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RESEARCH

REPORT

ON INCIDENCE ALGEBRAS AND TRIANGULAR MATRICES

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RRR 35-2002 NOVEMBER 2002

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Abstract. Incidence algebras of finite pre-ordered sets are shown to correspond exactly to matrix algebras over the commutative ring of diagonal matrices. Among these the conjugates by permutation matrices of the algebra of upper triangular matrices are precisely the incidence algebras of linear orders, and their various intersections are the poset incidence algebras.

1 Introduction

The theory of Möbius functions, including the classical Möbius function of number theory and the combinatorial inclusion-exclusion formula, is set in the context of incidence algebras (see Rota [R2]. Roman [R1], Stanley [S]). The *incidence algebra* of a finite poset $(S, \leq_r)$ over a field **F** is the set of all maps

$$a : S^2 \to \mathbf{F} \tag{1}$$

such that $a(x, y) = 0$ unless $x \leq_r y$. The notation $\leq_r$ stands for an arbitrary *partial order* on S (reflexive, transitive and antisymmetric binary relation). Addition in the incidence algebra is defined by

$$(a + b)(x, y) = a(x, y) + b(x, y) \tag{2}$$

and multiplication is defined by

$$(a \cdot b)(x, y) = \sum_{x \leq_r z \leq_r y} a(x, z) \cdot b(z, y) \tag{3}$$

This is the setting for the theory of Möbius inversion presented in [R1, R2, S] where in fact not only finite but also locally finite posets are considered, for which (3) is still well defined. The incidence algebra is thus a (generally non commutative) ring with the Kronecker delta $d : S^2 \to \mathbf{F}$ as multiplicative identity, and the Möbius function m of the poset is the multiplicative inverse of the zeta function $z : S^2 \to F$ given by

$$z(x, y) = 1 \text{ if } x \leq_r y$$

$$z(x, y) = 0 \text{ otherwise}$$

There is no loss of generality in assuming that $S = \{1, ..., n\}$ and the members (1) of the incidence algebra are then in fact nxn matrices with entries in **F,** (2) is obviously matrix addition, and among members of an incidence algebra the multiplication (3) is easily seen to coincide with the usual matrix product. The incidence algebra contains all matrices of the form

$$\begin{pmatrix} b & & \\ & \ddots & \\ & & b \end{pmatrix}$$

where $b \in$ **F**, which constitute a commutative subring isomorphic to **F**, and then the incidence algebra is an algebra over **F** in the sense of Lang [L]. In fact the incidence algebra contains all diagonal matrices, which constitute a larger but still commutative subring, and thus the incidence algebra is a matrix algebra over the algebra of nxn diagonal matrices with entries in **F** that we call the *diagonal matrix algebra* over **F**. While the diagonal matrix algebra is obviously the smallest poset incidence algebra over **F,** note that if $n > 1$ then there is no largest poset incidence algebra over **F**.

2 Incidence algebras of pre-orders

A *pre-order* r is a reflexive and transitive binary relation on a set S. If S is finite then without loss of generality we may assume that $S = \{1,..., n\}$. Given a field **F** the *incidence algebra* $A(r)$ of a pre-order set (S, r) over F is the set of maps $a : S^2 \to \mathbf{F}$ such that $a(x, y) = 0$ unless xry. These maps are in fact matrices and addition and multiplication in $A(r)$ are defined as matrix sum and product. The closure of $A(r)$ under both sum and product is easy to verify and as for poset incidence algebras, the product of a and b is also given by

$$(a, b)(i, j) = \sum_{irkrj} a(i,k)b(k, j)$$

Observe that the set U (respectively L) of all nxn upper (respectively lower) triangular matrices is just the incidence algebra of the linear order $1 \le ... \le n$ (respectively of its dual $n \le^* ... \le^* 1$).

Recall that for any nxn matrix M and non-singular nxn matrix P, the incidence matrix $M' = PMP^{-1}$ is called the *conjugate of* M *by* P. We are interested in the case where P is a permutation matrix. Since permutation matrices form a multiplicative group, the relation *" M_1 is a conjugate of M_2 by some permutation matrix"* is an equivalence relation. Observe also that every lower triangular matrix is the conjugate of some upper triangular matrix by the permutation matrix

$$\begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix} \qquad \textbf{(4)}$$

Recall finally that each pre-order on S being a subset of S^2, the set of pre-orders on S is naturally ordered by inclusion and constitutes a lattice (in fact this is a closure system, as pointed out in [F]).

THEOREM *The map associating to each pre-order* r *on* $\{1,2,...,n\}$ *its incidence algebra* $A(r)$ *over a field* ***F*** *is an order isomorphism from the lattice of pre-orders on* $\{1,2,...,n\}$ *to the lattice of* nxn *matrix algebras over the diagonal matrix algebra over* ***F***. *For any such incidence algebra* $A(r)$ *the following conditions are equivalent:*

i) r *is a linear order,*

ii) $A(r)$ *is the conjugate, by some permutation matrix, of the algebra U of upper triangular matrices,*

iii) $A(r)$ *is the conjugate, by some permutation matrix, of the algebra L of lower triangular matrices.*

Proof. As in the preliminary discussion, denote $\{1,2,...,n\}$ by S.

Obviously if $r_1 \subseteq r_2$ then $A(r_1) \subseteq A(r_2)$. Conversely if $r_1 \not\subset r_2$, then let $i r_1 j$ without $i r_2 j$. Let M_{ij} be the matrix whose only non-zero entry, equal to 1, is in position i, j. Then

$$M_{ij} \in A(r_1) - A(r_2)$$

and therefore $A(r_1) \not\subset A(r_2)$. This shows that $r \to A(r)$ establishes an order isomorphism between the lattice of pre-orders and some subset of the lattice of all nxn matrix algebras. It remains to show that the range of this embedding includes all matrix algebras A containing the diagonal algebra. Let indeed A be any such matrix algebra. Let

$$r = \{(i, j) \in S^2 : \exists a \in A \text{ such that } a(i, j) \neq 0\}.$$

We claim that r is a pre-order and $A(r) = A$.

Claim1. The relation r *is a preorder*. The relation r is reflexive because the identity matrix belongs to A and therefore $(i, j) \in r$ for all $i \in S$. To prove transitivity, for each $i \in S$ consider the diagonal matrix

$$D_i = \begin{pmatrix} d_{i1} & & & & \\ & \cdot & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & d_{in} \end{pmatrix} \qquad \textbf{(5)}$$

where the deltas are Kronecker's deltas. The matrices $D_1, D_2, ..., D_n$ belong to A. Suppose now that irk and krj. Let $a_{ik}, a_{kj} \in A$ with $a_{ik}(i,k) \neq 0$, $a_{kj}(k,j) \neq 0$. Then $a = D_a a_{ik} D_k a_{kj} D_j$ belongs to A and $a(i,j) \neq 0$, and thus irj, proving the transitivity of r.

Claim 2. The incidence algebra $A(r)$ *coincides with* A. The inclusion $A \subseteq A(r)$ is obvious. For the converse inclusion it is enough to show that for every $(i,j) \in r$, the algebra A contains the matrix M_{ij} whose only non-zero entry, equal to 1 is entry (i,j). But we know that A contains some matrix M whose entry (i,j) is not zero, say it is c. The matrix M may have other non-zero entries. Consider the diagonal matrices D_i and D_j defined as in (5).We have $M_{ij} = c.D_i.M.D_j$ and therefore M_{ij} belongs to $A(r)$ as claimed, concluding the proof that $A(r) = A$.

The two claims above being proved, the isomorphism stated in the theorem is now established.

As for the second statement of the theorem, the equivalence of (ii) and (iii) follows from the preliminary discussion about conjugation by the permutation matrix (4). To show that (i) implies (ii) , let r be a linear order on S, and let s be unique order isomorphism $(S, \leq) \to (S, r)$, where $\leq$ is the natural order on $S = \{1,2,...,n\}$. Let P be the permutation matrix whose only non-zero entries, equal to 1, are the entries in positions $(i, s(i))$, $i = 1,2,...,n$. It is not difficult to see that

$$A(r) = P.U.P^{-1} \qquad \textbf{(6)}$$

That proves (ii). In the other direction assume that we have (ii) for a preorder r, i.e. thus (6) holds for some permutation matrix P. Let $s : S \to S$ be the permutation defined by P, i.e. $s(i) = i$ if entry i, j is 1 in P.Then the incidence algebra of the linear order $\leq^s$ in which

$$s(1) \leq^s \ldots \leq^s s(n)$$

is $P.U.P^{-1}$, and therefore r must coincide with $\leq^s$, and thus it is a linear order. •

The following is then a straightforward consequence of the well-known fact that every partial order is the intersection of its linear extensions. (See e.g. Davey and Priestley [DP], or [F])

COROLLARY. *For any incidence algebra* $A(r)$ *over a field* $\boldsymbol{F}$ *of a pre-order* r *on the finite set* $\{1,2,...,n\}$ *the following conditions are equivalent :*

i) r *is a partial order on* $\{1,2,...,n\}$.

ii) $A(r)=\bigcap_{P\in\Sigma} P.U.P^{-1}$

where U is the algebra of nxn *upper triangular matrices over F and S is some non-empty set of nxn permutation matrices over* $\boldsymbol{F}$. •

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