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DUAL METHODS FOR THE NUMERICAL SOLUTION OF THE UNIVARIATE POWER MOMENT PROBLEM

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DUAL METHODS FOR THE NUMERICAL SOLUTION OF THE UNIVARIATE POWER MOMENT PROBLEM

Andràs Prékopa Gabriela Alexe

Abstract. The purpose of this paper is twofold. First to present a brief survey of some of the basic results related to the univariate moment problem, including Prékopa's dual approach for solving the discrete moment problem. Second we propose a new method for solving the continuous power moment problem when some higher order divided differences of the objective function are nonnegative. The proposed method combines Prékopa's dual approach for solving the discrete moment problem with a cutting-plane type procedure for solving linear semi-infinite programming problems.

1. Introduction

The classical moment problem received a considerable attention since it was formulated in the mid 1800's, and has important applications. This paper has two objectives: first, to present a survey of the basic results related to the univariate moment problem, with a focus on the linear programming approach proposed by Prékopa [Pre88], for solving the problem in discrete case, and second, to propose a new method for the solution of the continuous univariate moment problem when some higher order divided differences of the objective function are nonnegative. The proposed method combines the cutting-plane discretization method for solving linear semi-infinite programming problems [GL98] with Prékopa's dual method for the discrete case, and its efficiency is analyzed on several numerical examples.

This paper is organized as follows: In section 2 we present a brief history of the classical moment problem. In section 3 we present the general formulation of the univariate power moment problem and some of the fundamental results related to the feasibility and the bounding problem. In section 4 we present some basic results in the duality theory of the moment problem. Section 5 is dedicated to Prékopa's dual algorithm for the discrete moment problem. In section 6 we describe a cutting-plane algorithm for solving the continuous univariate moment problem, and focus on a specific variant of it that uses the dual algorithm of Prékopa as a subroutine. In section 7 we present various computational results which indicate the efficiency of the cutting-plane method.

2. A brief history of the moment problem

The concept of "moment problem" was introduced as a feasibility problem on the positive real axis in 1894-1895 by Stieltjes The concept of "moment problem" was introduced as a feasibility problem on the positive real axis in 1894-1895 by Stieltjes [Sti86], [Sti94], [Sti14], [Kje93] in connection with the analytic behavior of continued fractions; Stieltjes adopted the terminology "moment" from mechanics. However, bounding problems related to moments had already been considered by Bienaymé and Chebyshev [Che74], [Che87] since 1853-1854 [Kje93]. The bounding moment problem frequently appears in the literature as "Chebyshev type inequalities".

In early 1900's, Markov [Mar84] obtained significant results for both the feasibility and the bounding moment problems. Later, in 1920, Hamburger [Ham19,20] extended the Stieltjes moment problem to the real axis, and established the moment problem as a theory of its own. In the same time, Hausdorff [Hau21], [Hau23] defined the *Hausdorff moment problem* on a finite interval in connection with convergence-preserving matrices; this new approach for the moment problem was the first one not related to continued fractions. Two years later, in 1922, Nevanlinna [Nev22] extended the Hamburger moment problem to the complex functions. Riesz in 1923 [Rie23], was the first who extended the moment problem in functional analysis, by observing the connection between the moment problem and the space of bounded linear functionals on $C([a,b])$.

Starting with the mid 1900's, the duality theory for the moment problem was developed independently by Isii [Isi63], [Isi65], and Karlin [KS66], in connection with the linear semi-infinite programming; however, the use of the duality theory for solving the bounding moment problem was proposed earlier by Markov (1884 [Mar84]) and Riesz (1911 [Rie09, 23]). Fundamental results in the duality theory for the moment problem were obtained by Haar

[Haa26], Charnes, Cooper and Kortanek [CCK62], [CCK65] etc. In the mid 1900's, one of the most comprehensive studies dedicated to the use of the duality theory for solving the moment problem was written by Kemperman in 1968 [Kem68].

At the end of 1980's, Prékopa ([Pre88], [Pre90_1], [Pre90_2]) and Samuels and Studden ([SS89]) independently introduced and studied the *univariate discrete moment problem*, motivated by the fact that the sharp Bonferroni bounds, as well as other probability bounds, can be obtained as optimum values of discrete moment problems. Closed form formulas based on these results have been obtained by Boros and Prékopa in 1989 [BP89]. Few years later, Prékopa ([Pre92], [Pre95], [BP89]) introduced and studied the *multivariate discrete moment problem*. Although they address the same problem, the methodologies for solving the discrete moment problem used by Samuels and Studden, and Prékopa are completely different. Samuels and Studden use the classical approach for the general moment problem, and determine the solutions in closed form whenever possible; their method is applicable only to small size problems. Prékopa is the first who uses the linear programming methodology in moment theory, and it turns out that in the special case of the discrete moment problem, linear programming techniques provide us with more general and simpler algorithmic solutions than the classical ones. Moreover, the linear programming approach for the discrete moment problem allows for the efficient solution of solving efficiently large size moment problems, for which the classical methodology cannot give solutions, due to numerical difficulties.

Among the most important monographs dedicated to the moment problem are those of Krein and Nudelman [KN77], Karlin and Studden [KS66], and Prékopa [Pre95].

3. The univariate power moment problem: feasibility and boundedness

In this section we briefly present the definition of the general moment problem, and some classical results for the case of the univariate power moment problem. For a comprehensive presentation of the moment problem, see [KS66], [KN77], and [Pre95].

Let Ω be a Borel space, $\{u_k\}: u_0(z),\ldots,u_m(z), m \geq 1$, be a sequence of measurable and integrable functions on Ω, and let $\{m_k\}: m_0 = 1, m_1,\ldots,m_m$ be a (finite) sequence of real numbers. In a general formulation, the *moment problem* consists of the following two main problems:

(1) *The Feasibility Problem*. Find necessary and sufficient conditions that $\{m_k\}$ is a *generalized moment sequence* with respect to $u_0(z),\ldots,u_m(z)$, i.e. there exists a probability measure P on Ω, such that $\int_\Omega u_k(z)dP = m_k, k = 0,\ldots,m$. If such a measure P exists, then any random variable X on Ω having the probability distribution P is called a *feasible representation* of the moment sequence $\{m_k\}$.

(2) *The Bounding Problem*. Let f be a real function defined on Ω. Find the *best* lower and upper bounds for $E[f] = \int_\Omega f(z)dP$, where P is a feasible representation of $\{m_k\}$.

Formally, the bounding problem consists of solving the following programs:

$$\inf(\sup) E[f(X)] = \int_\Omega f(Z)dP$$
$$\textit{subject to}$$
$$\int_\Omega u_k(Z)dP = m_k, k = 0,\ldots,m.$$

We shall denote by $(P_{\inf})$ and $(P_{\sup})$ the problems of finding the infimum and supremum bounds in (1), respectively.

The general moment problem is called *determinate* if $\{m_k\}$ has a unique feasible representation *P*, and *indeterminate* otherwise.

The general moment problem can be restricted to finite probability measures, due to the following result of Richter, Tchakaloff, and Rogosinski:

Theorem 1 ([Ric57], [Tch57], [Rog62]). *If the general moment problem is feasible, i.e. if there exists a probability measure P such that* $\int_{\Omega} z^k dP = m_k, k = 0,...,m$, *then there exists a finite probability measure P' on* Ω *having at most m+1 points, such that* $\int_{\Omega} z^k dP' = m_k, k = 0,...,m$.

In most practical applications Ω is a subset of $R^n, n \geq 1$. If Ω is an interval (finite, semi-infinite or infinite) of R^n, then the moment problem is called *discrete* ([KS66], [Pre88], [SS89]).

If Ω is a compact real interval, the system of functions $u_k(z), k = 0,...,m$, is called a *Chebyshev system* of order *m* if for every elements $z_0 < ... < z_m$ in Ω, the determinant $\begin{vmatrix} u_0(z_0) & ... & u_0(z_m) \\ ... & & \\ u_m(z_0) & ... & u_m(z_m) \end{vmatrix}$ is nonnegative. A Chebysev system is called *positive* if the determinants occurring in its definition are positive, for any choice of $z_0,...,z_m$. Among the major references for the Chebyshev systems theory in general context we mention Karlin and Studden [KS66], and Krein and Nudleman [KN77]. Johnson and Taaffe [JT88] present various applications of Chebyshev systems on probability distributions.

In particular, the system $u_k(z) = z^k, k = 0,...,m$, is a *positive Chebyshev system*, and the corresponding moment problem is called the *power moment problem*. We mention that the power moment problem can be defined in general on an arbitrary Borel space.

When $\Omega = \{z_0,...,z_n\}$ is a discrete (finite) set, the moment problem is called the *discrete* (*finite*) *moment problem*. The discrete moment problem is of particular interest, especially from the computational point of view, and was closely studied by Prékopa in [Pre88], [Pre90_1]. Taking into account that the discrete moment problem can be formulated as a linear program

$$\begin{aligned} &\min(\max E[f(X)]) = \sum_{i=0}^{n} f_i x_i \\ &s.t. \sum_{i=0}^{n} z_i^k x_i = m_k, k = 0,...,m \qquad (2) \\ &x_i \geq 0, i = 0,...,n, \end{aligned}$$

where $f_i = f(z_i), i - 0,...,n$, Prékopa [Pre90_2] presented a very efficient and simple dual metod for solving it in the case when *f* satisfies some higher order convexity conditions. A main advantage of Prékopa 's dual method is that it can be applied efficiently for solving large size moment problems.

Closely related to the moment problem is the *binomial moment problem*, introduced and studied by Prékopa in [Pre88], [Pre95]. In the binomial moment problem, where the possible values of the random variable X are $\binom{z_i}{x_i}, i = 0,\dots,n$. The corresponding linear programs are

$$\begin{aligned}
&\min(\max) E[f(X)] = \sum_{i=0}^{n} f_i x_i \\
&s.t. \sum_{i=0}^{n} \binom{z_i}{k} x_i = S_k, k = 0,\dots,m \\
&x_i \ge 0, i = 0,\dots,n.
\end{aligned} \tag{3}$$

The discrete and the binomial problems can be transformed into each other. The transformation is based on the Stirling numbers $s(l,k)$ and $S(l,k)$ of the first and second kind, defined by the equations

$$(Z)_l = \sum_{k=0}^{l} s(l,k) z^k,$$

$$Z^l = \sum_{k=0}^{l} S(l,k)(z)_k,$$

where $(z)_l = z(z-1)\dots(z-l+1)$. More precisely, if the discrete and the binomial moment problems are written in the form

$$\begin{aligned}
&\min(\max) \sum_{i=0}^{n} f_i x_i \\
&s.t. Ax = b \\
&x \ge 0
\end{aligned}$$

where

$$A = \begin{pmatrix} 1 & \dots & 1 & 1 \\ z_1 & \dots & z_{n-1} & z_n \\ \dots & \dots & \dots & \dots \\ z_1^n & \dots & z_{n-1}^n & z_n^n \end{pmatrix}, \ b = \begin{pmatrix} m_0 \\ m_1 \\ \dots \\ m_n \end{pmatrix}, \tag{4}$$

and

$$\begin{aligned}
&\min(\max) \sum_{i=0}^{n} f_i x_i \\
&\textit{subject to } \ A'x = b' \\
&x \ge 0,
\end{aligned} \tag{5}$$

where

$$A' = \begin{pmatrix} 1 & \dots & 1 & 1 \\ \binom{z_1}{1} & \dots & \binom{z_{n-1}}{1} & \binom{z_n}{1} \\ \dots & \dots & \dots & \dots \\ \binom{z_1}{n} & \dots & \binom{z_{n-1}}{n} & \binom{z_n}{n} \end{pmatrix}, \ b' = \begin{pmatrix} S_0 \\ S_1 \\ \dots \\ S_n \end{pmatrix},$$

then

$$A' = T_1 A$$
$$b' = T_1 b$$

$$A = T_2 A'$$
$$b = T_2 b'$$

where

$$T_1 = \begin{pmatrix} s_{00} & & & \\ s_{10} & s_{11} & & \\ \ldots & \ldots & \ldots & \\ s_{m0} & s_{m1} & \ldots & s_{mm} \end{pmatrix}, \quad T_2 = \begin{pmatrix} S_{00} & & & \\ S_{10} & S_{11} & & \\ \ldots & \ldots & \ldots & \\ S_{m0} & S_{m1} & \ldots & S_{mm} \end{pmatrix},$$

and $s_{lk} := \frac{s(l.k)}{l!}, S_{lk} := \frac{S(L,k)}{k!}$.

Various practical applications of the discrete and binomial moment problems are related to finding sharp probability bounds and are presented in [Pre90_1], [Pre92], [Pre95].

In this paper we restrict our study to the univariate power moment problem with a finite number of moments.

3.1. Some classical results for the feasibility problem

In the following first we suppose that Ω is a closed finite interval $[a,b]$. The fundamental feasibility results for the univariate power moment problem are based on the following two theorems:

(i) *The Riesz Representation Theorem*, which states that if $\{m_k\}$ is a moment sequence on $[a,b]$, then the point $(m_0,\ldots,m_m)$ is in the conic hull K of the curve $\{(1,z,\ldots,z^m): z \in [a,b]\}$, and

(ii) *Farkas Theorem*, which states that the dual K^* of K is the set P_+ of polynomials $P(z) = \sum_{k=0}^{m} a_k z^k$ which are nonnegative on Ω.

The representation of the nonnegative polynomials is characterized in the theorem of Markov and Lukács (see, e.g., [KN77], [Pre95]):

Theorem 2 (Markov and Lukács) *Any algebraic polynomial P(t) of degree £ n, nonnegative on* $[a,b]$*, admits the following representation:*

$$P(t) = \left(\sum_{k=0}^{n} x_k t^k\right)^2 + (b-t)(t-a)\left(\sum_{k=0}^{n} y_k t^k\right)^2, \text{ for } n = 2n \text{ ;}$$

$$P(t) = (t-a)\left(\sum_{k=0}^{n} x_k t^k\right)^2 + (b-t)\left(\sum_{k=0}^{n} y_k t^k\right)^2, \text{ for } n = 2n + 1 .$$

A sequence $\{m_k\}: m_0,\ldots,m_m$ is called *positive* (respectively *strictly positive*) on the finite interval $\Omega = [a,b]$ ([KN77], [Pre95]) if $\sum_{k=0}^{m} a_k z^k \geq 0$ for every $a \leq z \leq b$ implies $\sum_{k=0}^{m} a_k m_k \geq 0$ (respectively $\sum_{k=0}^{m} a_k m_k > 0$).

The basic feasibility result for the power moment problem states that

Theorem 3 ([KN77]) $\{m_k\}$ *is a power moment sequence if and only if it is positive*.

Another classical characterization of the feasibility of the continuous moment problem on a finite interval $\Omega = [a,b]$.

Theorem 4 ([KN77]) *A sequence* $\{m_k\}$ *of real numbers is a moment sequence on* $[a,b]$ *if and only if the quadratic forms* $f = \sum_{i,j=0}^{n} m_{i+j} x_i x_j$ *and* $F = \sum_{i,j=0}^{n-1} [(a+b)m_{i+j+1} - ab m_{i+j} - m_{i+j+2}] x_i x_j$ *when* $m = 2n$ *is even (respectively,* $g = \sum_{i,j=0}^{n} (m_{i+j+1} - a m_{i+j}) x_i x_j$ *and* $G = \sum_{i,j=0}^{n-1} [b m_{i+j} - m_{i+j+1}] x_i x_j$ *when* $m = 2n+1$*) are nonnegative. The moment sequence is determined if and only if at least one of the forms f or F (respectively, g or G) is singular.*

Feasibility conditions can also be derived from the classical results regarding the closed form of the optimal solution or the bounding moment problem.

Let $P:\begin{pmatrix} z_0 \ldots z_N \\ p_0 \ldots p_N \end{pmatrix}$ be a finite feasible distribution (or representation) for a moment sequence $\{m_k\}$ defined on a finite interval $\Omega = [a,b]$. The values z_i are called *seeds*, and the probabilities p_i are called *weights*. The seed z_i is called a *boundary seed* if $z_i \in \{a,b\}$. The *index* of $P:\begin{pmatrix} z_i \\ p_i \end{pmatrix}_{i=0,\ldots,N}$ equals the number of nonboundary seeds z_i plus the number of weights. P is called *canonical* if its index is $\le m+2$, and *principal* if its index is $m+1$.

A *lower principal* representation is a principal representation having all seeds in (a,b) if m is even, or having the seed a and all other seeds in (a,b), if m is odd.

An *upper principal* representation is a principal representation having the seeds a,b and all other seeds in (a,b) if m is even, or having the seed b and all other seeds in (a,b), if m is odd.

It is known (see [KN77 or [Pre95] for details), that

If the moment problem on a finite interval Ω *is feasible, then it has a lower and an upper principal representation. Moreover, the seeds of the lower (respectively, upper) representation can be computed as follows:*

The seeds of the lower principal representation are the roots of the polynomials

- $\begin{vmatrix} m_0 & \ldots & m_{N-1} & 1 \\ m_1 & \ldots & m_N & z \\ \ldots & \ldots & \ldots & \ldots \\ m_N & \ldots & m_{2N-1} & z^N \end{vmatrix}$, where $m+1=2N$, if $m+1$ is even
- $(z-a)\begin{vmatrix} m'_0 & \ldots & m_{N-1} & 1 \\ m'_1 & \ldots & m_{N-2} & z \\ \ldots & \ldots & \ldots & \ldots \\ m'_{N-2} & \ldots & m_{2N-5} & z^{N-2} \end{vmatrix}$, where $m'_k = m_{k+1} - a m_k$, and $m+1=2N-1$, if $m+1$ is odd.

Similarly, the seeds of the upper principal representation can be determined as the roots of the polynomials

- $(b-z)(z-a)\begin{vmatrix} m'_0 & \dots & m'_{N-3} & 1 \\ m'_1 & \dots & m'_{N-2} & z \\ \dots & \dots & \dots & \dots \\ m'_{N-2} & \dots & m'_{2N-5} & z^{N-2} \end{vmatrix}$, where $m'_0=1$, $m'_k=(a+b)m_k-abm_{k+1}-m_{k+2}$, and $m+1=2(N-1)$, if $m+1$ is odd
- $(b-z)\begin{vmatrix} m'_0 & \dots & m'_{N-2} & 1 \\ m'_1 & \dots & m'_{N-1} & z \\ \dots & \dots & \dots & \dots \\ m'_{N-1} & \dots & m'_{2N-3} & z^{N-1} \end{vmatrix}$, where $m'_k=bm_k-m_{k+1}$, and $m+1=2N-1$, if $m+1$ is even.

For numerical applications we may always assume that Ω is a finite interval. A recent practical method due to Prékopa and Szedmák for checking if a moment sequence is feasible, is described in [PS02]. For the case when Ω is a semi-infinite or an infinite interval, the feasibility conditions can be expressed in terms of the semi-definite positiveness of certain matrices (see Vandenberghe [VB96], [VB99]). For example, if $\Omega=[0,+\infty)$, the feasibility conditions are equivalent to the fact that the circular matrices

$$R_{2n}=\begin{pmatrix} 1 & m_1 & \dots & m_n \\ m_2 & m_3 & \dots & m_{n+1} \\ \dots & \dots & \dots & \dots \\ m_n & m_{n+1} & \dots & m_{2n} \end{pmatrix}$$

and

$$R_{2n+1}=\begin{pmatrix} m_1 & m_2 & \dots & m_{n+1} \\ m_2 & m_3 & \dots & m_{n+2} \\ \dots & \dots & \dots & \dots \\ m_{n+1} & m_{n+2} & \dots & m_{2n+1} \end{pmatrix}$$

are positive semidefinite, while in case when $\Omega=R$, the feasibility conditions are equivalent to the fact that the circular matrix $R_{2\left[\frac{n}{2}\right]}$ is positive semidefinite (here [x] is the integral part of x).

If $\Omega=\{z_0,\dots z_n\}$ is a discrete real set, then as we show it in section 5, the dual method of Prékopa for the discrete moment bounding problem is also an elegant and efficient error-free method for the discrete moment feasibility problem.

We conclude this part by explicitly presenting the feasibility conditions for the moment problem, when m=1,2,3. These conditions can be derived either by applying the dual method of Prékopa, or by applying the fundamental feasibility theorem for the moment problem, although the first method is more elegant.

Proposition 1. (i) *For m=1,* m_1 *is the expectation of a discrete distribution with spectrum* $z_0<\dots<z_n$ *if and only if* $z_0\le m_1\le z_n$.

(ii) *For m=2, a sequence* $\{m_0,m_1,m_2\}$ *is the power moment sequence of a discrete distribution with spectrum* $z_0<\dots<z_N$ *if and only if*

$$(m_1-z_i)(z_{i+1}-m_1)\le m_2-m_1^2\le(m_1-z_0)(z_N-m_1)$$

where i is such that $z_i \le m_1 < z_{i+1}$.

(*iii*) *For m=3, a sequence* $\{m_0, m_1, m_2, m_3\}$ *is the power moment sequence of a discrete distribution with spectrum* $z_0 < ... < z_N$ *if and only if*

$$\begin{cases} z_0 \le m_1 < z_n \\ m_2 - m_1^2 \ge (m_1 - z_i)(z_{i+1} - m_2) \\ (m_1 - z_0)(m_3 - z_0 m_2) \ge (m_2 - z_0 m_1)^2 \\ z_0\left(m_2 - (z_j + z_{j+1})m_1 + z_j z_{j+1}\right) \le m_3 - (z_j + z_{j+1})m_2 + z_j z_{j+1} m_1 \\ (z_N - m_1)(z_N m_2 - m_3) \ge (-z_N m_1 + m_2)^2 \\ z_N\left(m_2 - (z_k + z_{k+1})m_1 + z_k z_{k+1}\right) \ge m_3 - (z_k + z_{k+1})m_2 + z_k z_{k+1} m_1 \end{cases}$$

where i, j, and k are chosen such that

$$z_i \le m_1 \le z_{i+1}, z_j \le \frac{m_2 - z_0 m_1}{m_1 - z_0} \le z_{j+1}, z_k \le \frac{m_1 z_N - m_2}{z_N - m_1} \le z_{k+1}.$$

Due to the strong relationship between the power and binomial moments, the feasibility conditions for the binomial moment problem can be formulated in a similar way.

Regarding the number of feasible distributions associated with a power moment sequence we mention the following result of Krein and Nudelman:

Theorem 5. ([KN77], [Pre99]) *A moment sequence* $\{m_k\}$ *admits a unique feasible distribution* P_0 *if and only if it is singularly positive. If the sequence* $\{m_k\}$ *is strictly positive, it admits infinitely many feasible distributions. In particular, a power moment sequence admits infinitely many feasible distributions.*

3.2. Some classical results for the bounding problem

First we assume that $\Omega = [a,b]$. Let $\{m_k\}$ be a moment sequence on $[a,b]$ and *f* a real function defined on $[a,b]$. The classical bounding moment problem has the following formulations:

(*i*) Find lower and upper bounds for $\int_a^b f(z)dP$, where *P* is a probability distribution of a feasible representation of (m).

(*ii*) (*Chebyshev-Markov Inequalities*) Find lower and upper bounds for $\int_a^x f(z)dP$, where $x \in [a,b]$, and *P* is the probability distribution of a feasible representation of $\{m_k\}$ containing x in its support.

The next result describes the optimal solution of the bounding problem (*i*) when some conditions are satisfied.

Theorem 6. ([KN77], [Pre95]). *Let* $\{m_k\}$ *be a feasible moment sequence and f a real function defined on* $[a,b]$*, such that the following conditions hold:*

Condition 1. The function f is continuous and the set of functions $\{1, z, ..., z^m, f(z)\}$ *is a positive Chebyshev system on* $[a,b]$.

Condition 2. $\{m_k\}$ *is an interior point of the set*

$M_{m+1} := \left\{(v) : v_k = \int_a^b z^k dP, \text{ and } P \text{ probability distribution on } [a,b]\right\}.$

Then the optimal solutions for the infimum (respectively supremum) bounded moment problems (i) are attained and correspond to a lower (upper) principal representation. Every optimal solution is a principal representation.

Regarding the bounding problem (*ii*), the following classical result is presented in Karlin and Nudelman [KN77] and Prékopa [Pre95].

Theorem 7. *If the sequence* $\{m_k\}$ *is strictly positive, then for any point* x *of* $[a,b]$ *there exists a unique canonical representation* P_x *whose seed is* x. *If* x *is an interior point of* $[a,b]$, *the representation is unique. If* $x = a$ *or* $x = b$, *then* P_x *is principal.*

According to [KN77], the seeds of the canonical representation P_x can be determined as roots of some orthogonal polynomials, as described below. As in [KN77], we consider the following recurrent sequences of polynomials

Case 1. If m+1 is odd (m+1=2n +1),

-

$$D_0 = \frac{1}{\sqrt{m_0}},$$

$$D_k = \frac{1}{\sqrt{\det\|m_{i+j}\|_{i,j=0}^{k}}\sqrt{\det\|m_{i+j}\|_{i,j=0}^{k-1}}}\det\left\|m_j m_{j+1} \ldots m_{j+k-1} t^j\right\|_{j=0}^{k},\ k = 1,\ldots,m,$$

$$E_0 = \frac{1}{\sqrt{m'_0}},$$

$$E_k = \frac{1}{\sqrt{\det\|m'_{i+j}\|_{i,j=0}^{k}}\sqrt{\det\|m'_{i+j}\|_{i,j=0}^{k-1}}}\det\left\|m'_j\ m'_{j+1} \ldots m'_{j+k-1}\ t^j\right\|_{j=0}^{k},\ k = 1,\ldots,m-1,$$

where $m'_k = (a+b)m_{k+1} - abm_k - m_{k+2}$.

Case 2. If m+1 is even (m+1=2n),

$$F_0 = \frac{1}{\sqrt{m'_0}},$$

$$F_k = \frac{1}{\sqrt{\det\|m'_{i+j}\|_{i,j=0}^{k}}\sqrt{\det\|m'_{i+j}\|_{i,j=0}^{k-1}}}\det\left\|m'_j\ m'_{j+1} \ldots m'_{j+k-1}\ t^j\right\|_{j=0}^{k},\ k = 1,\ldots,m-1,$$

where $m'_k = m_{k+1} - am_k$,

$$G_0 = \frac{1}{\sqrt{m'_0}},$$

$$G_k = \frac{1}{\sqrt{\det\left\|m'_{i+j}\right\|_{i,j=0}^{k}}\sqrt{\det\left\|m'_{i+j}\right\|_{i,j=0}^{k-1}}}\det\left\|m'_j\ m'_{j+1}\ldots m'_{j+k-1}\ t^j\right\|_{j=0}^{k},\ k=1,\ldots,m-1,$$

where $m'_k = bm_k - m_{k+1}$.

Let F be the functional defined by $F\left(\sum_{k=0}^{m} a_k z^k\right) = \sum_{k=0}^{m} a_k m_k$, where a_k are D_k, E_k, F_k, G_k, $k \geq 1$, are orthogonal with respect to $dF, (b-t)(a-t)dF, (t-a)dF$, and $(t-b)dF$, respectively.

The canonical representation P_x that has maximal weight $p(x)$ at a prescribed seed $x \in [a,b]$ can be constructed as follows (see [KN77]):

Case 1. m+1 odd (m+1=2n +1)

$$p(x) = \min\left\{\frac{1}{\sum_{k=0}^{n} D_k^2(x)}, \frac{1}{(b-x)(x-a)\sum_{k=0}^{n-1} E_k^2(x)}\right\} = \min\left\{p_1^{2n+1}(x), p_2^{2n+1}(x)\right\}.$$

The seeds of P_x are the roots of the polynomial

$$Q(t) = \begin{cases}(t-x)\sum_{k=0}^{n} D_k(x)D_k(t), & if \quad p(x) = p_1^{2n+1}(x)\\ (t-x)(b-t)(t-a)\sum_{k=0}^{n-1} E_k(x)E_k(t), & if \quad p(x) = p_2^{2n+1}(x).\end{cases}$$

Case 2. m+1 even (m+1=2n)

$$p(x) = \min\left\{\frac{1}{(x-a)\sum_{k=0}^{n-1} F_k^2(x)}, \frac{1}{(b-x)\sum_{k=0}^{n-1} G_k^2(x)}\right\} = \min\left\{p_1^{2n}(x), p_2^{2n}(x)\right\}$$

The seeds of P_x are the roots of the polynomial

$$Q(t) = \begin{cases}(t-x)(t-a)\sum_{k=0}^{n-1} F_k(x)F_k(t), & if \quad p(x) = p_1^{2n}(x)\\ (t-x)(b-t)\sum_{k=0}^{n-1} G_k(x)G_k(t), & if \quad p(x) = p_2^{2n}(x).\end{cases}$$

The weights p_k of P_x, corresponding to the seeds different from x, can be computed as the nonnegative solutions of a Vandermonde system.

Given a function $f : \Omega \to R$, we recall (see [Jor47] and [Ore95]) for a detailed presentation) that the k^{th} order divided difference of f with respect to some base points $z_i < z_{i+1} < \ldots < z_{i+k}$ in Ω is equal to

$$[z_i,\ldots,z_{i+k}; f] := \frac{\begin{vmatrix} 1 & \ldots & 1 \\ z_i & \ldots & z_k \\ \ldots & \ldots & \ldots \\ z_i^{k-1} & \ldots & z_{i+k}^{k-1} \\ f(z_i) & \ldots & f(z_{i+k}) \end{vmatrix}}{\begin{vmatrix} 1 & \ldots & 1 \\ z_i & \ldots & z_{i+k} \\ \ldots & \ldots & \ldots \\ z_i^{k-1} & \ldots & z_{i+k}^{k-1} \\ z_i^k & \ldots & z_{i+k}^k \end{vmatrix}}, \text{ for } k \geq 1.$$

It is well known that a sufficient condition for *f* to have positive (nonnegative) derivatives of order *k* is that *f* has positive (nonnegative) derivatives of order *k* in $\Omega = [a,b]$.

Theorem 8. ([KN77], [Pre95]) *Let* $\{m_k\}$ *be a power moment sequence on* $[a,b]$, *and let* x *be an interior point of* $[a,b]$. *Let* f *be a continuous function on* $[a,b]$, *such that all divided differences up to order n are nonnegative. Then the integral* $\int_a^{x-0} f(t)dP$ *(respectively* $\int_a^{x+0} f(t)dP$ *attains its minimum (maximum) for the canonical representation* X_x *having the probability distribution* P_x, *i.e.,*

$$\int_a^{x-0} f(t)dP_x \le \int_a^{x-0} f(t)dF(t) \le \int_a^{x+0} f(t)dF(t) \le \int_a^{x+0} f(t)dP_x \ .$$

A similar result can be formulated for the case when $\Omega = [0,\infty)$, (see e.g., [Pre95], and also [KS66] for details). In the case when Ω is discrete, and *f* has the divided differences of order *m*+1 nonnegative, then the optimal solutions of (*i*) can be efficiently computed in linear time, by using the dual method of Prékopa ([Pre99], [Pre00]), which will be described in section 5.

4. Duality theory for the univariate power moment problem

Another classical way to solve the bounding moment problem, initiated by Markov (1884 [Mar84]) and Riesz (1911 [Rie09]), is based on *duality theory*. The duality theory approach consists of associating with the moment problem, called " primal", another problem, called "dual", which in general is easier to solve than the primal. The dual problem provides us with bounds for the optimal value of the primal, and in some situations, the dual has the same optimal value as the primal. The duality approach is detailed in many references, including [Kem68], [HK93], [GL98], and [Sha01].

Let consider the bounding power moment problem (1) defined in Section 3 (where $u_k(z) = z^k, k = 0,...,m$). The *dual* of the power moment problem (1) is the problem

$$\begin{aligned} D_{\sup}(D_{\inf}): \sup(\inf)\sum_{k=0}^{m} m_k x_k \\ \textit{subject to} \\ \sum_{k=0}^{m} z^k x_k \le (\ge) f(z), z \in \Omega \end{aligned} \qquad (6)$$

where the unknown variables are $x_k, k = 0,...,m$ (the dual of the **inf** form of the primal is in the **sup** form, and conversely). Whenever necessary we will specify which primal-dual pair we consider; we will refer to the primal problem in either form as (*P*), and to its corresponding dual as (*D*).

Clearly, solving the dual problem (6) is equivalent to finding those *m*-degree polynomials of the random variable *X*, the expectations of which provide us with the best bounds for *f*(*X*) and bound *f* below and above, respectively.

When Ω is finite, (*P*) is a linear programming problem, and (*D*) is its dual, as defined in linear programming theory.

When Ω is infinite, the dual of the power moment problem is a *linear semi-infinite programming* (LSIP) *problem*, i.e., an optimization problem with linear objective and linear constraints, in which the number of constraints is infinite and the number of variables is finite. The moment problem (P) is the *dual* of (D) *in the sense of Haar*. In fact, the moment problem (P) is also a semi-infinite programming problem (not necessarily linear), having an infinite number of unknown variables, and a finite number of constraints.

Let $q(P)$ (respectively $n(P)$) be the optimal value of the **inf** (respectively **sup**) form of the primal problem (1), with the convention that $q(P)=+\infty (n(P)=-\infty)$ if (P) is infeasible; similarly, let $q(D)$ (respectively $n(D)$) be the optimal value of the **sup** (respectively **inf**) dual problem, with the convention that $q(D)=-\infty (n(D)=+\infty)$ if (D) is infeasible. As in the finite case, the weak duality theorem holds.

Theorem 9. (The Weak Duality Theorem, [Pre95], [GL98]) *If (P) is feasible, then (D) is either infeasible, or it is feasible and bounded, and* $q(D)\le q(P)\le n(P)\le n(D)$.

Proof. The inequalities are satisfied if (D) is infeasible. Assuming that (D) is feasible, it is enough to prove that $q(D)\le q(P)$. Let $x_k, k=0,...,m$ be a feasible solution for (D) and let P be a feasible distribution for (P). Then $\sum_{k=0}^{m} m_k x_k = \sum_{k=0}^{m} x_k \int_\Omega z^k dP = \int_\Omega \sum_{k=0}^{m} z^k x_k dP \le \int_\Omega f(z) dP$ and we deduce that (D) is bounded and $q(D)\le q(P)$.

In contrast with the finite case, the strong duality theorem and the complementary slackness property can fail in semi-infinite programming. For example, consider the problem ([GL98])

$$(P) \sup(-z_2)$$
$$\textit{subject to}$$
$$-z_1\begin{pmatrix}-1\\0\end{pmatrix}+z_2\begin{pmatrix}-1\\0\end{pmatrix}+\sum_{r\ge 3}\begin{pmatrix}-1/r\\1\end{pmatrix}=\begin{pmatrix}0\\1\end{pmatrix}$$
$$z_i \ge 0, i\ge 1.$$

Its dual is

$$(D) \inf x_2$$
$$\textit{subject to}$$
$$x_1 \le 0$$
$$-x_2 \le 1$$
$$k^{-1}x_1 - x_2 \le 0, k\ge 3.$$

Both the primal and the dual problems are feasible and have finite optima. However, the optimal values are -1 and 0, respectively, and therefore there exists a duality gap $d(P,D)=0-(-1)=1$

In the following we shall present several known sufficient conditions in which the strong duality theorem holds for the pair ((P),(D)).

Consider the following cones associated with the problem (P):

- $M_{m+1} := \left\{\int_\Omega z^k dP \mid P \text{ runs over the probability distributions on } \Omega\right\}$,
- $M_{m+2} := \left\{\left(\int_\Omega z^k dP, \int_\Omega f(z) dP\right) \mid P \text{ runs over the probability distributions on } \Omega\right\}$,
- $MH_{m+2} := \left\{\left(\int_\Omega z^k dP, \int_\Omega (f(z)+1) dP\right) \mid P \text{ runs over the probability distributions on } \Omega\right\}$

  (see [Pre95], [Kem68] and [HK93]).

We denote the closure of a set X by $\overline{X}$, and its interior by *int*(X).

Theorem 10. ([Isi65], [KS66], [Pre95]). *Assume that* $m=(m_0,...,m_m)\in \operatorname{int}\left(\overline{M_{m+1}}\right)$ *and both the primal problem (P) and its dual (D) are consistent. Under these conditions,*

(*i*) *both the primal and the dual problems are bounded;*

(*ii*) *the optimum value of the dual is attained;*

(*iii*) *the optimum value of the primal is the same as the optimum value of the dual.*

Theorem 11. ([Kem68], [Pre95]) *Assume that* $m=(m_0,...,m_m)\in \operatorname{int}\left(\overline{M_{m+1}}\right)$. *Then the following assertions are equivalent:*

(i) The primal problem (P) has an optimal solution;

(ii) There exists a feasible solution $x=(x_0,...,x_m)$ *of the corresponding dual problem (D), such that* $m=(m_0,...,m_m)$ *is an element of the convex hull of the vectors* $u(z), z\in Z=\left\{z\in\Omega: \sum z^k x_k = f(z)\right\}$.

According to [HK93], condition (*ii*) in the above theorem implies that x is an optimal solution of the dual problem, and that the optimal values of the primal and dual problems are the same. Clearly, (P) is feasible (or *consistent*) if and only if $m=(m_0,...,m_m)\in M_{m+1}$.

According to [HK93], the primal problem (P) is called *superconsistent* if m is an interior point of M_{m+1}. The dual problem (D) is called *superconsistent* if f is a continuous function and the set of feasible solutions of (D) has a non-empty interior. It is easy to see that superconsistency implies consistency for both (P) and (D). Moreover,

Lemma 1. ([CCK62]) *If (D) is superconsistent, then* M_{m+2} *is closed.*

Since M_{m+1} is the projection of M_{m+2} on the first component, it follows that if (D) is superconsistent, then M_{m+1} is closed, too.

Theorem 12. ([Gla79], [HK93])

(i) Suppose that (P) is superconsistent and bounded. Then there is no duality gap, and the optimum of (D) is attained.

(ii) Suppose that (D) is bounded and M_{m+2} *is closed. Then there is no duality gap, and the optimum of (P) is attained.*

Theorem 13. ([HK93]) *Assume that f is continuous. If any of the problems (P), (D) is bounded and superconsistent, then the other one attains the optimum, and there is no duality gap.*

The primal-dual pair ((P),(D)) is said to be in *perfect duality* if either there is no duality gap, or both problems are unbounded.

Theorem 14. ([HK93]) *If the dual problem is consistent, then the following conditions are equivalent:*

(i) For every moment sequence $\{m_k\}$, *the problems (P) and (D) are in perfect duality;*

(ii) The cone MH_{m+2} *is closed.*

If the function f is continuous, the superconsistency property of (P) or (D) implies that whenever the strong duality theorem holds, it remains stable under small perturbations of the moment sequence m and the function f.

5. Prékopa's dual algorithm for the discrete power moment problem

The *discrete bounding* moment problem consists of finding sharp lower and upper bounds for $E\left[f(X)\right]$, where X is a feasible representation for a given moment sequence $\{m_k\}$ and a given real function f on the discrete space $\Omega=\{z_0,...,z_n\}$. Prékopa [Pre88], [Pre90_1] is the first who proposes the use of the linear programming techniques for the solution of the discrete moment problem. Clearly, the discrete bounding moment problem can be presented as the following program:

$$\begin{aligned} &P_{\min}(P_{\max}):\min(\max)\sum_{i=0}^{n} f_i x_i \\ &\textit{subject to } \ Ax=b \\ &x\geq 0, \end{aligned} \tag{7}$$

where

$$A=\begin{pmatrix} 1 & ... & 1 & 1 \\ z_0 & ... & z_{n-1} & z_n \\ ... & ... & ... & ... \\ z_0^n & ... & z_{n-1}^n & z_n^n \end{pmatrix},\quad b=\begin{pmatrix} m_0 \\ m_1 \\ ... \\ m_n \end{pmatrix},$$

and $f_i=f(z_i), i=0,...,n$.

Since the Vandermonde systems are in ill-conditioned ([BBM01], [BP70], Mor01]), solving the discrete moment problem (7) using a primal approach is computationally difficult. Prékopa's idea ([Pre88]) was to use the dual algorithm of Lemke, a major advantage of it being that when the objective function has nonnegative divided differences of order m+1, the dual feasible bases could be determined with no computational effort.

In the following we present Prékopa's dual approach for the solution of problem (7).

Let B be a basis of A, and $I=\{i_0,...,i_m\}$ the corresponding set of basic indices. B is called *dual feasible* if

$$f_B^t B^{-1} a_j \leq f_j, \text{ for every } j\in\{0,...,n\}-I,$$

in case of the minimization problem, and

$$f_B^t B^{-1} a_j \geq f_j, \text{ for every } j\in\{0,...,n\}-I,$$

in case of the maximization problem. B is called *dual non-degenerate* if $f_B^t B^{-1} a_j \neq f_j$, for every $j\in\{0,...,n\}-I$. B is called *primal feasible* if $B^{-1}b\geq 0$.

In [Pre88], [Pre95] it is shown that $\begin{pmatrix} 1 & f_B^t \\ 0 & B\end{pmatrix}^{-1}=\begin{pmatrix} 1 & -f_B^t B^{-1} \\ 0 & B^{-1}\end{pmatrix}$, and consequently, $\begin{pmatrix} 1 & f_B^t \\ 0 & B\end{pmatrix}\begin{pmatrix} f_i - f_B^t B^{-1} a_j \\ B^{-1} a_j\end{pmatrix}=\begin{pmatrix} f_j \\ a_j\end{pmatrix}$.

Applying Cramer's rule, we derive $f_i - f_B^t B^{-1} a_j = \dfrac{\begin{vmatrix} f_j & f_B^t \\ a_j & B \end{vmatrix}}{|B|}$.

On the other hand, by developing the determinant $\begin{vmatrix} f_j & f_B^t \\ a_j & B \end{vmatrix}$ according to the first row, we see that

$$f_i - f_B^t B^{-1} a_j = f_j - L_I(z_j),\ j \in \{0,...,n\} - I\ ,$$

where

$$L_I(z) = \sum_{i=0}^{m} f_i \frac{\prod_{k\in I-i}(z - z_k)}{\prod_{k\in I-i}(z_i - z_k)}$$

is the *Lagrange polynomial* of degree *m* of *f*, corresponding to the points $z_i, i \in I$.

Moreover,

$$f(z) - L_I(z) = \prod_{i\in I}(z - z_i)[z, z_i, i \in I; f].$$

Combining the above formulas, it follows that

$$f_j - f_B^t B^{-1} a_j = \prod_{i\in I}(z - z_i)[z, z_i, i \in I; f],\ j \in \{0,...,n\} - I. \tag{8}$$

Corresponding to problems (5), it is proved that *B* is a primal (dual) feasible basis for problem (7) if and only if T_1B is a primal (dual) feasible basis for problem (5), and we have that

$$f_j - f_B^t (T_1 B)^{-1} a_j (T_1^{-1} a_j) = f_j - f_B^t B^{-1} a_j\ .$$

A direct consequence of (8) is the next important result (theorem 3.1 in [Pre90_1]):

Theorem 15. *Suppose that all (m+1)st divided differences of the function f(z),* $z \in \{z_0,...,z_n\}$ *are positive. Then, in (3) all bases are dual nondegenerate and the dual feasible bases have the following structure:*

	m+1 even	*m*+1 odd
min problem	{*j*, *j*+1, …, *k*, *k*+1}	{0, *j*, *j*+1, …, *k*, *k*+1}
max problem	{0, *j*, *j*+1, …, *k*, *k*+1, *n*}	{*j*, *j*+1, …, *k*, *k*+1, *n*}

where in parentheses the numbers are arranged in increasing order.

Thus, in case when the (*m*+1)st divided differences of *f* are positive, the dual feasible bases can be determined in closed form, with no computational effort. For this special case, Prékopa's dual algorithm for solving (3) can be applied as follows (see [Pre00]) for a detailed presentation):

Prékopa's dual algorithm for the discrete moment problem

Step 1. Pick any dual feasible basis in agreement with the above result; let $I = \{i_0,...,i_m\}$ be the set of basic indices.

Step 2. Determine the corresponding primal feasible solution $x_i = (B^{-1}b)_i$, for $i \in I$, and $x_i = 0$ for $i \in \{0,...,n\} - I$. Taking account of the formula:

$$\begin{vmatrix} a_0 & 1 & \ldots & 1 \\ a_1 & x_1 & \ldots & x_m \\ \ldots & \ldots & \ldots & \ldots \\ a_m & x_1^m & \ldots & x_m^m \end{vmatrix} = \begin{vmatrix} 1 & \ldots & 1 \\ x_1 & \ldots & x_m \\ \ldots & \ldots & \ldots \\ x_1^{m-1} & \ldots & x_m^{m-1} \end{vmatrix} \sum_{j=0}^{m} (-1)^j a_j S_{m-j} ,$$

where $S_j := \sum_{1 \le i_1 \le \ldots \le i_j \le m} z_{i_1} \ldots z_{i_j}$, $j = 0,\ldots,m$, we obtain that

$$x_{i_k} = \frac{\begin{vmatrix} m_0 & 1 & 1 & 1 \\ m_1 & z_{i_0} & \ldots & z_{i_m} \\ \ldots & \ldots & \ldots & \ldots \\ m_m & z_{i_0}^m & \ldots & z_{i_m}^m \end{vmatrix}_{ik}}{\begin{vmatrix} 1 & 1 & 1 & 1 \\ z_{i_k} & z_{i_0} & \ldots & z_{i_m} \\ \ldots & \ldots & \ldots & \ldots \\ z_{i_k} & z_{i_0}^m & \ldots & z_{i_m}^m \end{vmatrix}} = \frac{(-1)^{m-k} \sum_{j=0}^{m} (-1)^j m_j S_{m-j}}{\prod_{j=0}^{i_k-1} \left(z_{i_k} - z_j\right) \prod_{j=i_{k+1}}^{i_m} \left(z_j - z_{i_k}\right)} ,$$

for every $i_k \in I$.

If $x_{i_k} \ge 0$, for every $i_k \in I$, then B is a primal-dual feasible basis, and therefore the current solution is optimal. Go to **Step 4.**

If $x_{i_k} < 0$, for some i_k then the i_k th vector of B is candidate for outgoing. Go to **Step 3**.

Step 3. Include that vector in the basis that restores the dual feasible basis structure and go to **Step 2**.

Step 4. Stop. The optimal value $f_B^t B^{-1} b$ is a lower (upper) bound for $E\left[f\left(X\right)\right]$, depending on the type of the optimization problem (min or max, respectively).

The dual algorithm can be also applied in the more general case when the objective function *f* has nonnegative divided differences of order *m*+1, if some anti-cycling rule (e.g., lexicographic) is applied whenever dual degeneracy occurs.

An important property of the discrete power moment problems (1) is that the optimal basis does not depend on the coefficients of the objective function. An immediate consequence of this fact is that the dual algorithm can be applied for the feasibility of the discrete power moment problem by taking an arbitrary objective function with positive divided differences of order *m*+1 (e.g., *f*(*z*) = exp (*z*)).

Due to the strong relationship between binomial and power moment problems, similar results can be obtained for the case of the binomial moment problem, too (see [Pre00]).

Another approach for problem (3) (see [Pre00]), especially useful in the case when we want to evaluate $P\left(A_1 \cup \ldots \cup A_n\right) = E\left[f\left(X\right)\right]$, where $f\left(z_0\right) = 0$ and $f\left(z_i\right) = 1, i = 1,\ldots,n$ (note that in this case *f* does not have the property that all *m*+1th divided differences are positive) is the following:

remove the variable x_0 *and the equation* $\sum_{i=0}^{n} x_i = 1$ *from the constraints set in (4).*

This way we end up with the linear program

$$\min(\max)\sum_{i=1}^{n} x_i$$
$$\textit{subject to } \sum_{i=1}^{n}\binom{i}{k}x_i = S_k, k = 1,\ldots,m \qquad (9)$$
$$x_i \geq 0, i = 1,\ldots, n.$$

It is easy to see that the optimal values of the minimization problems (9) and (7) are equal, and that the optimal value of the maximization problem (7) equals $\min(1, n_{\max})$, where $n_{\max}$ is the optimal value of the maximization problem (9). This last approach for solving binomial moment problems was applied in [BP89] in particular to obtain closed forms for the bounds of problems (7) with $f(z_i) = 1, i = 1,\ldots,n$, and for $m \in \{2,3,4\}$].

Moreover, in [Pre88], [Pre95], , beside the case when all divided differences of order m+1 of f are positive, there were considered three other special cases for the objective function f, namely:

- $f_r = 1, f_i = 0, i \neq r$, for some $0 \leq r \leq n$, which provides sharp bounds for $P(X = z_r)$
- $f_0 = \ldots = f_{r-1} = 0, f_r = \ldots = f_n = 1$, for some $0 \leq r \leq n$, which provides sharp bounds for $P(X \geq z_r)$.
- $f_i = \binom{i}{t}, i = 0,\ldots, n$, where t is an integer such that $m+1 \leq t \leq n$. If $z_i = i, i = 0,\ldots, n$ then (2) provides us with sharp bounds for S_t, based on the knowledge of S_1,...., S_m.

A generalization of the power and binomial discrete moment problems can be defined (see [Pre00]) with respect to Chebyshev systems of functions. In the case when the function $(-1)^{m+1} f$ is strictly convex of order m+1 with respect to the Chebysev system, then the dual algorithm can be applied in a similar way as above.

6. A cutting-plane algorithm for the continuous power moment problem

In this section we present a method for solving the dual of the continuous bounding power moment problem in the case when the objective function has positive divided differences of order m+1. The method provides us with a lower (respectively, an upper) bound for the infimum (respectively, supremum) of the bounding power moment problem, by combining a cutting plane discretization-type method for the dual of the bounding moment problem with Prékopa's dual method for the discrete moment problem.

In what follows we restrict ourselves to the bounding power moment problem (1) defined on a finite interval $\Omega = [a,b]$ (where $u_k(z) = z^k$), and assume that the objective function f is continuous and has positive divided differences of order m+1.

As presented in section 3.2, if f has positive divided differences of order m+1 and m is an interior point of the cone M_{m+1}, then the optimal solution of problem (1) can be presented in closed form, as the lower (respectively, upper) principal representation for the moment sequence $(m_0,\ldots,m_m)$. . However, to find the closed form the lower (upper) principal representation is known to be computationally expensive, while the dual problem can be solved more efficiently.

As remarked in the previous section, the dual (6) of the bounding power moment problem (1) is a linear semi-infinite problem (LSIP). Among several known numerical methods for solving LSIP (e.g. discretization, local reductions, exchange, simplex-like, and descent methods), the discretization methods by grid and by cutting planes are considered the most efficient (see [GL98], [HK93] and [Pre95]).

Formally, a discretization-type method for solving (2) consists of constructing a sequence of finite grids $(T_r)_{r\geq 0}$ of the interval $[a,b]$ such that $T_r \subseteq T_{r+1}$, solving the corresponding finite subproblems $(D_{\sup})_r$, and expecting that for the optimal solution we have $q(D_{\sup}) = \lim_r q(D_{\sup})_r$. In the discretization by grid methods the sequence $(T_r)_r$ is predetermined (e.g., $T_r = \{a + k(b-a)/2^r : r = 0,...,2^n\}$ while in the cutting plane methods $(T_r)_r$ is inductively constructed, such that T_{r+1} is obtained by adding to T_r a set of feasible cuts. A major drawback of the grid discretization methods is that they imply the solution of large scale linear programming problems, which are not needed in the cutting-plane methods. On the other hand, the cutting plane methods have a slow (linear) convergence rate, if some regularity assumptions are not satisfied. Moreover, the selection of the feasible cuts involve the solution of a nonlinear program.

In what follows we assume that the objective function f is continuous and has positive divided differences of order m+1, and $m = \mathrm{int}(M_{m+1})$ (*).

For simplicity, we restrict ourselves to consider only the primal-dual pair $(P_{\inf}, D_{\sup})$.

Let $e > 0$ be given. The method we propose for the solution of the bounding moment problem consists of the following steps:

Step 1. Initiate k:=0. Choose a finite grid $T^k : a = z_0 < ... < z_i < ... < z_{n_k} = b, i = 0,...,n_k$ of the interval $[a,b]$. Define the restricted problems $(P^k_{\inf})$ and $(D^k_{\sup})$ to the grid T^k.

Step 2. Optimize $(P^k_{\inf})$ using Prékopa's dual method for the discrete moment problem. If $(D^k_{\sup})$ is infeasible, then output "$(D^k_{\sup})$ is inconsistent" and stop. Otherwise, let x_k be the optimal solution of $(P^k_{\inf})$, and B_k be the corresponding primal-dual optimal basis; go to *Step* 3.

Step 3. Construct the Lagrange polynomial $L_I(z) = f^t B_k^{-1} \begin{pmatrix} 1 \\ z \\ ... \\ z^m \end{pmatrix}$ and compute $s := \inf_{z\in[a,b]}(f(z) - L_I(z))$.

Step 4. Add S_{new} to the grid division, update k:=k+1, and go to **Step 3**.

Compared to the general cutting plane algorithm, this specialized method for objective functions having positive divided differences of order m+1 offers two main advantages:

First, by applying Prékopa's dual method in **Step 3**, the complexity of finding the optimal solution x^k is reduced, since the selection of the dual feasible basis requires practically no computation.

Second, let us remark that in **Step 3**, since B_k is the optimal dual feasible basis for $(P^k_{\inf})$, it follows that $f(z) - L_I(z) \geq 0$ for all $z \in [a,b] - \bigcup_{i_1} [z_{i_1}, z_{i_1+1}]$, and consequently,

$$s = \min_{i_1} \inf_{z\in[z_{i_1}, z_{i_1+1}]} (f(z) - L_I(z)),$$

where $\{i_1, i_1+1\}$ are consecutive pairs in the optimal basis B_k. In this way, the initial nonlinear program in **Step 3** is split into $[(m+1)/2]$ subproblems on smaller subdomains $[z_{i_1}, z_{i_1+1}]$. If the size of these subdomains is small enough, then we expect that the computational difficulty of solving the nonlinear programs to be substantially reduced.

The correctness of the algorithm presented above follows from the general result that if $e > 0$, then the cutting plane algorithm for a consistent and *continuous* *LSIP* terminates in a finite number of steps, and at termination it produces an "e - optimal solution" for the *LSIP* (see e.g., [GL98]).

Let $q_e(D)$ (respectively $n_e(D)$) be the optimal solutions provided by the cutting plane method described above for solving $(P_{\inf})$ (respectively $(P_{\sup})$). The assumptions (*) imply that there is no duality gap between (*P*) and (*D*). Moreover, the optimal solution corresponding to $q_e(D)$ (respectively $n_e(D)$) is a feasible solution for the problem $(D_{e,\sup})$ (respectively $(D_{e,\inf})$), obtained from $(D_{\sup})$ (respectively $(D_{\inf})$), by replacing *f* with $f+e$ (respectively, with $f-e$). Finally, the optimal value of $(D_{e,\sup})$ equals $q(P)+e$, and the optimal value of $(D_{e,\inf})$ equals $n(P)-e$.

Therefore, the following relations hold:

$$q_e(D)-e \le q(D)=q(P)\le n(P)=n(D)\le n_e(D)+e .$$

Next we illustrate the cutting plane algorithm on a very simple example.

Example. Consider the bounding moment problem $(P_{\inf})$ corresponding to *m*=3, $f(z)=\exp(z), z\in \Omega=[-3.3]$, and $m_0=1, m_1=0, m_2=4, m_3=0$ (In this example, the moments are generated by a uniform discrete distribution). Let $e=0.15$.

Step 1. Initialize *k*=0 and consider the initial grid T^0:{-3,-2,-1,0,1,2,3} and the corresponding restricted problem $(P^0_{\inf})$.

Step 2. Apply Prékopa's dual method for solving $(P^0_{\inf})$:

- Start with the dual feasible basis *B* corresponding to the consecutive pairs {-2,-1} and {0,1}. The current solution

$$x=B^{-1}b=\begin{pmatrix}1&1&1&1\\-2&-1&0&1\\4&1&0&1\\-8&-1&0&1\end{pmatrix}^{-1}\begin{pmatrix}1\\0\\4\\0\end{pmatrix}=\begin{pmatrix}0\\2\\-3\\2\end{pmatrix}$$

  has one negative entry, correponding to the index 3. Declare the 3rd index outgoing, and restore *B* as corresponding to the pairs {-2,-1} and {1,2}.

- Compute the current solution

$$x=B^{-1}b=\begin{pmatrix}1&1&1&1\\-2&-1&1&2\\4&1&1&4\\-8&-1&1&8\end{pmatrix}^{-1}\begin{pmatrix}1\\0\\4\\0\end{pmatrix}=\begin{pmatrix}1/2\\0\\0\\1/2\end{pmatrix},$$

  which is primal-dual feasible. Define

$$x^0 := \begin{pmatrix} 0 \\ 0.5 \\ 0 \\ 0 \\ 0 \\ 0.5 \\ 0 \end{pmatrix}$$

and go to **Step 3**.

Step 3. Compute the Lagrange polynomial

$$L_{I_0(z)} = \begin{pmatrix} e^{-2} & e^{-1} & e & e^2 \end{pmatrix} \begin{pmatrix} -\frac{1}{6}+\frac{1}{12}z+\frac{1}{6}z^2-\frac{1}{12}z^3 \\ \frac{2}{3}-\frac{2}{3}z-\frac{1}{6}z^2+\frac{1}{6}z^3 \\ \frac{2}{3}+\frac{2}{3}z-\frac{1}{6}z^2-\frac{1}{6}z^3 \\ -\frac{1}{6}-\frac{1}{12}z+\frac{1}{6}z^2+\frac{1}{12}z^3 \end{pmatrix}$$

$$= 0.21274Z^3 0.13971z^2 + 0.96246z + 0.80338.$$

Compute

$$s = \min\left\{ \inf_{z\in[-2,-1]} \left(e^z \text{-} \left(0.21274z^3 + 0.73971z^2 + 0.96246z + 0.80338\right)\right) \right.$$

$$\left. \inf_{z\in[1,2]} \left(e^z \text{-} \left(0.21274z^3 + 0.73971z^2 + 0.96246z + 0.80338\right)\right) \right\} = \text{-}0.155355463.$$

Since $s < -e$, determine S_{new} = {1.607}, update k:=k+1, and the define $T^1 := T^0 \cup \{1.607\}$. Consider the restricted problem $\left(P^1_{\inf}\right)$ and go to **Step 2**.

Step 2. Apply Prékopa's dual method for solving $\left(P^1_{\inf}\right)$:

- Start with the dual feasible basis B corresponding to the consecutive pairs {-2,-1} and {0, 1.61}. The current solution

$$x = B^{-1}b = \begin{pmatrix} 1 & 1 & 1 & 1 \\ -2 & -1 & 0 & 1.607 \\ 4 & 1 & 0 & 1.607^2 \\ -8 & -1 & 0 & 1.607^3 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 0 \\ 4 \\ 0 \end{pmatrix} = \begin{pmatrix} 0.33657 \\ 0.60299 \\ \text{-}0.73367 \\ 0.79411 \end{pmatrix}$$

has one negative entry, corresponding to the index 3. Declare the 3_{rd} index outgoing, and restore B as corresponding to the pairs {-2,-1} and {1.607, 2}.

- Compute the current solution

$$x = B^{-1}b = \begin{pmatrix} 1 & 1 & 1 & 1 \\ -2 & -1 & 1.607 & 2 \\ 4 & 1 & 1.607^2 & 4 \\ -8 & -1 & 1.607^3 & 8 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 0 \\ 4 \\ 0 \end{pmatrix} = \begin{pmatrix} 0.5 \\ 2\times 10^{-10} \\ 0 \\ 0.5 \end{pmatrix},$$

which is primal-dual feasible.

- Define

$$x^1 = \begin{pmatrix} 0 \\ 0.5 \\ 2\times 10^{-10} \\ 0 \\ 0 \\ 0 \\ 0.5 \\ 0 \end{pmatrix}$$

and go to **Step 3**.

Step 3. Compute the Lagrange polynomial

$$L_{I_1}(z) = \begin{pmatrix} e^{-2} & e^{-1} & e^{1.607} & e^{2} \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ -2 & -1 & 1.607 & 2 \\ 4 & 1 & 1.607^2 & 4 \\ -8 & -1 & 1.607^3 & 8 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ z \\ z^2 \\ z^3 \end{pmatrix} =$$

$$= 0.63522 + 0.79431z + 0.78174z^2 + 0.25478z^3$$

Compute

$$s = \min\left\{ \inf_{z\in[-2,-1]} \left(e^z - \left(0.63522 + 0.79431z + 0.78174z^2 + 0.25478z^3\right)\right),\right.$$

$$\left. \inf_{z\in[1.607,2]} \left(e^z - \left(0.63522 + 0.79431z + 0.78174z^2 + 0.25478z^3\right)\right)\right\} = -0.121165274$$

Since $s > -0.15$, we declare that the optimal solution for the dual is

$$(y_0, y_1, y_2, y_3) = (0.63522,\ 0.79431,\ 0.78174,\ 0.25478)$$

and the dual optimal value $q_e(D)$ is 3.7622.

7. Computational results

The cutting plane algorithm was implemented in *Delphi* v5.0, and was run on a 1.7MHz processor. The program uses alternatively two main procedures: (*i*) the procedure *Discrete_Dual* for solving the discrete bounding moment problem by using Prékopa's dual method, and (*ii*) the procedure *Nonlinear* for finding the feasible cuts. The *Nonlinear* procedure has the following options:

(1) to use the MATLAB optimization function *fminbnd*, which finds a (local) minimum of a continuous univariable function on a given interval, based on golden section search and parabolic interpolation,

(2) to use the MATLAB optimization function *fmincon*, which finds a (local) minimum of a constrained multivariable function, based on a Sequential Quadratic Programming (SQP) method for medium-scale optimization), or based on an interior-reflective Newton method for large scale optimization),

(3) or to use a simple discretization heuristic, which defines a fine auxiliary grid (with step 10^{-2} or 10^{-3}) of the interval on which the residual function is defined, and selects the feasible cuts as those grid points for which the restriction of the residual function attains its minimum (respectively maximum) value. Since the residual function is continuous, this heuristic performs well for small size residual intervals.

Due to the high numerical instability of the Vandermonde matrices, the *Discrete_Dual* procedure stalls frequently when using the usual arithmetic precision provided by Delphi (Extended real in the range Extended 3.6 x 10^{4951} .. 1.1 x 10^{4932}, which has 19-20 significant digits and uses 10 bytes). In order to avoid stalling, it is necessary to use an *Arithmetic Precision Library*, which was implemented in Delphi v 5.0 as an auxiliary tool for the program, and which can work with real numbers having theoretically an arbitrary number of significant digits, but practically (due to processing time limitations) can work with up to 30 significant digits. The optimization procedures in MATLAB (Student Version v5.3) may work only with double-precision (up to 16 significant digits). For solving the primal bounding moment problem, a state-of-the-art implementation of the interior point algorithm for LP (BPMPD) is provided by Maros and Meszaros at `http://www.sztaki.hu/~meszaros/bpmpd/`.

We applied the cutting plane method described above on several numerical examples. In the following we describe some of our computational results.

For the numerical example, we considered

(*i*) the moments $m_0,...,m_m$ generated by a discrete normal distribution function of the form $\begin{pmatrix} n \\ p_n \end{pmatrix}_{n\in Z}$, where $p_n = Ke^{-(n-a)^2/2h}, a>0, h>0$, and $\sum_n p_n = 1$.

(*ii*) the objective function $f(z) = c_1\left(\frac{l_1}{l_1+z}\right)^{a_1} + c_2\left(\frac{l_2}{l_2+z}\right)^{a_2} + c_3\left(\frac{l_3}{l_3+z}\right)^{a_3}, \quad z\in[-L,L]$ where c_i, l_i and a_i are positive, i=1,2,3, and $L > \max(l_i)$.

It is easy to check that f is continuous, has derivatives of arbitrary order, and $f^{(2k)}(z)>0, f^{(2k+1)}(z)<0$, for every k=0,1,… . Therefore, $(-1)^{m+1} f$ has positive divided differences of order (m+1).

We applied the cutting plane method with $e = 0.0001$ for finding the bounds:

$$P_{\inf}(P_{\sup}): \inf(\sup)\int_{[-50,50]}(-1)^{m+1} f(z)dP$$

$$\text{subject to } \int_{[-50,50]} z^k dP = m_k, k = 0,...,m\ .$$

We considered first the particular case when a=5, h=1, and c_i=1, $l_i = 51$, and $a_i = i$, i=1,2,3. The values of the first 15 moments corresponding to this particular case are presented in **Table 1**, and the graph of the corresponding objective function f is presented in **Figure 1**.

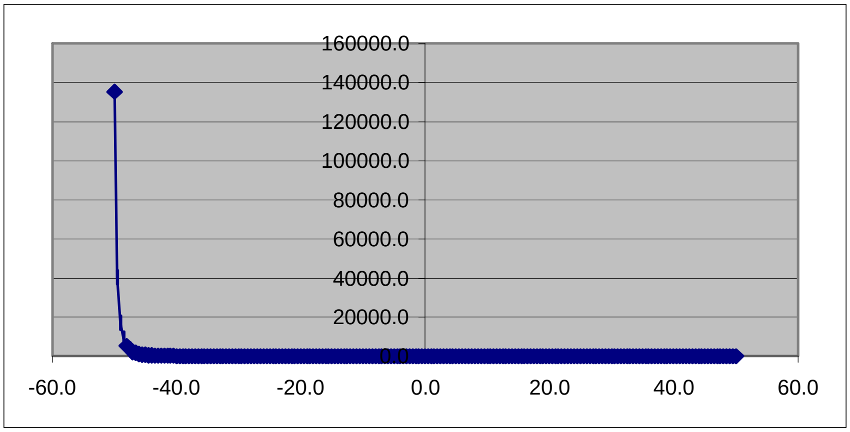

Figure 1

We started the cutting-plane algorithm with the initial grid

$$T^0 = \{-50, \ldots, -1, 0, 1, \ldots, 50\}.$$

and solved the nonlinear optimization programs using the simple discretization heuristic with the grid step 10^{-3}. The arithmetic precision was set to 25 significant digits. We allowed at most 100 iterations for solving the restricted bounding problems with the procedure *Dual_Discrete*.

The lower and upper bounds obtained for the expectation $\int_{[-50,50]} f(z)dP$ are presented in **Table 1.**

Order	Moment	Lower bound (lb)	Upper bound (ub)	Gap (ub-lb)
1	5.00000000	2.4954616891399400	3.3097514110341300	0.8142897218941900
2	25.49897913	2.4964695380118800	2.4979784129112200	0.0015088748993399
3	132.48468696	2.4966963091803100	2.4967847003572000	0.0000883911768903
4	700.60388259	2.4967124689016800	2.4967673963512100	0.0000549274495296
5	3767.64923772	2.4967182057431400	2.4967589375100000	0.0000407317668603
6	20588.64885440	2.4967188390393500	2.4967557723341100	0.0000369332947598
7	114246.82459629	2.4967195543538700	2.4967534716667500	0.0000339173128800
8	643342.06226407	2.4967217714838700	2.4967526101013900	0.0000308386175201
9	3674192.10354070	2.4967287895160000	2.4967509926495800	0.0000222031335797
10	21269629.55648070	2.4967287895241000	2.4967509826495800	0.0000221931254800
11	124740450.69093800	2.4967287943184600	2.4967509925835300	0.0000221982650701
12	740783444.13225300	2.4967288018799700	2.4967509925787700	0.0000221906987998
13	4452629486.31506000	2.4967288055719800	2.4967509924963900	0.0000221869244101
14	27077366231.20050000	2.4967288069012400	2.4967499893874800	0.0000211824862397
15	166533873705.49300000	2.4967288136886200	2.4967499846846400	0.0000211709960198

Table 1

We remark that the gap between the upper and lower bounds is less then 10^{-4}, and it decreases when the value of *m* increases.

Next, we changed the l_i values to $l_i = 100$, *i*=1,2,3, and we repeated the computation for this particular case. The graph of the objective function *f* is presented in **Figure 2** below, and the bounds obtained for the expectation of *f* are presented in **Table 2**.

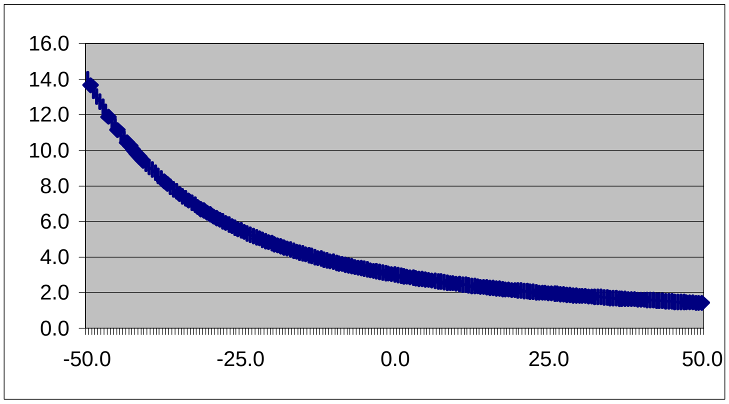

Figure 2

Order	Moment	Lower bound (lb)	Upper bound (ub)	Gap (ub-lb)
1	5.00000000	2.5674572321576200	2.8686770666277900	0.3012198344701690
2	25.49897913	2.6778437867810900	2.7983148690939400	0.1204710823128500
3	132.48468696	2.6978891786997700	2.7867961562121300	0.0889069775123601
4	700.60388259	2.7174416946292000	2.7579224568988000	0.0404807622696000
5	3767.64923772	2.7236489875279300	2.7436512913619100	0.0200023038339752
6	20588.64885440	2.7236569535558100	2.7343648654189500	0.0107079118631401
7	114246.82459629	2.7236607127676700	2.7300646870918000	0.0064039743241278
8	643342.06226407	2.7236639073260300	2.7236752485369100	0.0000113412108793
9	3674192.10354070	2.7236641989336900	2.7236747300045200	0.0000105310708274
10	21269629.55648070	2.7236649141947900	2.7236681101341900	0.0000031959394002
11	124740450.69093800	2.7236653132964000	2.7236653250416200	0.0000000117452199
12	740783444.13225300	2.7236653147917600	2.7236653247846900	0.0000000099929296
13	4452629486.31506000	2.7236653234326400	2.7236653236075600	0.0000000001749201
14	27077366231.20050000	2.7236653243993100	2.7236653245296400	0.0000000001303300
15	166533873705.49300000	2.7236653243373400	2.7236653243425700	0.0000000000052300

Table 2

We remark that in this case, the gap between the upper and lower bounds are smaller than 10^{-6} and it decreases with an exponential rate for values of *m* larger than 10.

8. Conclusions

Implementing the combination between the classical cutting plane method for (LSIP) and Prékopa's dual method for solving the discrete bounding moment problem has two main advantages: it provides a robust method for solving the linear optimization problems having a Vandermonde constraint matrix, and also, it decreases the computational effort.

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