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RRR 47-2004, DECEMBER, 2004

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ON STRONG UNIMODALITY OF MULTIVARIATE DISCRETE DISTRIBUTIONS

Ersoy Subasi Mine Subasi András Prékopa

Abstract. A discrete function f defined on $\mathbb{Z}^n$ is said to be logconcave if $f(\lambda\mathbf{x}+(1-\lambda)\mathbf{y}) \geq [f(\mathbf{x})]^{\lambda}[f(\mathbf{y})]^{1-\lambda}$ for $\mathbf{x}$, $\mathbf{y}$, $\lambda\mathbf{x}+(1-\lambda)\mathbf{y} \in \mathbb{Z}^n$. A more restrictive notion is strong unimodality. Following Barndorff-Nielsen (1973) a discrete function $p(\mathbf{z}), \mathbf{z} \in \mathbb{Z}^n$ is called strongly unimodal if there exists a convex function $f(\mathbf{x}),\ \mathbf{x} \in \mathbb{R}^n$ such that $f(\mathbf{x}) = -\log p(\mathbf{x}),$ if $\ \mathbf{x} \in \mathbb{Z}^n$. In this paper sufficient conditions are given that a discrete function is strongly unimodal. Six sufficient conditions are given for the case of $n = 3$ and one for the general case. A three-dimensional example shows that the logconcavity of a discrete function does not imply strong unimodality, in general. Examples are presented.
Key words: Strong Unimodality, Logcancavity, Discrete Logconcavity.

1 Introduction

A nonnegative function f defined on a convex subset A of the space $\mathbb{R}^n$ is said to be *logconcave* if for every pair $\mathbf{x}, \mathbf{y} \in A$ and $0 < \lambda < 1$ we have the inequality

$$f(\lambda\mathbf{x} + (1-\lambda)\mathbf{y}) \geq [f(\mathbf{x})]^{\lambda}[f(\mathbf{y})]^{(1-\lambda)}.$$

If f is positive valued, then $\log f$ is a concave function on A. If the inequality holds strictly for $\mathbf{x} \neq \mathbf{y}$, then f is said to be strictly logconcave.

The notion of a logconcave probability measure was introduced in Prékopa (1971). A probability measure P, defined on $\mathbb{R}^n$, is said to be logconcave if for every pair of nonempty convex sets $A, B \subset \mathbb{R}^n$ (any convex set is Borel measurable) and we have the inequality

$$P(\lambda A + (1-\lambda)B) \geq [P(A)]^{\lambda}[P(B)]^{(1-\lambda)},$$

where the + sign refers to Minkowski addition of sets, i.e.,

$$\lambda A + (1-\lambda)B = \{\lambda\mathbf{x} + (1-\lambda)\mathbf{y} | \mathbf{x} \in A, \mathbf{y} \in B\}.$$

The above notion generalizes in a natural way to nonnegative valued measures. In this case we require the logconcavity inequality to hold for finite $P(A), P(B)$.

In 1912 Fekete introduced the notion of an *r-times positive sequence*. The sequence of nonnegative elements $\ldots, a_{-2}, a_{-1}, a_0, \ldots$ is said to be *r-times positive* if the matrix

$$A = \begin{bmatrix} \ddots & \ddots & \ddots & & \\ \ddots & a_0 & a_1 & a_2 & \\ \ddots & a_{-1} & a_0 & a_1 & \ddots \\ & a_{-2} & a_{-1} & a_0 & \ddots \\ & & \ddots & \ddots & \ddots \end{bmatrix}$$

has no negative minor of order smaller than or equal to r.

Twice-positive sequences are those for which we have

$$\begin{vmatrix} a_i & a_j \\ a_{i-t} & a_{j-t} \end{vmatrix} = a_i a_{j-t} - a_j a_{i-t} \geq 0. \tag{1.1}$$

for every $i < j$ and $t \geq 1$. This holds if and only if $a_i^2 \geq a_{i-1}a_{i+1}$. Fekete (1912) also proved that the convolution of two r-times positive sequences is r-times positive. Twice-positive sequences are also called *logconcave sequences*. For this, Fekete's theorem states that the convolution of two logconcave sequences is logconcave.

A discrete probability distribution, defined on the real line, is said to be logconcave if the corresponding probability function is logconcave.

In what follows we present our results in terms of probability functions. They generalize in a straightforward manner for more general logconcave functions.

Let $\mathbb{Z}^n$ designate the set of lattice points in the space. The convolution of two logconcave distributions on $\mathbb{Z}^n$ is no longer logconcave in general, if $n \geq 2$.

Consider a discrete probability function $p(\mathbf{z}), \mathbf{z} \in \mathbb{Z}^n$ is called strongly unimodal if there exists a convex function $f(\mathbf{x}),\ \mathbf{x} \in \mathbb{R}^n$ such that $f(\mathbf{x}) = -\log p(\mathbf{x})$ if $\mathbf{x} \in \mathbb{Z}^n$. (Barndorff-Nielsen, 1973). If $p(\mathbf{z}) = 0$, then by definition $f(\mathbf{z}) = \infty$. This notion is not a direct generalization of that of the one-dimensional case, i.e., of formula (1.1). However in case of $n = 1$ the two notions are the same (see,e.g., Prékopa 1995). It is trivial that if p is strongly unimodal, then it is logconcave.

The joint probability function of a finite number of mutually independent discrete random variables, where each has a logconcave probability function is strongly unimodal.

Pedersen (1975) gave the following two sufficient conditions for a discrete distribution on $\mathbb{Z}^2$ to be strongly unimodal. Let p be a discrete probability function on $\mathbb{Z}^2$. It is sufficient for p to be strongly unimodal if it satisfies one of the following conditions (a) or (b):

(a) $$p_{i-1j}p_{ij-1} \geq p_{ij}p_{i-1j-1}$$
$$p_{i-1j}p_{ij} \geq p_{ij-1}p_{i-1j+1}$$
$$p_{ij}p_{ij-1} \geq p_{i-1j}p_{i+1j-1},$$

(b) $$p_{ij}p_{i-1j-1} \geq p_{i-1j}p_{ij-1}$$
$$p_{ij}p_{i-1j} \geq p_{ij+1}p_{i-1j-1}$$
$$p_{ij}p_{ij-1} \geq p_{i+1j}p_{i-1j-1},$$

where p_{ij} denotes the value of p on $(i,j) \in \mathbb{Z}^2$.

Pedersen (1975) also proved that the trinomial probability function is logconcave and the convolution of any finite number of these distributions with possibly different parameter sets is also logconcave.

A function $f(\mathbf{z}), \mathbf{z} \in \mathbb{R}^n$ is said to be polyhedral (simplicial) on the bounded convex polyhedron $K \subseteq \mathbb{R}^n$ if there exists a subdivision of K into n-dimensional convex polyhedra (simplices), with pairwise disjoint interiors such that f is continuous on K and linear on each subdividing polyhedron (simplex). Prékopa and Li (1995) presented a dual method to solve a linearly constrained optimization problem with convex, polyhedral objective function, along with a fast bounding technique, for the optimum value. Any $f(x)$, defined by the use of a strongly unimodal probability function $p(x)$, is a simplicial function and can be used in the above-mentioned methodology.

In Section 2 we give sufficient conditions for a discrete distribution on $\mathbb{Z}^3$ to be strongly unimodal. In Section 3 we give a counterexample, for the case of $n = 3$, to show that logconcavity does not imply strong unimodality. In Section 4 we give sufficient condition for a discrete distribution on $\mathbb{Z}^n$ to be strongly unimodal. In Section 5 we present four examples for strongly unimodal distributions on $\mathbb{Z}^n$.

2 Sufficient Conditions for a Discrete Distribution on $\mathbb{Z}^3$ to be Strongly Unimodal

In this section we give sufficient conditions for a discrete probability function defined on $\mathbb{Z}^3$ that ensure its strong unimodality. The function f defined on $\mathbb{R}^3$ that we fit to the values of $-\log p(.)$ is piecewise linear. We accomplish the job in such a way that we subdivide $\mathbb{R}^3$ into simplices with disjoint interiors such that the function $f(\mathbf{x})$ is linear on each of them. First we subdivide $\mathbb{R}^3$ into unit cubes and then subdivide each cube into six simplices with disjoint interiors. In each cube the same type of subdivision is used. On each simplex we define $f(\mathbf{x})$ by the equation of the hyperplane determined by the values of $-\log p(\mathbf{x})$ at the vertices. Next we ensure that $f(\mathbf{x})$ is convex on any neighboring simplices. The resulting function $f(\mathbf{x})$ is convex on the entire space.

Any cube in $\mathbb{R}^3$ can be subdivided into simplices with disjoint interiors (such that the vertices of the simplices are those of the cube) in six different ways. In view of this we subdivide $\mathbb{R}^3$ into simplices in six different ways as follows:

Subdivision 1.
Let $T_{1c}(i,j,k), \quad c = 1,2,...,6$ be the simplices in $\mathbb{R}^3$ defined by

$$\begin{aligned}
T_{11}(i,j,k) &= conv\{(i,j,k),(i+1,j,k),(i+1,j+1,k),(i+1,j+1,k+1)\},\\
T_{12}(i,j,k) &= conv\{(i,j,k),(i+1,j,k),(i+1,j,k+1),(i+1,j+1,k+1)\},\\
T_{13}(i,j,k) &= conv\{(i,j,k),(i,j+1,k),(i+1,j+1,k),(i+1,j+1,k+1)\},\\
T_{14}(i,j,k) &= conv\{(i,j,k),(i,j+1,k),(i,j+1,k+1),(i+1,j+1,k+1)\},\\
T_{15}(i,j,k) &= conv\{(i,j,k),(i,j,k+1),(i+1,j,k+1),(i+1,j+1,k+1)\},\\
T_{16}(i,j,k) &= conv\{(i,j,k),(i,j,k+1),(i,j+1,k+1),(i+1,j+1,k+1)\}.
\end{aligned}$$

Subdivision 2.
Let $T_{2c}(i,j,k), \quad c = 1,2,...,6$ be the simplices in $\mathbb{R}^3$ defined by

$$\begin{aligned}
T_{21}(i,j,k) &= conv\{(i,j,k),(i,j,k+1),(i+1,j,k+1),(i+1,j+1,k+1)\},\\
T_{22}(i,j,k) &= conv\{(i,j,k),(i,j,k+1),(i,j+1,k+1),(i+1,j+1,k+1)\},\\
T_{23}(i,j,k) &= conv\{(i,j,k),(i,j+1,k),(i,j+1,k+1),(i+1,j+1,k+1)\},\\
T_{24}(i,j,k) &= conv\{(i,j,k),(i+1,j,k),(i+1,j,k+1),(i+1,j+1,k+1)\},\\
T_{25}(i,j,k) &= conv\{(i,j,k),(i+1,j,k),(i,j+1,k),(i+1,j+1,k+1)\},\\
T_{26}(i,j,k) &= conv\{(i+1,j,k),(i,j+1,k),(i+1,j+1,k),(i+1,j+1,k+1)\}.
\end{aligned}$$

Subdivision 3.
Let $T_{3c}(i,j,k), \quad c = 1,2,...,6$ be the simplices in $\mathbb{R}^3$ defined by

$$\begin{aligned}
T_{31}(i,j,k) &= conv\{(i,j,k),(i,j,k+1),(i+1,j,k+1),(i,j+1,k+1)\},\\
T_{32}(i,j,k) &= conv\{(i,j,k),(i+1,j,k+1),(i,j+1,k+1),(i+1,j+1,k+1)\},\\
T_{33}(i,j,k) &= conv\{(i,j,k),(i+1,j+1,k),(i,j+1,k+1),(i+1,j+1,k+1)\},
\end{aligned}$$

$$
\begin{aligned}
T_{34}(i,j,k) &= conv\{(i,j,k),(i,j+1,k),(i+1,j+1,k),(i,j+1,k+1)\},\\
T_{35}(i,j,k) &= conv\{(i,j,k),(i+1,j,k+1),(i+1,j,k),(i+1,j+1,k+1)\},\\
T_{36}(i,j,k) &= conv\{(i,j,k),(i+1,j+1,k),(i+1,j,k),(i+1,j+1,k+1)\}.
\end{aligned}
$$

Subdivision 4.
Let $T_{4c}(i,j,k), \quad c = 1,2,...,6$ be the simplices in $\mathbb{R}^3$ defined by

$$
\begin{aligned}
T_{41}(i,j,k) &= conv\{(i,j,k),(i,j,k+1),(i+1,j,k+1),(i,j+1,k+1)\},\\
T_{42}(i,j,k) &= conv\{(i,j,k),(i+1,j,k+1),(i,j+1,k+1),(i+1,j+1,k+1)\},\\
T_{43}(i,j,k) &= conv\{(i,j,k),(i,j+1,k),(i,j+1,k+1),(i+1,j+1,k+1)\},\\
T_{44}(i,j,k) &= conv\{(i,j,k),(i,j+1,k),(i+1,j+1,k),(i+1,j+1,k+1)\},\\
T_{45}(i,j,k) &= conv\{(i,j,k),(i+1,j,k),(i+1,j,k+1),(i+1,j+1,k+1)\},\\
T_{46}(i,j,k) &= conv\{(i,j,k),(i+1,j,k+1),(i+1,j,k),(i+1,j+1,k+1)\}.
\end{aligned}
$$

Subdivision 5.
Let $T_{5c}(i,j,k), \quad c = 1,2,...,6$ be the simplices in $\mathbb{R}^3$ defined by

$$
\begin{aligned}
T_{51}(i,j,k) &= conv\{(i,j,k),(i,j,k+1),(i+1,j,k+1),(i,j+1,k+1)\},\\
T_{52}(i,j,k) &= conv\{(i,j,k),(i+1,j,k+1),(i,j+1,k+1),(i+1,j+1,k+1)\},\\
T_{53}(i,j,k) &= conv\{(i,j,k),(i+1,j+1,k),(i,j+1,k+1),(i+1,j+1,k+1)\},\\
T_{54}(i,j,k) &= conv\{(i,j,k),(i,j+1,k),(i,j+1,k+1),(i+1,j+1,k)\},\\
T_{55}(i,j,k) &= conv\{(i,j,k),(i+1,j+1,k),(i+1,j,k+1),(i+1,j+1,k+1)\},\\
T_{56}(i,j,k) &= conv\{(i,j,k),(i+1,j,k),(i+1,j+1,k),(i+1,j,k+1)\}.
\end{aligned}
$$

Subdivision 6.
Let $T_{6c}(i,j,k), \quad c = 1,2,...,6$ be the simplices in $\mathbb{R}^3$ defined by

$$
\begin{aligned}
T_{61}(i,j,k) &= conv\{(i,j,k),(i,j,k+1),(i+1,j,k+1),(i,j+1,k+1)\},\\
T_{62}(i,j,k) &= conv\{(i,j,k),(i+1,j,k+1),(i,j+1,k+1),(i+1,j+1,k+1)\},\\
T_{63}(i,j,k) &= conv\{(i,j,k),(i,j+1,k),(i,j+1,k+1),(i+1,j+1,k+1)\},\\
T_{64}(i,j,k) &= conv\{(i,j,k),(i,j+1,k),(i+1,j+1,k),(i+1,j+1,k+1)\},\\
T_{65}(i,j,k) &= conv\{(i,j,k),(i+1,j,k),(i,j+1,k),(i+1,j+1,k+1)\},\\
T_{66}(i,j,k) &= conv\{(i+1,j,k),(i+1,j+1,k),(i,j+1,k),(i+1,j+1,k+1)\}.
\end{aligned}
$$

Let $C_t, \ ,t = 1,2,...,6$ be the collection of the simplices $T_{tc}(i,j,k), c = 1,2,...,6,\ (i,j,k) \in \mathbb{Z}^3$. Let p be the probability function of a discrete probability distribution defined on $\mathbb{R}^3$ and p_{ijk} the value of p at $(i,j,k) \in \mathbb{Z}^3$.

Theorem 1. *If p satisfies one of the following conditions $(a),(b),(c),(d),(e),(f)$ for all $(i,j,k) \in \mathbb{Z}^3$, then it is strongly unimodal.*

(a) *C_1 is the collection of the simplices $T_{1c}(i,j,k), c = 1,2,...,6$ and*

(1) $p_{i+1jk}p_{ij+1k} \le p_{ijk}p_{i+1j+1k},$

(2) $p_{i+1jk}p_{ijk+1} \le p_{ijk}p_{i+1jk+1}$,

(3) $p_{ij+1k}p_{ijk+1} \le p_{ijk}p_{ij+1k+1}$,

(4) $p_{i+1j+1k}p_{i+1jk+1} \le p_{i+1jk}p_{i+1j+1k+1}$,

(5) $p_{i+1j+1k}p_{ij+1k+1} \le p_{ij+1k}p_{i+1j+1k+1}$,

(6) $p_{i+1jk+1}p_{ij+1k+1} \le p_{ijk+1}p_{i+1j+1k+1}$,

(7) $p_{i-1jk}p_{i+1j+1k+1} \le p_{ijk}p_{ij+1k+1}$,

(8) $p_{ij-1k}p_{i+1j+1k+1} \le p_{ijk}p_{i+1jk+1}$,

(9) $p_{ijk-1}p_{i+1j+1k+1} \le p_{ijk}p_{i+1j+1k}$,

(10) $p_{ijk}p_{i+2j+1k+1} \le p_{i+1jk}p_{i+1j+1k+1}$,

(11) $p_{ijk}p_{i+1j+2k+1} \le p_{ij+1k}p_{i+1j+1k+1}$,

(12) $p_{ijk}p_{i+1j+1k+2} \le p_{ijk+1}p_{i+1j+1k+1}$.

(b) *C_2 is the collection of the simplices $T_{2c}(i,j,k), c = 1,2,...,6$ and*

(13) $p_{i+1jk+1}p_{ij+1k+1} \le p_{ijk+1}p_{i+1j+1k+1}$,

(14) $p_{i+1jk}p_{ijk+1} \le p_{ijk}p_{i+1jk+1}$,

(15) $p_{ijk+1}p_{ij+1k} \le p_{ijk}p_{ij+1k+1}$,

(16) $p_{ij+1k+1}p_{i+1jk} \le p_{ijk}p_{i+1j+1k+1}$,

(17) $p_{i+1jk+1}p_{ij+1k} \le p_{ijk}p_{i+1j+1k+1}$,

(18) $p_{i+1j+1k}p_{ijk} \le p_{i+1jk}p_{ij+1k}$,

(19) $p_{ij-1k}p_{i+1j+1k+1} \le p_{ijk}p_{i+1jk+1}$,

(20) $p_{i+1j+2k+1}p_{ijk} \le p_{ij+1k}p_{i+1j+1k+1}$,

(21) $p_{i-1j+1k}p_{i+1j+1k+1} \le p_{ij+1k}p_{ij+1k+1}$,

(22) $p_{i+2j+1k+1}p_{ijk} \le p_{i+1jk}p_{i+1j+1k+1}$,

(23) $p_{i+1j-1k}p_{i+1j+1k+1} \le p_{i+1jk}p_{i+1jk+1}$,

(24) $p_{i-1jk}p_{i+1j+1k+1} \le p_{ijk}p_{ij+1k+1}$,

(25) $p_{i+1j-1k}p_{ijk} \le p_{i+1jk}p_{ij+1k}$.

(c) *C_3 is the collection of the simplices $T_{3c}(i,j,k), c = 1,2,...,6$ and*

(26) $p_{ijk+1}p_{i+1j+1k+1} \le p_{ij+1k+1}p_{i+1jk+1}$,

(27) $p_{i+1j+1k}p_{i+1jk+1}p_{ij+1k+1} \le p_{ijk}p^2_{i+1j+1k+1}$,

(28) $p_{ij+1k+1}p_{ij+1k} \le p_{ij+1k+1}p_{i+1j+1k+1}$,

(29) $p_{i+1jk}p_{ij+1k+1} \le p_{ijk}p_{i+1j+1k+1}$,

(30) $p_{i+1j+1k}p_{i+1jk+1} \le p_{i+1jk}p_{i+1j+1k+1}$,

(31) $p_{i-1jk}p_{i+1jk+1} \le p_{ijk}p_{ijk+1}$,

(32) $p_{i+2jk+1}p_{ijk} \le p_{i+1jk}p_{i+1jk+1}$,

(33) $p_{i+2jk+1}p_{ijk} \le p_{i+1jk}p_{i+1j+1k}$,

(34) $p_{i-1jk}p_{i+1j+1k} \le p_{ijk}p_{ij+1k}$.

(d) *C_4 is the collection of the simplices $T_{4c}(i,j,k), c = 1,2,...,6$ and*

(35) $p_{ijk+1}p_{i+1j+1k+1} \le p_{ij+1k+1}p_{i+1jk+1}$,

(36) $p_{i+1jk+1}p_{ij+1k} \le p_{ijk}p_{i+1j+1k+1}$,

(37) $p_{i+1j+1k}p_{ij+1k+1} \le p_{ij+1k}p_{i+1j+1k+1}$,

(38) $p_{i+1jk}p_{ij+1k+1} \le p_{ijk}p_{i+1j+1k+1}$,

(39) $p_{i+1jk}p_{ij+1k} \le p_{ijk}p_{i+1j+1k}$,

(40) $p_{i+1j+1k}p_{i+1jk+1} \le p_{i+1jk}p_{i+1j+1k+1}$,

(41) $p_{ij-1k}p_{i+1j+1k+1} \le p_{ijk}p_{ijk+1}$,

(42) $p_{i-1jk}p_{i+1jk+1} \le p_{ijk}p_{ijk+1}$,

(43) $p_{ij+2k+1}p_{ijk} \le p_{ij+1k}p_{ij+1k+1}$,

(44) $p_{i-1jk}p_{i+1j+1k+1} \le p_{ijk}p_{ij+1k+1}$,

(45) $p_{ijk}p_{i+1j+2k+1} \le p_{ij+1k}p_{i+1j+1k+1}$,

(46) $p_{ijk}p_{i+2jk+1} \le p_{i+1jk}p_{i+1jk+1}$,

(47) $p_{ij-1k}p_{i+1j+1k+1} \le p_{ijk}p_{i+1jk+1}$,

(48) $p_{ijk}p_{i+2j+1k+1} \le p_{i+1jk}p_{i+1j+1k+1}$.

(e) *C_5 is the collection of the simplices $T_{5c}(i,j,k), c = 1,2,...,6$ and*

(49) $p_{ijk+1}p_{i+1j+1k+1} \le p_{ij+1k+1}p_{i+1jk+1}$,

(50) $p_{i+1j+1k}p_{i+1jk+1}p_{ij+1k+1} \le p_{ijk}p^2_{i+1j+1k+1}$,

(51) $p_{i+1j+1k}p_{ij+1k} \le p_{ij+1k+1}p_{i+1j+1k+1}$,

(52) $p_{i+1jk}p_{i+1j+1k+1} \le p_{i+1j+1k}p_{i+1jk+1}$.

(f) *C_6 is the collection of the simplices $T_{6c}(i,j,k), c = 1,2,...,6$ and*

(53) $p_{ijk+1}p_{i+1j+1k+1} \le p_{ij+1k+1}p_{i+1jk+1}$,

(54) $p_{ij+1k}p_{i+1jk+1} \le p_{ijk}p_{i+1j+1k+1}$,

(55) $p_{i+1jk}p_{ij+1k+1} \le p_{ijk}p_{i+1j+1k+1}$,

(56) $p_{i+1j+1k}p_{ijk} \le p_{i+1jk}p_{ij+1k}$,

(57) $p_{ijk}p_{i+1j+1k+2} \le p_{i+1jk+1}p_{ij+1k+1}$,

(58) $p_{i-1jk}p_{i+1jk+1} \le p_{ijk}p_{ijk+1}$,

(59) $p_{ij-1k}p_{i+1j+1k+1} \le p_{ijk}p_{ijk+1}$,

(60) $p_{ijk}p_{ij+2k+1} \le p_{ij+1k}p_{ij+1k+1}$,

(61) $p_{i-1j+1k}p_{i+1j+1k+1} \le p_{ij+1k}p_{ij+1k+1}$,

(62) $p_{ijk}p_{i+2jk+1} \le p_{i+1jk}p_{i+1jk+1}$,

(63) $p_{i+1j-1k}p_{i+1j+1k+1} \le p_{i+1jk}p_{i+1jk+1}$,

(64) $p_{i-1j+1k}p_{i+1jk} \le p_{ijk}p_{ij+1k}$.

Proof. We prove the sufficiency of (a). The proof of the sufficiency of (b), (c), (d), (e) and (f) can be done in a similar way.

We designate by $L(c,i,j,k),\ (i,j,k) \in \mathbb{Z}^3,\ c = 1,2,...,6$ the linear function on $\mathbb{R}^3$ that coincides with $-\log p(.)$ on the vertices of $T_{1c}(i,j,k)$ and define

$$f(\mathbf{y}) = \begin{cases} L(c,i,j,k) & \text{if } \mathbf{y} \in T_{1c}(i,j,k), (i,j,k) \in \mathbb{Z}^3 \\ \infty & \text{if } \mathbf{y} \notin C_1 \end{cases}$$

Obviously, f coincides on $\mathbb{Z}^3$ with $-\log p(.)$.

Claim: Conditions (1), ..., (12) ensure that the restriction of f to any two neighboring simplices $T_{1c}(i,j,k),\ (i,j,k) \in \mathbb{Z}^3$ with a common face is convex.

Proof of the claim: Right at the outset we need to define a function f and then for that f we have to prove that for any two neighboring simplices it satisfies the convexity property. On each simplex we define a linear function. In case of any simplex a linear piece is determined by the vertices of the simplex and the corresponding values of $-\log p(.)$. The collection of these linear pieces form the function f.

The function f is convex on any two neighboring simplices if for any

$$\mathbf{z}_0 = \begin{pmatrix} z_{01} \\ z_{02} \\ z_{03} \end{pmatrix},\quad \mathbf{z}_1 = \begin{pmatrix} z_{11} \\ z_{12} \\ z_{13} \end{pmatrix},\quad \mathbf{z}_2 = \begin{pmatrix} z_{21} \\ z_{22} \\ z_{23} \end{pmatrix},\quad \mathbf{z}_3 = \begin{pmatrix} z_{31} \\ z_{32} \\ z_{33} \end{pmatrix},\quad \mathbf{y}_0 = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

such that $\mathbf{z}_0, \mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3 \in T_{1c}(i,j,k),\ (i,j,k) \in \mathbb{Z}^3$ and y is the vertex of a neighboring simplex which is not belong to the current one, we have the relation

$$\frac{\begin{vmatrix} f(\mathbf{y}) & f(\mathbf{z}_0) & f(\mathbf{z}_1) & f(\mathbf{z}_2) & f(\mathbf{z}_3) \\ 1 & 1 & 1 & 1 & 1 \\ y_1 & z_{01} & z_{11} & z_{21} & z_{31} \\ y_2 & z_{02} & z_{12} & z_{22} & z_{32} \\ y_3 & z_{03} & z_{13} & z_{23} & z_{33} \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 1 & 1 \\ z_{01} & z_{11} & z_{21} & z_{31} \\ z_{02} & z_{12} & z_{22} & z_{32} \\ z_{03} & z_{13} & z_{23} & z_{33} \end{vmatrix}} \ge 0. \tag{2.1}$$

If $\mathbf{z}_0,\ \mathbf{z}_1,\ \mathbf{z}_2,\ \mathbf{z}_3,\ y$ are the vertices of the simplices given in Subdivision 1, then we obtain inequalities (1), (2), ..., (12). In order to obtain one of the inequalities given in condition (a) we consider the neighboring simplices $T_{11}(i,j,k)$ and $T_{12}(i,j,k)$. We take

$\mathbf{z}_0 = (i,j,k),\ \mathbf{z}_1 = (i{+}1,j,k),\ \mathbf{z}_2 = (i{+}1,j{+}1,k),\ \mathbf{z}_3 = (i{+}1,j{+}1,k{+}1)$ and $\mathbf{y} = (i{+}1,j,k{+}1)$.

In this case inequality (2.1) can be written as

$$\frac{\begin{vmatrix} f(\mathbf{y}) & f(\mathbf{z}_0) & f(\mathbf{z}_1) & f(\mathbf{z}_2) & f(\mathbf{z}_3) \\ 1 & 1 & 1 & 1 & 1 \\ i+1 & i & i+1 & i+1 & i+1 \\ j & j & j & j+1 & j+1 \\ k+1 & k & k & k & k+1 \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 1 & 1 \\ i & i+1 & i+1 & i+1 \\ j & j & j+1 & j+1 \\ k & k & k & k+1 \end{vmatrix}} \geq 0, \tag{2.2}$$

where $f = -\log p(.)$.

One can easily show that the determinant in the denominator in (2.2) is equal to 1. In order to guarantee the convexity of f the numerator in (2.2) must be nonnegative. Therefore we must have

$$\begin{vmatrix} f(\mathbf{y}) & f(\mathbf{z}_0) & f(\mathbf{z}_1) & f(\mathbf{z}_2) & f(\mathbf{z}_3) \\ 1 & 1 & 1 & 1 & 1 \\ i+1 & i & i+1 & i+1 & i+1 \\ j & j & j & j+1 & j+1 \\ k+1 & k & k & k & k+1 \end{vmatrix} =$$

$$\begin{vmatrix} f(\mathbf{y})-f(\mathbf{z}_0) & f(\mathbf{z}_0) & f(\mathbf{z}_1)-f(\mathbf{z}_0) & f(\mathbf{z}_2)-f(\mathbf{z}_1) & f(\mathbf{z}_3)-f(\mathbf{z}_2) \\ 0 & 1 & 0 & 0 & 0 \\ 1 & i & 1 & 0 & 0 \\ 0 & j & 0 & 1 & 0 \\ 1 & k & 0 & 0 & 1 \end{vmatrix} \geq 0$$

It follows that $f(\mathbf{y}) + f(\mathbf{z}_2) \geq f(\mathbf{z}_1) + f(\mathbf{z}_3)$. This is, however, the same as

$$p(\mathbf{y})p(\mathbf{z}_2) \leq p(\mathbf{z}_1)p(\mathbf{z}_3) \quad \text{or} \quad p_{i+1jk+1}p_{i+1j+1k} \leq p_{i+1jk}p_{i+1j+1k+1}.$$

So we obtain the inequality (4) of condition (a). All other inequalities can be obtained by considering any neighboring simplices having a common face. Thus the claim is true.

As C_1 is the collection of the simplices $T_{1c}(i,j,k), (i,j,k) \in \mathbb{Z}^3,\ c = 1,2,...,6$ and f is convex on any two neighboring simplices, it is convex on the entire space. Thus p is strongly unimodal. □

Remark 1. If p is a probability function on $\{0,1\}^3$, then the conditions obtained from condition (a) of Theorem 1 for p to be strongly unimodal are as follows:

(i) $p_{101}p_{011} \le p_{001}p_{111}$,

(ii) $p_{100}p_{001} \le p_{000}p_{101}$,

(iii) $p_{001}p_{010} \le p_{000}p_{011}$,

(iv) $p_{011}p_{110} \le p_{010}p_{111}$,

(v) $p_{100}p_{010} \le p_{000}p_{110}$,

(vi) $p_{101}p_{110} \le p_{100}p_{111}$.

We can obtain similar conditions from (b), (c), (d), (e) and (f) of Theorem 1.

Remark 2. A function $f : X = X_1 \times X_2 \times ... \times X_n \to [0,\infty)$ is said to be *multivariate totally positive of order* 2, MTP_2, if for all $\mathbf{x},\ \mathbf{y} \in X$.

$$f(\mathbf{x} \vee \mathbf{y})f(\mathbf{x} \wedge \mathbf{y}) \ge f(\mathbf{x})f(\mathbf{y}),$$

where

$$\mathbf{x} \vee \mathbf{y} = (\max(x_1,y_1), \max(x_2,y_2), ... \max(x_n,y_n)),$$

$$\mathbf{x} \wedge \mathbf{y} = (\min(x_1,y_1), \min(x_2,y_2), ... \min(x_n,y_n)).$$

It is easy to see that if for all $a, b \in C_1$ we have $p_a p_b \le p_{a\vee b}p_{a\wedge b}$, i.e., if p is MTP_2 on C_1, then the conditions (1), ... , (6) in Theorem 1 are satisfied.

3 Logconcavity does not imply strong unimodality: A Counterexample in $\mathbb{Z}^3$

Let ξ_1 and ξ_2 be two discrete random variables with support set $\mathcal{S}$, where

$$\mathcal{S} = \{(0,0,0), (1,0,0), (0,1,0), (0,0,1)\}.$$

Let

$$Pr\{\xi_1 = (0,0,0)\} = p_1,\ Pr\{\xi_1 = (1,0,0)\} = q_1,$$
$$Pr\{\xi_1 = (0,1,0)\} = r_1,\ Pr\{\xi_1 = (0,0,1)\} = s_1,$$
$$Pr\{\xi_2 = (0,0,0)\} = p_2,\ Pr\{\xi_2 = (1,0,0)\} = q_2,$$
$$Pr\{\xi_2 = (0,1,0)\} = r_2,\ Pr\{\xi_2 = (0,0,1)\} = s_2,$$

and all other probabilities equal to 0. We consider the convolution $\xi_1 + \xi_2$. Let p be the probability function of $\xi_1 + \xi_2$. The random variable $\xi_1 + \xi_2$ has the support set:

$$\tilde{\mathcal{S}} = \{(0,0,0), (1,0,0), (0,1,0), (0,0,1), (1,1,0), (1,0,1), (0,1,1), (2,0,0), (0,2,0), (0,0,2)\}$$

and the corresponding nonzero probabilities are:

$$
\begin{aligned}
Pr\{\xi_1 + \xi_2 = (0,0,0)\} &= p_1 p_2, \\
Pr\{\xi_1 + \xi_2 = (1,0,0)\} &= p_1 q_2 + p_2 q_1, \\
Pr\{\xi_1 + \xi_2 = (0,1,0)\} &= p_1 r_2 + p_2 r_1, \\
Pr\{\xi_1 + \xi_2 = (0,0,1)\} &= p_1 s_2 + p_2 s_1, \\
Pr\{\xi_1 + \xi_2 = (1,1,0)\} &= q_1 r_2 + q_2 r_1, \\
Pr\{\xi_1 + \xi_2 = (1,0,1)\} &= q_1 s_2 + q_2 s_1, \\
Pr\{\xi_1 + \xi_2 = (0,1,1)\} &= r_1 s_2 + r_2 s_1, \\
Pr\{\xi_1 + \xi_2 = (2,0,0)\} &= q_1 q_2, \\
Pr\{\xi_1 + \xi_2 = (0,2,0)\} &= r_1 r_2, \\
Pr\{\xi_1 + \xi_2 = (0,0,2)\} &= s_1 s_2.
\end{aligned}
$$

We show that the probability function p is logconcave. Let $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \tilde{\mathcal{S}}$, $\mathbf{z} = \lambda \mathbf{x} + (1-\lambda)\mathbf{y} \in \tilde{\mathcal{S}}$, $0 < \lambda < 1$. We show that

$$p(\mathbf{z}) \geq p(\mathbf{x})^{\lambda} p(\mathbf{y})^{1-\lambda}.$$

If $\mathbf{x} = (0,0,0)$, $\mathbf{y} = (0,1,0)$, $\lambda = 1/2$ then $\mathbf{z} = (0,2,0)$ and we have:

$$p(0,0,0) = p_1 p_2, \quad p(0,1,0) = p_1 r_2 + p_2 r_1, \quad p(0,2,0) = r_1 r_2.$$

It follows that

$$
\begin{aligned}
p(0,1,0)^2 - p(0,0,0)p(0,2,0) &= (p_1 r_2 + p_2 r_1)^2 - p_1 p_2 r_1 r_2 \\
&= p_1^2 r_2^2 + p_2^2 r_1^2 + p_1 p_2 r_1 r_2 \geq 0.
\end{aligned}
$$

The logconcavity of p for other points in $\tilde{\mathcal{S}}$ can be shown similarly. Thus p is logconcave.

The probability function p does not satisfy any of the conditions presented in Theorem 1. We show it in connection with condition (a). The others can be handled similarly. Let us recall the inequality (4) of condition (a):

$$p_{i+1j+1k} p_{i+1jk+1} \leq p_{i+1jk} p_{i+1j+1k+1}.$$

If we take $(i+1, j+1, k) = (1,1,0)$, then we obtain $(i+1, j, k+1) = (1,0,1)$, $(i+1, j, k) = (1,0,0)$ and $(i+1, j+1, k+1) = (1,1,1)$. Therefore, the value on the right hand side of inequality (4) is equal to

$$p_{i+1jk} p_{i+1j+1k+1} = p_{100} p_{111} = 0$$

and the value on the left hand side of inequality (4) is equal to

$$p_{i+1j+1k} p_{i+1jk+1} = p_{110} p_{101} = (q_1 r_2 + q_2 r_1)(q_1 s_2 + q_2 s_1).$$

Thus, (4) becomes

$$(q_1 r_2 + q_2 r_1)(q_1 s_2 + q_2 s_1) \leq 0.$$

This is, however, a contradiction, since all q_j, r_j and s_j probabilities are positive. Therefore p does not satisfy (4) and condition (a) is not satisfied. Thus p is not strongly unimodal.

4 A Sufficient Condition for a Discrete Distribution on $\mathbb{Z}^n$ to be Strongly Unimodal

In this section we give a sufficient condition for a discrete probability function defined on $\mathbb{Z}^n$ that ensures its strong unimodality. The function f defined on $\mathbb{R}^n$ that we fit to the values of $-\log p(.)$ is piecewise linear. In view of this we need a subdivision of $\mathbb{R}^n$ into non-overlapping convex polyhedra such that $f(\mathbf{x}) = -\log p(\mathbf{x}),\ \mathbf{x} \in \mathbb{Z}^n$ is linear on each of them. We consider one special subdivision of $\mathbb{R}^n$ into simplices and give a sufficient condition for a discrete function on $\mathbb{Z}^n$ to be strongly unimodal. Let $\mathcal{S}_1, ..., \mathcal{S}_{n!}$ designate the subdividing simplices of $\mathbb{R}^n$ with disjoint interiors defined as follows:

$$\mathcal{S}_1 = \{(i_1, i_2, ..., i_n), (i_1+1, i_2, ..., i_n), (i_1+1, i_2+1, i_3, ..., i_n), ..., (i_1+1, ..., i_n+1)\},$$

$$\mathcal{S}_2 = \{(i_1, i_2, ..., i_n), (i_1+1, i_2, ..., i_n), (i_1+1, i_2, i_3+1, ..., i_n), ..., (i_1+1, ..., i_n+1)\},$$

$$\vdots$$

$$\mathcal{S}_n = \{(i_1, i_2, ..., i_n), (i_1+1, i_2, ..., i_n), (i_1+1, i_2, ..., i_{n-1}, i_n+1), ..., (i_1+1, ..., i_n+1)\},$$

$$\mathcal{S}_{n+1} = \{(i_1, i_2, ..., i_n), (i_1, i_2+1, ..., i_n), (i_1+1, i_2+1, i_3, ..., i_n), ..., (i_1+1, ..., i_n+1)\},$$

$$\vdots$$

$$\mathcal{S}_{n!} = \{(i_1, i_2, ..., i_n), (i_1, ..., i_{n-1}, i_n+1), (i_1+1, i_2, ..., i_{n-1}, i_n+1), ..., (i_1+1, ..., i_n+1)\}.$$

Note that $|\mathcal{S}_1| = ... = |\mathcal{S}_{n!}| = n+1$ and $\mathcal{S}_i$ and $\mathcal{S}_j$ have a common facet if they have n common vertices.

The sufficiency condition for an n-dimensional discrete probability function to be strongly unimodal is given by the use of any two neighboring simplices with one common facet.

Let p be the probability function of a discrete distribution on $\mathbb{Z}^n$ and $p(i_1, i_2, ..., i_n)$ the value of p at $(i_1, i_2, ..., i_n) \in \mathbb{Z}^n$.

Let C denote the collection of all simplices $\mathcal{S}_1, ..., \mathcal{S}_{n!}$, vertices of which are lattice points.

Theorem 2. *Suppose that p satisfies the following conditions for all $(i_1, i_2, ..., i_n) \in \mathbb{Z}^n$:*

$$p(i_1+\varepsilon_1, ..., i_n+\varepsilon_n)p(i_1+\delta_1, ...i_n+\delta_n) \le$$

$$p(\min\{i_1+\varepsilon_1, i_1+\delta_1\}, ..., \min\{i_n+\varepsilon_n, i_n+\delta_n\})p(\max\{i_1+\varepsilon_1, i_1+\delta_1\}, ..., \max\{i_n+\varepsilon_n, i_n+\delta_n\}) \tag{4.1}$$

where

$$\varepsilon_1 + ... + \varepsilon_n = n-1$$
$$\sum_{j=1}^n \varepsilon_j\delta_j = n-2, \quad n \ge 2$$
$$\varepsilon_j, \delta_j \in \{0,1\}, \quad j = 1, ..., n$$

and

$$\begin{array}{r}
p(i_1-1,i_2,...,i_n)p(i_1+1,...,i_n+1) \le p(i_1,...,i_n)p(i_1,i_2+1,...,i_n+1)\\
\vdots\\
p(i_1,i_2,...,i_n-1)p(i_1+1,...,i_n+1) \le p(i_1,...,i_n)p(i_1+1,...,i_{n-1}+1,i_n)\\
p(i_1+2,i_2+1,...,i_n+1)p(i_1,...,i_n) \le p(i_1+1,i_2,...,i_n)p(i_1+1,...,i_n+1)\\
\vdots\\
p(i_1+1,...,i_{n-1}+1,i_n+2)p(i_1,...,i_n) \le p(i_1,...,i_{n-1},i_n+1)p(i_1+1,...,i_n+1)
\end{array} \tag{4.2}$$

Then p is strongly unimodal.

Proof. We use the similar idea which is used in the proof of Theorem 1.

Let $L(c,i_1,i_2,...,i_n),(i_1,i_2,...,i_n) \in \mathbb{Z}^n,\ c = 1,2,...,n!$ denote the linear function on $\mathbb{R}^n$ which coincides on the vertices of C with $-\log p(.)$ and define

$$f(\mathbf{y}) = \begin{cases} L(c,i_1,i_2,...,i_n) & \text{if } \mathbf{y} \in \mathcal{S}_c(i_1,i_2,...,i_n),(i_1,i_2,...,i_n) \in \mathbb{Z}^n \\ \infty & \text{if } \mathbf{y} \notin C \end{cases}$$

It is easy to see that f coincides on $\mathbb{Z}^n$ with $-\log p(.)$.

Claim:Conditions given in Theorem 2 ensure that the restriction of f to any two neighboring simplices $\mathcal{S}_i$ and $\mathcal{S}_j$ with a common facet is convex.

Proof of the claim: On each simplex we define linear piece by the equation of the hyperplane determined by the vertices of the simplex and the corresponding values of $-\log p(.)$. Similar to three-dimensional case the collection of these linear pieces form the function f. We also need to show that the function f is convex on any two neighboring simplices $\mathcal{S}_i$ and $\mathcal{S}_j$.

For the sake of simplicity we consider $\mathcal{S}_1$ and $\mathcal{S}_2$. Let

$$\mathbf{z}_0 = \begin{pmatrix} z_{01} \\ \vdots \\ z_{0n} \end{pmatrix}, \quad \mathbf{z}_1 = \begin{pmatrix} z_{11} \\ \vdots \\ z_{1n} \end{pmatrix}, ..., \quad \mathbf{z}_n = \begin{pmatrix} z_{n1} \\ \vdots \\ z_{nn} \end{pmatrix}, \quad \mathbf{y} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$$

and assume that $\mathbf{z}_0,\mathbf{z}_1,...,\mathbf{z}_n \in \mathcal{S}_1$ and $\mathbf{y} \in \mathcal{S}_2$. The function f is convex on these two neighboring simplices if the following condition is satisfied for any $\mathbf{z}_0,\mathbf{z}_1,...,\mathbf{z}_n \in \mathcal{S}_1$ and $\mathbf{y} \in \mathcal{S}_2$:

$$\frac{\begin{vmatrix} f(\mathbf{y}) & f(\mathbf{z}_0) & f(\mathbf{z}_1) & f(\mathbf{z}_2) & \dots & f(\mathbf{z}_n) \\ 1 & 1 & 1 & 1 & \dots & 1 \\ y_1 & z_{01} & z_{11} & z_{21} & \dots & z_{n1} \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ y_n & z_{0n} & z_{1n} & z_{2n} & \dots & z_{nn} \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 1 & \dots & 1 \\ z_{01} & z_{11} & z_{21} & \dots & z_{n1} \\ \vdots & \vdots & \vdots & & \vdots \\ z_{0n} & z_{1n} & z_{2n} & \dots & z_{nn} \end{vmatrix}} \ge 0 \tag{4.3}$$

where $f = -\log p(.)$.

Let us take

$$\mathbf{z}_0 = (i_1, i_2, ..., i_n),\ \mathbf{z}_1 = (i_1{+}1, i_2, ..., i_n),\ \mathbf{z}_2 = (i_1{+}1, i_2{+}1, i_3, ..., i_n), ..., \mathbf{z}_n = (i_1{+}1, i_2{+}1, ..., i_n{+}1)$$

and $\mathbf{y} = (i_1+1, i_2, i_3+1, i_4, ..., i_n)$. Then (4.3) can be written as

$$\frac{\begin{vmatrix} f(\mathbf{y}) & f(\mathbf{z}_0) & f(\mathbf{z}_1) & f(\mathbf{z}_2) & \dots & f(\mathbf{z}_n) \\ 1 & 1 & 1 & 1 & \dots & 1 \\ i_1+1 & i_1 & i_1+1 & i_1+1 & \dots & i_1+1 \\ i_2 & i_2 & i_2 & i_2+1 & \dots & i_2+1 \\ i_3+1 & i_3 & i_3 & i_3 & \dots & i_3+1 \\ i_4 & i_4 & i_4 & i_4 & \dots & i_4+1 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ i_n & i_n & i_n & i_n & \dots & i_n+1 \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 1 & \dots & 1 \\ i_1 & i_1+1 & i_1+1 & \dots & i_1+1 \\ i_2 & i_2 & i_2+1 & \dots & i_2+1 \\ i_3 & i_3 & i_3 & \dots & i_3+1 \\ i_4 & i_4 & i_4 & \dots & i_4+1 \\ \vdots & \vdots & \vdots & & \vdots \\ i_n & i_n & i_n & \dots & i_n+1 \end{vmatrix}} \geq 0 \tag{4.4}$$

where $f = -\log p(.)$.

It is easy to show that the determinant in the denominator in (4.4) is equal to 1. Therefore the convexity of the function f is ensured if the determinant in the numerator of (4.4) is nonnegative, i.e.,

$$\begin{vmatrix} f(\mathbf{y}) & f(\mathbf{z}_0) & f(\mathbf{z}_1) & f(\mathbf{z}_2) & \dots & f(\mathbf{z}_n) \\ 1 & 1 & 1 & 1 & \dots & 1 \\ i_1+1 & i_1 & i_1+1 & i_1+1 & \dots & i_1+1 \\ i_2 & i_2 & i_2 & i_2+1 & \dots & i_2+1 \\ i_3+1 & i_3 & i_3 & i_3 & \dots & i_3+1 \\ i_4 & i_4 & i_4 & i_4 & \dots & i_4+1 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ i_n & i_n & i_n & i_n & \dots & i_n+1 \end{vmatrix} \geq 0.$$

One can easily show that this is the same as

$$\begin{vmatrix} f(\mathbf{y})-f(\mathbf{z}_0) & f(\mathbf{z}_0) & f(\mathbf{z}_1)-f(\mathbf{z}_0) & f(\mathbf{z}_2)-f(\mathbf{z}_1) & \dots & f(\mathbf{z}_n)-f(\mathbf{z}_n-1) \\ 0 & 1 & 0 & 0 & \dots & 0 \\ 1 & i_1 & 1 & 0 & \dots & 0 \\ 0 & i_2 & 0 & 1 & \dots & 0 \\ 1 & i_3 & 0 & 0 & \dots & 0 \\ 0 & i_4 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & i_n & 0 & 0 & \dots & 1 \end{vmatrix} \geq 0.$$

From the inequality given above we obtain $f(\mathbf{y})+f(\mathbf{z}_2) \geq f(\mathbf{z}_1)+f(\mathbf{z}_3)$. This is equivalent to $p(\mathbf{y})p(\mathbf{z}_2) \leq p(\mathbf{z}_1)p(\mathbf{z}_3)$ or

$$p(i_1{+}1, i_2, i_3{+}1, i_4, ..., i_n)p(i_1{+}1, i_2{+}1, i_3, ..., i_n) \leq p(i_1{+}1, i_2, ..., i_n)p(i_1{+}1, i_2{+}1, i_3{+}1, i_4, ..., i_n).$$

So we have obtained one of the inequalities in (4.1). All other inequalities can be obtained in a similar way. Therefore we see that inequalities (4.1) and (4.2) ensure the convexity of f on any two neighboring simplices.

As C is the collection of $\mathcal{S}_1, ..., \mathcal{S}_{n!}$ and f is convex on any two neighboring simplices $\mathcal{S}_i$ and $\mathcal{S}_j$, it is convex on the entire space. Thus p is strongly unimodal. □

5 Examples

The properties of the distributions presented in the examples of this section can be found in Johnson, Kotz and Balakrishnan (1997).

Example 1. The negative multinomial distribution has the following probability function:

$$p(x_1, x_2, ..., x_n) = \frac{(k-1+\sum_{i=1}^n x_i)!}{(k-1)!} p_0^k \prod_{i=1}^n \frac{p_i^{x_i}}{x_i!}$$
$$x_i = 0, 1, 2, ... \quad i = 1, 2, ..., n$$
$$\sum_{i=0}^n p_i = 1,\ 0 < p_i < 1,\ i = 1, 2, .., n.$$

Note that conditions (4.1) are equivalent to the MTP_2 conditions. This can be shown in a similar way which is used in Section II for the sufficiency conditions for a discrete probability function on $\mathbb{Z}^3$ to be strongly unimodal. Since the negative mulinomial distribution is MTP_2 (Karlin and Rinott, 1980), p satisfies the conditions (4.1) of Theorem 2.

The negative multinomial distribution is strongly unimodal if

$$n-1-2x_1+x_2+...+x_n \geq 0,$$
$$\vdots$$
$$n-1+x_1+x_2+...+x_{n-1}-2x_n \geq 0.$$

Example 2. The multivariate hypergeometric distribution has the following probability function:

$$p(x_1, x_2, ..., x_{n-1}) = \frac{\prod_{i=1}^{n-1} \binom{m_i}{x_i} \binom{m - m_1 - ... - m_{n-1}}{k - x_1 - ... - x_{n-1}}}{\binom{m}{k}}$$

$$0 \le x_i \le m_i, \quad i = 1, 2, ..., n-1$$

$$\sum_{i=1}^{n-1} x_i \le k, \quad \sum_{i=1}^{n-1} m_i \le m.$$

One can easily show that p satisfies conditions (4.1) and (4.2) of Theorem 2. Thus, the multivariate hypergeometric distribution is strongly unimodal.

Example 3. The multivariate negative hypergeometric distribution has probability function:

$$p(x_1, x_2, ..., x_n) = \frac{k!\Gamma(m)\Gamma(m - m_1 - ... - m_{n-1} + k - x_1 - ... - x_{n-1})}{\Gamma(k+m)\Gamma(m - m_1 - ... - m_{n-1})(k - x_1 - ... - x_{n-1})!} \prod_{i=1}^{n-1} \frac{\Gamma(m_i + x_i)}{\Gamma(m_i) x_i!}$$

$$0 \le x_i \le m_i, \quad i = 1, 2, ..., n-1$$

$$\sum_{i=1}^{n-1} x_i \le k, \quad \sum_{i=1}^{n-1} m_i \le m.$$

Since p satisfies conditions (4.1) and (4.2) of Theorem 2, it is strongly unimodal.

Example 4. Consider the Dirichlet (or Beta)-compound multinomial distribution

$$\text{Multinomial}(k; p_1, ..., p_{n-1}) \bigwedge_{p_1,...,p_{n-1}} \text{Dirichlet}(\alpha_1, ..., \alpha_{n-1})$$

The probability mass function of this compound distribution is:

$$p(x_1, x_2, ..., x_{n-1}) = \frac{k!\Gamma(\alpha)\Gamma(\alpha_n + k - x_1 - ... - x_{n-1})}{\Gamma(k+\alpha)\Gamma(\alpha_n)(k - x_1 - ... - x_{n-1})!} \prod_{i=1}^{n-1} \frac{\Gamma(\alpha_i + x_i)}{\Gamma(\alpha_i) x_i!}.$$

$$\alpha_n = \alpha - \sum_{i=1}^{n-1} \alpha_i, \qquad \sum_{i=1}^{n-1} x_i \le k, \qquad x_i \ge 0.$$

The function p satisfies conditions (4.1) and (4.2) of Theorem 2. Thus, it is strongly unimodal.

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