU. Then, by theorem 5, M5 can return weakly efficient projections for inefficient DMUs. This happens because, by theorem 2, the projected point of DMU 8 can be P_8 itself.

The projected point of DMU 9 or DMU 8, given by M5, can be a point belonging to the line between P_8 and P_1. These points are the basic optimal solutions to M5. Therefore, in that case, M5 returns alternative optimal solutions ($\alpha P_8 + (1-\alpha)P_1, \alpha \in [0..1]$).

Analyzing P_{10}, it is easy to see that, after solving M5, $U^* = 0.5$. Since $y_{10} - y_4 = U^* = 0.5$ then, by theorem 5, a possible projected point is P_4. Since, there are not weakly efficient DMUs in the output side, as it is DMU 8 in the input side, then the optimal solution of M5 applied to DMU 10 is unique (P_4).

5 An illustrative example with 2 inputs and an 1 output

In this section the application of the TCH model is made to an example with 10 DMUs, 2 inputs and 1 output. The factors' data, its envelopment surface and its efficient frontier are displayed in figure 5.

	P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9	P_{10}
Input$_1$	4,5	6	9	13,5	7,5	8,5	10,5	12	13	16,5
Input$_2$	135,7	145	133	93,9	150	100	24,5	120	175,5	65
Output$_1$	12,8	24,5	34	38,5	21	8,75	15,6	29,3	31,5	17,5

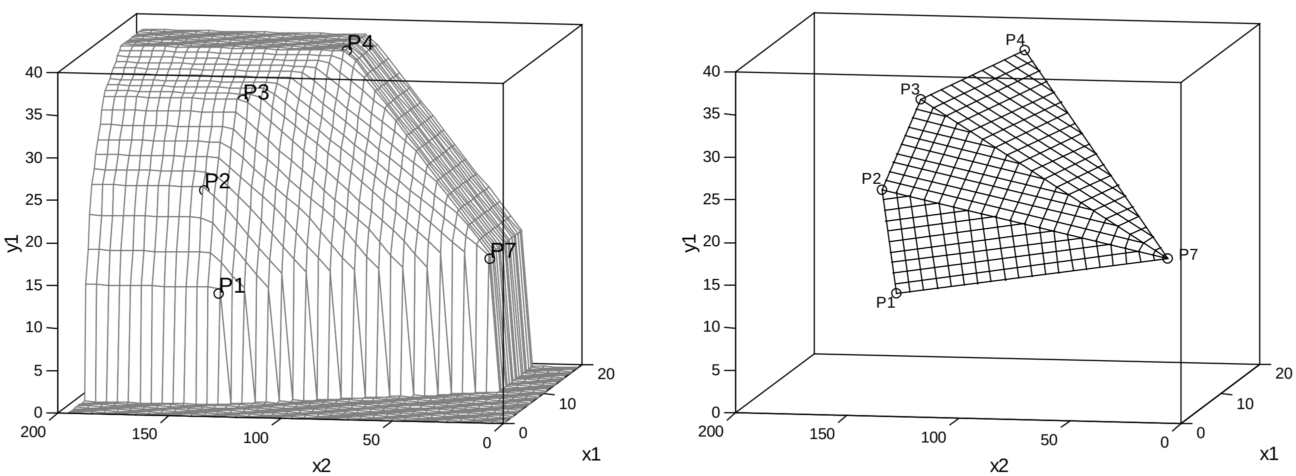

Figure 5: Example data, its envelopment surface and efficient frontier.

By assuming VRS, the efficient frontier is defined by the triangular facets whose vertices are the efficient DMUs' operation points: {P_1, P_2, P_7}, {P_2, P_3, P_7} and {P_3, P_4, P_7}. The remaining DMUs are inefficient and they are located in the interior of the envelopment surface.

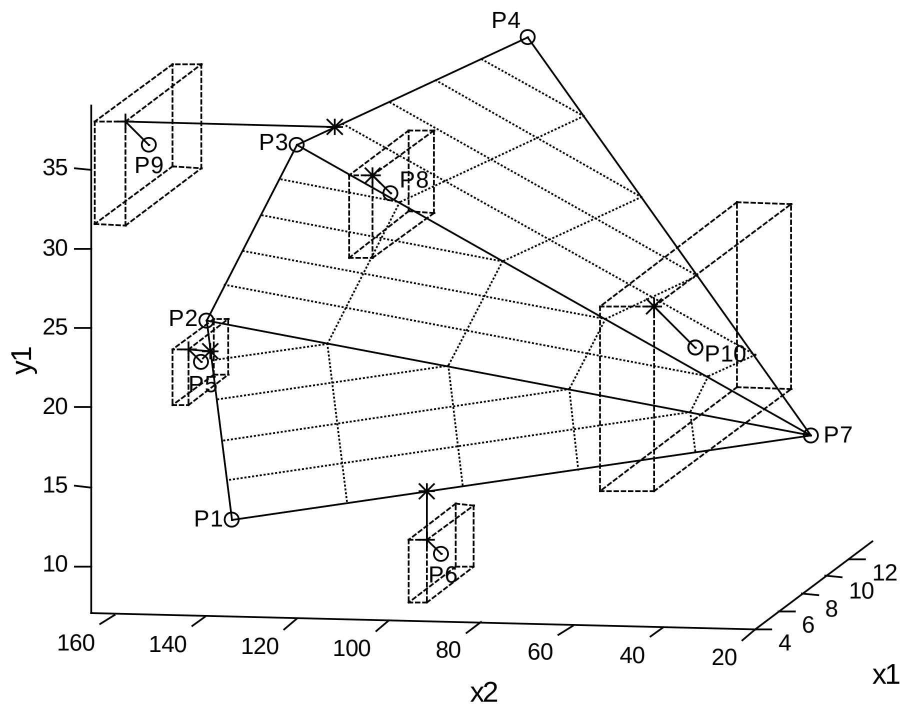

Figure 6: Inefficient DMUs projections based on the minimum L∞ distance to the efficient frontier.

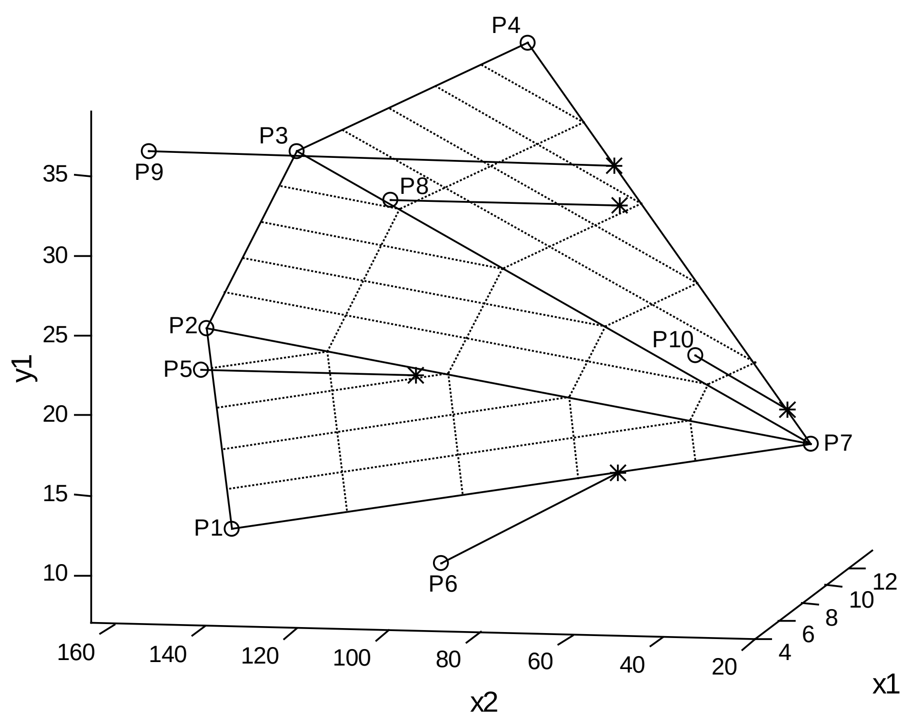

Figure 7: Inefficient DMUs projections based on the maximum L1 distance to the efficient frontier.

As it is well known, DEA basic models return an efficient projection for the inefficient DMUs. In figures 6 and 7, projections based on the minimization of the $L\infty$ distance and on the maximization of the L1 distance are displayed. By comparing these figures, it can be seen that the TCH model returns projections in which the projected point tends to be closer to the actual operation point of an inefficient DMU, with respect to the ADD model. This issue is particularly important for studies that attempt to find better efficiency measures (Pastor et al., 1999).

The results of the TCH and ADD models are presented in tables 1 and 2.

Table 1: Example TCH model optimal solutions.

DMU_k	U_k	λ_{1k}	λ_{2k}	λ_{3k}	λ_{4k}	λ_{7k}	s_{1k}^-	s_{2k}^-	s_{1k}^+
1	0	1	0	0	0	0	0	0	0
2	0	0	1	0	0	0	0	0	0
3	0	0	0	1	0	0	0	0	0
4	0	0	0	0	1	0	0	0	0
5	1.7273	0.1515	0.8485	0	0	0	1.7273	6.4091	1.7273
6	1.9676	0.6613	0	0	0	0.3387	1.9676	1.9676	4.9985
7	0	0	0	0	0	1	0	0	0
8	2.5932	0	0	0.8254	0.0483	0.1263	2.5932	2.5932	2.5932
9	3.2500	0	0	0.8333	0.1667	0	3.2500	49.0167	3.2500
10	5.8237	0	0	0.2137	0.1656	0.6207	5.8237	5.8237	5.8237

Table 2: Example ADD model optimal solutions.

DMU_k	λ_{1k}	λ_{2k}	λ_{3k}	λ_{4k}	λ_{7k}	s_{1k}^-	s_{2k}^-	s_{1k}^+
1	1	0	0	0	0	0	0	0
2	0	1	0	0	0	0	0	0
3	0	0	1	0	0	0	0	0
4	0	0	0	1	0	0	0	0
5	0.0364	0.6182	0	0	0.3455	0	46.9655	0
6	0.3333	0	0	0	0.6667	0	38.4333	5.9167
7	0	0	0	0	1	0	0	0
8	0	0	0.0754	0.5377	0.3869	0	50.0060	0
9	0	0	0	0.6943	0.3057	0.4170	102.8140	0
10	0	0	0	0.0830	0.9170	5.7511	34.7419	0

The projected points, for both models, are obtained by using (1). These projected points are presented in table 3.

Table 3: Example TCH and ADD projections.

				TCH model				ADD model			
DMU_k	x_{1k}	x_{2k}	y_{1k}	$\hat{x}_{1k}$	$\hat{x}_{2k}$	$\hat{y}_{1k}$	ED*	$\hat{x}_{1k}$	$\hat{x}_{2k}$	$\hat{y}_{1k}$	ED
1	4.50	135.70	12.80	4.50	135.70	12.80	0.00	4.50	135.70	12.80	0.00
2	6.00	145.00	24.50	6.00	145.00	24.50	0.00	6.00	145.00	24.50	0.00
3	9.00	133.00	34.00	9.00	133.00	34.00	0.00	9.00	133.00	34.00	0.00
4	13.50	93.90	38.50	13.50	93.90	38.50	0.00	13.50	93.90	38.50	0.00
5	7.50	150.00	21.00	5.77	143.59	22.73	6.86	7.50	103.03	21.00	46.97
6	8.50	100.00	8.75	6.53	98.03	13.75	5.72	8.50	61.57	14.67	38.88
7	10.50	24.50	15.60	10.50	24.50	15.60	0.00	10.50	24.50	15.60	0.00
8	12.00	120.00	29.30	9.41	117.41	31.89	4.49	12.00	69.99	29.30	50.01
9	13.00	175.50	31.50	9.75	126.48	34.75	49.24	12.58	72.69	31.50	102.81
10	16.50	65.00	17.50	10.68	59.18	23.32	10.08	10.75	30.26	17.50	35.21

*Euclidean distance betw een the projecction and the actual point of operation

As a final comment to this section, it must be remarked that the results of the TCH and M5 models are the same for this example, because no weakly efficient DMUs exist.

6 Conclusions and work currently underway

In this study, a new DEA model (TCH) has been presented which is based on the minimization of the Tchebycheff distance. This issue distinguishes this model from the additive model, which is based on the maximization of the L1 distance of the actual operation point to the observed extreme frontier. This issue is important since a better efficiency evaluation can be achieved with the proposed model with respect to the additive model.

Two models have been presented regarding the attainment of weakly efficient points (model M5) and efficient points (model TCH). Some results concerning weakly efficient points have been shown.

The TCH model assumes VRS, but by removing the convexity constraint $\sum_{j=1}^{n}\lambda_{jk}=1$ this model can be transformed into a CRS model.

This new model can be extended by using the weighted Tchebycheff augmented distance. The authors are currently working in this topic where attention will be paid to interactive aspects. Moreover, this distance function could minimize the problems associated with the dependence of results on units change.

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