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Abstract. In this paper we describe some structural properties of {claw, net}-free graphs. Using these properties, we find (among other results) necessary and sufficient conditions for a {claw, net}-free graph G with $s, t \in V(G)$, $s \neq t$, $e \in E(G)$, and disjoint paths S and T to have:

($c1$) a Hamiltonian st–path,
($c2$) a Hamiltonian s–path containing e,
($c3$) a Hamiltonian st–path containing e if G is 3–connected,
($c4$) a Hamiltonian cycle containing S if G is k–connected, $k \geq 2$, and $v(S) \leq k$,
($c5$) a Hamiltonian path containing e and S if G is $(k+1)$–connected, $k \geq 2$, and $v(S) \leq k$, and
($c6$) a Hamiltonian st–path containing S and T if G is k–connected, $k \geq 3$, $v(S) + v(T) \leq k$, and s, t are end–vertices of paths S and T, respectively. *Let G be a 3–connected {claw, net}-free graph, $\{s, t\} \subseteq V(G)$, $uv \in E(G - s)$, and $s \neq t$. Then G has Hamiltonian st–path in G containing uv if and only if $t \in \{u, v\}$ and $G - \{s, u, v\}$ is not connected. Therefore if G is 4–connected, then G has Hamiltonian st–path in G containing uv. Also if G is 4–connected, then every two edges in G belong to a Hamiltonian cycle.*

Keywords: claw, net, graph, {claw, net}-free graph, Hamiltonian cycle, Hamiltonian path, trace, polynomial-time algorithm.

1 Introduction

We consider simple undirected graphs. All notions on graphs that are not defined here can be found in [2, 11].

A graph G is called *H–free* if G has no induced subgraph isomorphic to a graph H. A *claw* is a graph having exactly four vertices and three edges that are incident to a (the) common vertex. A claw can be drawn as letter Y. A *net* is a graph obtained from a triangle by attaching to each vertex a new dangling edge. A graph G is called *{claw, net}-free* if G has no induced subgraph isomorphic to either a claw or a net.

There are many papers devoted to the study of Hamiltonicity of claw-free graphs, and, in particular, {claw, net}-free graphs (e.g. [1, 3, 4, 6, 7, 8, 9, 10]). The maximum independent vertex set problem for {claw, net}-free graphs was studied in [5].

In this paper we describe some structural properties of {claw, net}-free graphs. Using these properties, we establish some new Hamiltonicity results on {claw, net}-free graphs.

An *st–path* is a path with the end–vertices s and t. An *s–path* is a path with an end–vertex s. Let G be a {claw, net}-free graph, $\{s,t\} \subseteq V(G)$, $s \neq t$, $e = uv \in E(G)$, and S and T disjoint paths in G, where $V(S) \cap \{u,v\} = \emptyset$. We give necessary and sufficient conditions for G to have:

($c1$) a Hamiltonian st–path (see **3.6** and **6.7** below)'

($c2$) a Hamiltonian s–path containing e (**3.8** and **5.3**),

($c3$) a Hamiltonian st–path containing e if G is 3–connected (**7.5**),

($c4$) a Hamiltonian cycle containing S if G is k–connected, $k \geq 2$, $v(S) \leq k$ (**8.1**),

($c5$) a Hamiltonian path containing e and S if G is $(k+1)$–connected, $k \geq 2$, and $v(S) \leq k$ (**8.4**), and

($c6$) a Hamiltonian st–path containing S and T if G is k–connected, $k \geq 3$, $v(S) + v(T) \leq k$, and s, t are end–vertices of paths S and T, respectively (**8.9**).

¿From the above mentioned results we obtain the following corollaries.

1.1. (Corollary of **7.5** and strengthening of **3.3** [10]) *Let G be a 3–connected {claw, net}-free graph. Then for every two different vertices s, t and every edge e in G incident to no vertex in $\{s,t\}$ there is a Hamiltonian st–path in G containing e.*

1.2. (Corollary of **7.5**) *Let G be a {claw, net}-free graph. If G is 3–connected, then every two non-adjacent edges in G belong to a Hamiltonian cycle. If G is 4–connected, then every two edges in G belong to a Hamiltonian cycle.*

1.3. (Corollary of **8.7**) *Let G be a k–connected {claw, net}-free graph, $k \geq 5$, C be a cycle in G, $v(C) \leq k-2$, $s,t \in V(G)$, $s \neq t$, and $V(C) \cap \{s,t\} = \emptyset$. Then for every $e \in E(C)$ there is a Hamiltonian st–path in G containing $C - e$.*

1.4. (Corollary of **8.7**) *Let G be a k–connected {claw, net}-free graph, $k \geq 5$, C be a cycle in G, $v(C) \leq k-2$, $s,t \in V(G)$, $s \neq t$, and $V(C) \cap \{s,t\} = \emptyset$. Then for every $e \in E(C)$ there is a Hamiltonian st–path in G containing $C - e$ (and so, avoiding e).*

1.5. (Corollary of **8.9**) *Let G be a $(k+1)$–connected {claw, net}-free graph, $k \geq 2$, S and T two disjoint paths in G, s and t end–vertices of paths S and T, respectively, and $v(S) + v(T) \leq k$. Then G has a Hamiltonian st–path containing S and T.*

1.6. (Corollary of **8.1**) *In a k–connected {claw, net}-free graph, $k \geq 2$, every $(k-1)$-vertex path belongs to a Hamiltonian cycle.*

1.7. (see **8.6**) *Let G be a k–connected {claw, net}-free graph, $k \geq 2$, $e = uv$ an edge and L an xy–path in G, $v(L) \leq k-1$, and $\{u, v\} \cap V(L) \subset \{x, y\}$. Then G has a Hamiltonian path containing e and L, and, in particular, every k-vertex path belongs to a Hamiltonian path in G.*

Our proofs provide polynomial-time algorithms for solving the corresponding Hamiltonicity problems. In [1] a linear-time algorithm was given for finding a Hamiltonian path and a Hamiltonian cycle (if any) in a {claw, net}-free graph.

Some results of this paper were presented at the Discrete Mathematics Seminar at the University of Puerto Rico in November 1999 (see also [7, 8]).

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2 Main notions and notation

Let $V(G)$ and $E(G)$ denote the sets of vertices and edges of a graph G, respectively, $v(G) = |V(G)|$ and $e(G) = |E(G)|$.

Given a vertex x and a subgraph A of G, let $N_G(x, A)$ (or simply, $N(x, A)$) denote the set of vertices in A adjacent in G to a vertex x and let $\dot{N}_G(x, A)$ denote the subgraph of G induced by $N_G(x, A)$. Let $N_G(x, G) = N(x, G)$ and $d(x, G) = |N(x, G)|$.

Let $\kappa(G)$ denote the *vertex connectivity of a graph* G, i.e. $\kappa(G) = v(G) - 1$, if G is a complete graph, and $\kappa(G)$ is the size of a minimum vertex cut of G, otherwise. A graph G is called *k–connected* if $\kappa(G) \geq k$.

An *st–path* (*s–path*) is a path with the end–vertices s and t (an end-vertex s, respectively). If a and b are vertices of P, then aPb denote the subpath of P with the end-vertices a and b.

A cycle (a path) C of G is called *Hamiltonian*, if $V(C) = V(G)$. A Hamiltonian path of G is also called a *trace* of G. We introduce the term *track* of G for a Hamiltonian cycle of G.

Let $X, Y \subseteq V(G)$. We say that G is *Hamiltonian (X, Y)–connected* if G has a Hamiltonian xy–path for every distinct vertices $x \in X$ and $y \in Y$. If G is Hamiltonian $(V(G), V(G))$–connected, then G is called *Hamiltonian connected.*

Let H be a subgraph of G. A vertex x of H is called an *inner vertex of H* if x is adjacent to no vertices in $G - H$, and a *boundary vertex of H*, otherwise. Let $Bnd(H)$ and $In(H)$ denote the set of boundary and inner vertices of H, respectively.

A *block* of G is either an isolated vertex or a maximal connected subgraph H of G such that $H - v$ is connected for every $v \in V(H)$.

A block B of G is called an *end–block* of G if B has exactly one boundary vertex, and an *inner block*, otherwise.

We write $G = AXB$ if A and B are induced subgraphs of G, $V(A) \cap V(B) = X$, $E(A) \cap E(B) = \emptyset$, and so $E(A) \cup E(B) = E(G)$. If $X = \{x\}$, we write simply AxB instead of AXB. If B is an edge xy, we write simply Axy or yxA instead of AxB.

Two disjoint vertex subsets or subgraphs A and B of a graph G are called *adjacent* if there is $ab \in E(G)$ such that $a \in A$ and $b \in B$.

A complete subgraph of a graph G is also called a *clique* of G.

3 Some known results and preliminaries

3.1. [3] (Corollary of **3.5** below) *Every connected {claw, net}-free graph has a trace.*

3.2. [3] (Corollary of **3.9**) *Every 2–connected {claw, net}-free graph has a Hamiltonian cycle.*

3.3. [10] (Corollary of **6.7** and of **7.5**) *Every 3–connected {claw, net}-free graph is Hamiltonian connected.*

The following result in [8] allowed us to give short proofs of **3.1** and **3.2** and also to find some new Hamiltonicity results on {claw, net}-free graphs.

3.4. [8] *Let G be a connected {claw, net}-free graph and $z \in V(G)$. Suppose that $G - z$ has an xy–trace P and there exists $e_z = zp \in E(G)$ (and so G is connected). Let e_x and e_y be the end-edges of P. Then G has an ab–trace Q such that $\{a, b\} \subset \{x, y, z\}$, $e_z \in E(Q)$ and $\{e_x, e_y\} \cap E(Q) \neq \emptyset$.*

¿From **3.4** we have immediately the following strengthening of **3.1**.

3.5. [8] *Let G be a connected {claw, net}-free graph. Then*

$(a1)$ *G has a trace and*

$(a2)$ *if $uv \in E(G)$ and $G - v$ is connected, then uv belongs to a trace of G.*

Let $\mathcal{L}$ denote the set of 4-tuples $(G; s, t, uv)$ such that G is a graph, $\{s, t\} \subseteq V(G)$, $s \neq t$, $uv \in E(G)$, and one of the following holds:
$(l1)$ $\{s, t\}$ avoids one of the components of $G - \{u, v\}$,
$(l2)$ $\{s, t\} \cap \{u, v\} \neq \emptyset$, say $t = u$, and either $G - \{s, v\}$ is not connected and the component containing t has at least two vertices or there is $x \in V(G - \{u, v\})$ such that $\{s, v\}$ avoids one of the components of $G - \{t, x\}$.

Obviously if G has an st–trace containing uv, then $(G; s, t, uv) \notin \mathcal{L}$.

By using **3.4**, we proved the following strengthening of **3.5**.

3.6. [8] *Let G be a connected {claw, net}-free graph having $n+2 \geq 2$ blocks. Let A_j, $j = 1, 2$, be end–blocks of G, a_j the boundary vertex of A_j, $a'_j \in A_j - a_j$, and $\alpha_j \in E(A_j)$. Let B_i be an inner block of G and $\beta_i \in E(B_i)$. Let $U = \{\alpha_1, \alpha_2\} \cup \{\beta_i : i = 1, \ldots, n\}$. Then G has an $a'_1a'_2$–trace containing U if and only if*
(*c*1) $(A_j; a_j, a'_j, \alpha_j) \notin \mathcal{L}$ *for every $j \in \{1, 2\}$ and*
(*c*2) *if $v(B_i) \geq 3$ then β_i is an edge incident to an inner vertex in B_i, $i \in \{1, \ldots, n\}$.*

¿From **3.6**, we have, in particular:

3.7. [8] *Let G be a {claw, net}-free graph, $\kappa(G) = 1$, $s, t \in V(G)$, and $e \in E(G)$. Let A and B be the end–blocks of G with the boundary vertices a and b, respectively. Then*

(*c*1) *G has an st–trace if and only if s and t are inner vertices of different end–blocks of G, and so G is Hamiltonian $(V(A - a), V(B - b))$–connected, and*

(*c*2) *G has an st–trace containing e if and only if s and t are inner vertices of different end–blocks of G and $(G; s, t, e) \notin \mathcal{L}$.*

¿From **3.7** we have:

3.8. [8] *Let G be a {claw, net}-free graph, $\kappa(G) = 1$, $s \in V(G)$, and $e \in E(G)$. Then G has an s–trace containing e if and only if s is an inner vertex of an end-block in G and $(G; b, s, e) \notin \mathcal{L}$, where b is the boundary vertex of the end-block avoiding s.*

By using **3.4**, we proved the following strengthening of **3.2**.

3.9. [8] *Let G be a 2–connected {claw, net}-free graph and $uv \in E(G)$. Then*
(*a*1) *the following are equivalent:*
  (*c*1) *uv belongs to a track of G and*
  (*c*2) *$G - \{u, v\}$ is connected, and*
(*a*2) *if $G - \{u, v\}$ is not connected, then for every inner vertices s, t of two different end–blocks S, T of $G - v$ there is an st–trace of G containing uv.*

Obviously **3.9** is a strengthening of **3.2** because every 2–connected graph G has an edge uv such that $G - \{u, v\}$ is connected.

¿From **3.9** we have, in particular:

3.10. [8] *Let G be a 2–connected {claw, net}-free graph. Then every edge in G belongs to a trace of G.*

We will use the following corollary of **3.6**.

3.11. [8] *Let G be a 2–connected graph and $x'xGyy'$ a {claw, net}-free graph, and so $x \neq y$. Let $uv \in E(G)$ and $\{u, v\} \neq \{x, y\}$. Then $x'xGyy'$ has an $x'y'$–trace containing uv.*

Proof (uses **3.1** and **3.4**). Let $G' = G - v$. Since G is 2–connected, clearly G' is connected. By **3.1**, $x'xG'yy'$ has an $x'y'$–trace. By **3.4**, $x'xGyy'$ has an ab–trace containing uv, where $\{a, b\} \subset \{x, y, v\}$. If $\{a, b\} \neq \{x, y\}$, then clearly $x'xGyy'$ has no ab–trace. Therefore $x'xGyy'$ has an $x'y'$–trace containing uv. □

We need the following three simple facts.

3.12. *Let G be a 2–connected claw-free graph and $x \in V(G)$. Suppose that G is not a triangle and $\dot{N}(x, G)$ is a clique. Then there is an edge uv in $\dot{N}(x, G)$ such that $G - \{x, u, v\}$ is connected.*

Proof Since G is 2–connected, $|N(x, G)| \geq 2$.

(**p1**) Suppose that $|N(x, G)| \geq 3$, say $\{u, v, v'\} \subseteq N(x, G)$. Suppose that $G - \{u, v\}$ and $G - \{u, v'\}$ are not connected. Let C and C' be components of $G - \{u, v\}$ and $G - \{u, v'\}$, respectively, that do not contain x. Since G is 2–connected, there exist $c \in V(C)$ and $c' \in V(C')$ such that $cu, c'u \in E(G)$. Then $\{u; c, c', x\}$ induces a claw in G centered at u, a contradiction. Therefore we can assume that $G - \{u, v\}$ is connected. Since $N(x, G)$ induces in G a complete graph, $G - \{x, u, v\}$ is connected.

(**p2**) Now suppose that $|N(x, G)| = 2$, say $N(x, G) = \{u, v\}$. Since G is not a triangle, $G - \{u, v\} \neq \emptyset$. Suppose that $G - \{x, u, v\}$ is not connected. Then $G - \{u, v\}$ has at least three components. Since G is claw-free, $G - \{u, v\}$ has at most two components, a contradiction. □

3.13. *Let G be a 2–connected graph, Gzy a {claw, net}-free graph, $x \in V(G - z)$, $N(x, G)$ induce a clique in G, and $s \in V(G - \{x, z\})$. Then there is an edge pq in Q such that $G - x$ has an sy–trace containing pq, and so G has also an sz–trace containing pxq.*

Proof (uses **3.7** and **3.12**). If G is a triangle, then $V(G) = \{s, x, z\}$ and our claim is obviously true with $pq = sz$. So we assume that G is not a triangle. Then since G is 2–connected and $\dot{N}(x, G)$ is a clique, $G - x$ is 2–connected.

Suppose that $N(x, G - z)$ has exactly one vertex, say a. Since G is not a triangle, $G - \{z, a\}$ has a component C avoiding x, and so G has an edge cz, where $c \in V(C)$. Then $(z; c, x, y)$ induces a claw in Gzy centered at z, a contradiction.

Now suppose that $|N(x, G - z)| \geq 2$. Since $\dot{N}(x, G - z)$ is a clique in $G - z$, by **3.12**, there is an edge pq in $\dot{N}(x, G - z$ such that $G - \{x, pq\}$ is connected. By **3.7**, $G - x$ has an sz–trace P containing pq, say $P = sPpqPz$. Then $sPpxqPz$ is an sz–trace in G containing pxq. □

3.14. *Let H be a 2–connected graph, $\{x_1, x_2, z\}$ be a set of three different vertices in H, either H a triangle or $H - x_i$ is 2–connected for every $i \in \{1, 2\}$ and $x_1Hx_2ux_1$ {claw, net}-free. Then H has a zx_i–trace P_i for every $i \in \{1, 2\}$ such that if $x_1x_2 \in E(G)$, then $x_1x_2 \in E(P_i)$.*

Proof (uses **3.7**, and **3.12**). It is sufficient to prove our claim for $i = 1$. If H is a triangle, then our claim is obviously true. So we assume that H is not a triangle. Let $\{i, j\} = \{1, 2\}$, $G = x_1Hx_2ux_1$, $F_i = G - x_i$, and $H_i = H - x_i$. Then H_i is 2–connected and $F_i = ux_jH_i$. Since G is {claw, net}-free, each F_i is also {claw, net}-free, and so $\dot{N}(x_i, H_j) = Q_j$ is a clique. Since H_j is 2–connected, $|N(x_i, H_j)| \geq 2$.

Suppose that $x_1x_2 \notin E(G)$. Then $N(x_i, H_j) = N(x_i, H)$. Since G is claw-free and edge ux_i belongs to no triangle in G, clearly $G - ux_i = ux_jH$ is claw-free. By **3.12**, there is an edge st in Q_2 such that $H_2 - \{s, t\}$ is connected. Since $F_2 = ux_1H_2$ is {claw, net}-free and H_2 is 2–connected, by **3.7**, H_2 has a zx_1–trace P containing st, say $P = zPstPx_1$. Then $P = zPsx_2tPx_1$ is a zx_1–trace of H.

Now suppose that $x_1x_2 \in E(G)$. Since $F_1 = ux_2H_1$ is {claw, net}-free and H_1 is 2–connected, by **3.7**, H_1 has a zx_2–trace P. Then zPx_2x_1 is a zx_1–trace of G containing x_1x_2. □

4 On the structure of {claw, net}-free graphs

We start with the following simple observations.

4.1. *Let G be a connected claw-free graph and S a minimal vertex cut in G. Then $G - S$ has exactly two components.*

Proof Suppose, on the contrary, that A, B, and C are distinct components of $G - S$. Since S is a minimal vertex cut of G, a vertex s in S is adjacent to vertices, say a, b, and c, in A, B, and C, respectively. Then $\{s; a, b, c\}$ induces a claw in G centered at s, a contradiction. □

4.2. [8] *Let G be a graph. The following are equivalent:*

$(a1)$ *G is {claw, net}-free and*

$(a2)$ *G has no connected induced subgraph with at least three end–blocks.*

Proof Obviously $(a2) \Rightarrow (a1)$. We prove $(a1) \Rightarrow (a2)$. Let H be an induced subgraph of G with at least three end–blocks. We want to show by induction on $v(H)$ that H (and therefore also G) has either an induced claw or an induced net.

Suppose that H has an end-block B with $v(B) \geq 2$ and with the boundary vertex b. Let b' be a vertex in b adjacent to b and $H' = (H - (B - \{b, b'\})$. Then H' is an induced subgraph of G and $v(H') < v(H)$, and therefore we are done by induction.

Now suppose that every end-block of H has exactly two vertices. If H is not a tree, then there is $x \in V(H)$ such that $H - x$ is a connected graph with at least three end-blocks and again we are done by induction. If H is a tree, then since H has at least three leaves, H has an induced claw. □

A connected graph G is called a *chain* if G has either exactly one edge or exactly two end–blocks. If G is a chain and $e(G) \geq 2$, then $G = B_1x_1B_2 \dots x_{n-1}B_n$, where $n \geq 2$, each B_i is a block of G (and so B_i is either 2–connected or has one edge), $\{x_1, \dots, x_n\}$ are distinct vertices, and each $x_i = V(B_{i-1}) \cap V(B_i)$. A *brick* of a chain is a block with specified boundary vertices. Both $\ddot{B}_1 = B_1 x_1$ and $\ddot{B}_n = x_{n-1} B_k$ are *end-bricks* and each $\ddot{B}_i = x_{i-1} B_i x_i$, $2 \le i \le n-1$, is an *inner brick* of chain G. Let $\bar{B}_i = B_i \cup x_{i-1}x_i$ for $2 \le i \le n-1$ and $\bar{B}_j = B_j$ for $j \in \{1, k\}$. We say that a brick $\ddot{B}_i$ is *formed by* B_i and is *k–connected* if $\bar{B}_i$ is k–connected.

Obviously from **4.2** we have:

4.3. *Let G be a {claw, net}-free graph and $\kappa(G) = 1$. Then G is a chain.*

4.4. *Let H be a 2–connected graph non-isomorphic to a triangle and S a minimal vertex cut of H. Let a graph G be obtained from H by adding two new vertices x_1 and x_2, and two new edges x_1h_1 and x_2h_2, where each $h_i \in V(H)$ and $h_1 \ne h_2$. Suppose that G is {claw, net}-free. Then no component of $H - S$ avoids $\{h_1, h_2\}$, and so in particular, $H \cup h_1h_2$ is 3–connected.*

Proof (uses **4.1** and **4.2**). Suppose, on the contrary, there is a component C of $H - S$ such that $V(C) \cap \{h_1, h_2\} = \emptyset$, and so C is also a component of $G - S$. Since G is claw-free, $N(h_i, H)$ induces in H a clique and by **4.1**, $G - S$ has exactly two components. Let C' be the component of $G - S$ distinct from C. Since H is 2–connected, $|N(h_i, H)| \ge 2$. Therefore $S \cap \{h_1, h_2\} = \emptyset$, and so $\{h_1, h_2\} \subseteq V(C')$. Let $s \in S$. Since S is a minimal cut, s is adjacent to a vertex, say c, in C. Then $G - (C - c)$ is an induced subgraph of G with three end–blocks cs, x_1h_1, and h_2x_2. Since G is claw-free, by **4.2**, $G - (C - c)$ has at most two end–blocks, a contradiction. □

A graph G is called a *string* if G is a chain such that for every inner brick b_1Bb_2 of G, either $B \cup b_1b_2$ is 3–connected or B is a triangle or B has one edge.

4.5. *Suppose that G is a {claw, net}-free graph and $\kappa(G) = 1$. Then G is a string.*

Proof (uses **4.3** and **4.4**). By **4.3**, G is a chain. Let B be an inner block of G and b_1 and b_2 the two boundary vertices of B. Then G has two disjoint edges x_1b_1 and x_2b_2 in G such that $x_1b_1Bb_2x_2$ is an induced subgraph of G. Since G is {claw, net}-free, $x_1b_1Bb_2x_2$ is also {claw, net}-free. We can assume that B is 2–connected and not a triangle. Then by **4.4**, $(B \cup b_1b_2)$ is 3–connected. Therefore G is a string. □

A sequence $R = (x_0, B_0, x_1, B_1, \ldots x_{n-1}, B_{n-1}, x_n)$, is called a *circle representation of a graph G* if

$(r0)$ $k \ge 2$ and $x_0 = x_n$,

$(r1)$ each B_i is either a 2–connected graph or a one edge graph,

$(r2)$ $E(B_i) \cap E(B_j) = \emptyset$ if $i \ne j$, and $V(B_i) \cap V(B_j) = \emptyset$ if $|i - j| \ge 2$,

$(r3)$ if $n \ge 3$, then $V(B_{i-1}) \cap V(B_i) = x_i$, $1 \le i \le n$, and if $n = 2$, then $V(B_1) \cap V(B_2) = \{x_1, x_2\}$ and $v(B_1) \ge 3$, $v(B_2) \ge 3$,

$(r4)$ $\{x_1, \ldots, x_{n-1}\}$ are distinct vertices, and

$(r5)$ $G = x_0B_0x_1B_1 \ldots x_{n-1}B_{n-1}x_0$ (in this case we say that G is a *circle* $x_0B_0x_1B_1 \ldots x_{n-1}B_{n-1}x_0$).

We call each $\ddot{B}_i = x_iB_ix_{i+1}$ a *brick of* R. Let $r(R)$ denote the number of bricks in R. A brick in R is called *trivial*, if it has exactly one edge, and *non-trivial*, otherwise. Let $\bar{B}_i = B_i \cup x_ix_{i+1}$. We say that a brick $\ddot{B}_i$ is *formed by* B_i and is k*–connected* if $\bar{B}_i$ is k–connected.

Obviously, a graph G has a circle representation if and only if $\kappa(G) = 2$. A graph G with $\kappa(G) = 2$ may have different circle representations.

If G is a 2–connected graph, then let $S(G) = \{x \in V(G) : \kappa(G - x) = 1\}$. Obviously, G is 3–connected if and only if $S(G) = \emptyset$.

A circle representation R of G is called a *ring representation* if $r(R) = |S(G)|$, and so every two ring representations of G have the same number of bricks. Thus if R is a ring representations of G, then let $r(G) = r(R)$.

Obviously

4.6. *Let G be a 2–connected graph and $R = (x_0, B_0, x_1, B_1, \ldots x_{n-1}, B_n, x_0)$ a circle representation of G, and so $n \ge 3$.*

Then the following are equivalent:

$(a1)$ *R is a ring representation of G,*

$(a2)$ *$S(G) = \{x_0, \ldots, x_{n-1}\}$, and*

$(a3)$ *each brick $x_iB_ix_{i+1}$ is such that either B_i has one edge or B_i is a triangle or $B_i \cup x_ix_{i+1}$ is 3–connected.*

A 2–connected graph G is called a *ring* if G is either a triangle or has a ring representation with at list three bricks. Obviously, a cycle is a minimal ring.

It is easy to see the following.

4.7. *Suppose that G is a ring and $v(G) \ge 4$. Then G has a unique ring representation.*

The following claim provides a simple necessary condition for a {claw, net}-graph to be a ring.

4.8. *Suppose that*

$(h1)$ *G is a {claw, net}-graph,*

$(h2)$ *G has a circle representation R, and*

$(h3)$ *one of the following holds:*

$(h3.1)$ *$r(R) = 3$ and all three bricks of R are non-trivial,*

$(h3.2)$ *$r(R) = 4$ and R has at least two non-trivial bricks having a common vertex,*

$(h3.3)$ *$r(R) \geq 5$.*

Then G is a ring and the ring representation of G can be obtained from R by replacing non-trivial and not 2–connected bricks of R by their strings.

Proof (uses **4.5** and **4.7**). Let b_1Bb_2 be a brick of a circle representation R of G. By $(h3)$, there are two disjoint edges x_1b_1 and x_2b_2 in G such that $x_1b_1Bb_2x_2$ is an induced subgraph of G. By **4.5**, $x_1b_1Bb_2x_2$ is a string $x_1b_1B_1y_1 \ldots y_{k-1}B_kb_2x_2$. Let R' be obtained from R by replacing each non-trivial and not 2–connected brick b_1Bb_2 of R by its string $b_1B_1y_1 \ldots y_{k-1}B_kb_2$. Then R' is a ring representation of G, and so G is a ring. By **4.7**, R' is the only ring representation of G. □

A sequence $Q = (B_0, X_1, B_1, \ldots, X_n, B_n)$, where $k \geq 1$, is called a *2–chain representation* of a graph G if

$(q1)$ each B_i is a graph,

$(q2)$ $E(B_i) \cap E(B_j) = \emptyset$ if $i \neq j$, and $V(B_i) \cap V(B_j) = \emptyset$ if $|i - j| \geq 2$,

$(q3)$ $V(B_{i-1}) \cap V(B_i) = X_i = \{x_i, x'_i\}$, $x_i \neq x'_i$ for every $i \in \{1, \ldots, n-1\}$,

$(q4)$ $X_i \neq X_j$ for $i \neq j$, and

$(q5)$ $G = B_0X_1B_1 \ldots X_nB_n$.

Obviously, each X_i is a vertex 2-cut of G, and so $\cup\{X_i : i \in \{1, \ldots, n\}\} \subseteq S(G)$. Both $\ddot{B}_0 = B_0X_1$ and $\ddot{B}_1 = X_nB_n$ are called the *end–bricks* and each $\ddot{B}_i = X_{i-1}B_iX_i$, $1 \leq i \leq n-1$, is called an *inner brick* of the 2–chain representation Q of G. We also say that a brick $\ddot{B}_i$ is *formed by B_i*. Let $\bar{B}_0 = B_0 \cup x_1x'_1$, $\bar{B}_n = B_n \cup x_nx'_n$, and $\bar{B}_i = B_i \cup \{x_ix'_i, x_{i+1}x'_{i+1}\}$ for $1 \leq i \leq n-1$.

A brick $\ddot{B}_i$ is called *k–connected* (*a square, a triangle*) if $\bar{B}_i$ is k–connected (a square, a triangle, respectively).

Let G be a 2–connected graph. A 2–chain representation Q of G is called a *2–string representation* if $S(G)$ is obtained from $\cup\{X_i : i \in \{1, \ldots, n\}\}$ by adding the vertex set of each end–brick which is a square (if any).

A graph G is called a *2–string* if G has a 2–string representation.

¿From the definition of a 2–string representation we have:

4.9. *Let G be a 2–connected graph and $Q = (B_0, X_1, B_1, \ldots, X_n, B_n)$ a 2–chain representation of G, $n \geq 2$. Then the following are equivalent:*

$(a1)$ *Q is a 2–string representation of G,*

$(a2)$ *each $\bar{B}_i$ is either a triangle, or a square, or a 3–connected graph.*

¿From the definitions of a ring and a 2–string we have:

4.10. *Let G be a graph. The following are equivalent:*

$(a1)$ *G is both a ring and a 2–string and*

$(a2)$ *one of the following holds:*

$(a2.1)$ *G is a ring $x_2x_1B_1xB_2x_2$, where x_1B_1x and xB_2x_2 are non-trivial bricks (and so G is also a 2–string with three bricks, where the middle one is a triangle),*

$(a2.2)$ *G is a ring $x_3x_0B_1x_1x_2x_3x_0$, where $x_0B_1x_1$ is a non-trivial brick (and so G is also a 2–string with two bricks, one of which is a square),*

$(a2.3)$ *G is a ring $x_3x_0B_1x_1x_2B_3x_3$, where $x_0B_1x_1$ and $x_2B_2x_3$ are non-trivial bricks (and so G is also a 2–string with three bricks, where the middle one is a square).*

It is easy to show the following.

4.11. *If G has a 2–string representation, then G has only one 2–string representation up to the edges xy such that $G - \{x, y\}$ is not connected (which may belong to either of the two bricks containing $\{x, y\}$).*

It is easy to see that if a {claw, net}-free graph G has a minimal vertex cut, then G can be decomposed as follows.

4.12. *Let G be a connected graph and S a minimal vertex cut of G, and so $G = G_1SG_2$. Let K_S be the complete graph with the vertex set S and $F_i = G_i \cup K_S$. Suppose that G is {claw, net}-free. Then each F_i is {claw, net}-free. Also if $|S| = 2$ and $H_i = G_i \cup \{zs : s \in S\}$, where z is a new vertex, then each H_i is {claw, net}-free.*

It is also easy to prove the following.

4.13. *Let $G = B_0X_1B_1 \ldots X_kB_k$ be a {claw, net}-free 2–string, where $X_i = \{x_i, x'_i\}$. Let $0 \le i < j \le k$. Then*

$(a1)$ *if $j - i = 2$ and $\bar{B}_i \cap \bar{B}_j \neq \emptyset$, then $\bar{B}_{i+1}$ is a triangle and neither $\bar{B}_i$ nor $\bar{B}_j$ is a square,*

$(a2)$ *if $j - i \ge 3$, then $\bar{B}_i \cap \bar{B}_j = \emptyset$,*

$(a3)$ *if $\bar{B}_i$ is a square, then $\bar{B}_{i+1}$ is not a square, and*

$(a4)$ *if $\bar{B}_i$ is a square and either $\bar{B}_{i+1}$ is a triangle or $i \ge 3$, then $x_{i+1}x'_{i+1} \in E(G)$.*

By **4.12** and **4.13**, a {claw, net}-free 2–string can be decomposed as follows.

4.14. *Let $G = B_1X_1 \ldots X_{n-1}B_n$ be a {claw, net}-free 2–string, where $X_i = \{x_i, x'_i\}$. Let $F_i = B_1X_1 \ldots X_{i-1}B_iX_iF$ and $D_j = DX_{j-1}B_jX_j \ldots X_{n-1}B_n$, where F is the 3–path $x_ifx'_i$, D is the 3–path $x_{j-1}dx'_{j-1}$, $1 \le i < k$, and $1 < j \le n$. Then F_i and D_j are {claw, net}-free. Moreover,*

$(a1)$ *if $1 < r < n$ and $\bar{B}_r$ is either triangle or a square, then both F_{r-1} and D_{r+1} are 2–strings and*

$(a2)$ *if $1 \le r < n$ and both $\bar{B}_r$ and $\bar{B}_{r+1}$ are 3–connected, then both F_r and D_{r+1} are 2–strings.*

4.15. *Let A, B, C, and D be disjoint connected graphs, and D be 3–connected. Let $\{a'_1, a'_2\}$, $\{b'_1, b'_2\}$, and $\{c'_1, c'_2\}$ be pairs of distinct vertices in A, B, and C, respectively, and let $d(a'_i, A \cup a'_1a'_2) \ge 2$, $d(b'_i, B \cup b'_1b'_2) \ge 2$, and $d(c'_i, C \cup c'_1c'_2) \ge 2$ for every $i \in \{1, 2\}$. Let a_1a_2, b_1b_2, and c_1c_2 be distinct edges of D. Let the graph F be obtained from A, B, C, and D by identifying a_i with a'_i, b_i with b'_i, and c_i with c'_i for each $i \in \{1, 2\}$. Then F is not {claw, net}-free.*

Proof (uses **4.1** and **4.2**). Suppose, on the contrary, that F is {claw, net}-free. Then by **4.1**, $F - S$ has exactly two components for every minimal vertex cut S.

Suppose that $x = a_1 = b_1$. Since $d(x, A \cup a_1a_2) \geq 2$ and $d(x, B \cup b_1b_2) \geq 2$, there are edges ax and bx in F such that $a \in V(A - a_2)$ and $b \in V(B - b_2)$. Since D is 3–connected, there is an edge dx in F such that $d \in V(D - \{a_2, b_2\}$. Then the vertex set $\{x, a, b, c\}$ induces a claw in G centered at x, a contradiction.

Now suppose that no two edges in $\{a_1a_2, b_1b_2, c_1c_2\}$ are adjacent. By **4.2**, it is sufficient to show that there are $a \in \{a_1, a_2\}$, $b \in \{b_1, b_2\}$, and $c \in \{c_1, c_2\}$ such that $D - \{a, b, c\}$ has a component containing $\{a', b', c'\}$, where $\{a, a'\} = \{a_1, a_2\}$, $\{b, b'\} = \{b_1, b_2\}$, and $\{c, c'\} = \{c_1, c_2\}$. If $D - \{a_1, b_1, c_1\}$ has a component containing $\{a_2, b_2, c_2\}$, then we are done. So we assume that there is no such component, and so $D - \{a_1, b_1, c_1\}$ is not connected. Then $D - \{a_1, b_1, c_1\}$ has exactly two components, say D_1 and D_2. We can assume that $a_2 \in V(D_1)$ and $b_2, c_2 \in V(D_2)$. Then $D - \{a_2, b_1, c_1\}$ has a component containing $\{a_1, b_2, c_2\}$. □

4.16. *Let G be a 2–connected {claw, net}-free graph. Then G is either 3–connected, or a ring or a 2–string.*

Proof (uses **4.1**, **4.2**, **4.9**, **4.8**, **4.12**, and **4.15**). We prove our claim by induction on $v(G)$. If $v(G) = 3$, then the claim is obviously true. Let $v(G) \geq 4$. We can assume that G is not 3–connected, and so $\kappa(G) = 2$. Then G has a 2–vertex cut $X = \{x_1, x_2\}$, and so $G = AXB$. Let $e = x_1x_2$ and let $\bar{A} = A \cup e$ and $\bar{B} = B \cup e$. We can assume that $\bar{B}$ has no 2–vertex cut. If $\bar{A}$ also has no 2–vertex cut, then G is a 2–string. So let $\bar{A}$ have a 2–vertex cut. Since G is claw–free, by **4.1**, $G - X$ has exactly two components, and so X is not a vertex 2–cut of A. Since G is 2–connected, $\bar{A}$ is also 2–connected, and so $\kappa(\bar{A}) = 2$. Since G is {claw, net}-free, by **4.12**, $\bar{A}$ is also {claw, net}-free. Since $v(\bar{A}) < v(G)$, by the induction hypothesis, $\bar{A}$ is either a ring or a 2–string. If $\bar{A}$ is a ring, then G has a circle representation with at least two non-trivial bricks and therefore, by **4.8**, G is also a ring. So we assume that $\bar{A}$ is a 2–string. If e is an edge of an end–brick of $\bar{A}$, then G is a 2–string. So we assume that $e = x_1x_2$ is an edge of an inner brick, say $\{s_1, z_1\}H\{s_2, z_2\}$, and so there are two bricks $\ddot{H}_1$ and $\ddot{H}_2$ in 2–string $\bar{A}$ such that $V(H) \cap V(H_i) = \{s_i, z_i\}$, $i \in \{1, 2\}$. Since H is a brick of a 2–string, by **4.9**, $\bar{H} = H \cup \{s_1z_1, s_2z_2\}$ is either a triangle or a square or 3–connected.

Suppose that $\bar{H}$ is either a triangle (z, s_1, s_2, z), where $z = z_1 = z_2$, or a square $(s_1s_2z_2z_1s_1)$. We can assume that $s_1s_2 = x_1x_2$. Since $\bar{A}$ is not a ring, one of H_i's is not an end–brick of $\bar{A}$. Therefore we can assume that $\bar{A}$ has a brick $\dot{H}_3$ such that $H_2 \cap H_3 = \{s_3, z_3\} \subseteq V(G)$. Since G is claw-free, by **4.1**, $\{s_3, z_3\} \neq \{s_2, z_2\}$, say $s_3 \notin \{s_2, z_2\}$. Then $G - \{s_1, s_3\}$ is a connected graph having three end–blocks. By **4.2**, G is not {claw, net}-free, a contradiction.

Now suppose that $\bar{H}$ is 3–connected. Put in **4.15** $A = H_1 \cup s_1z_1$, $B = \bar{B}$, $C = H_2 \cup s_2z_2$, $D = \bar{H}$, $b_1b_2 = x_1x_2$, $a_1a_2 = s_1z_1$, and $c_1c_2 = s_2z_2$. Then $F = A \cup B \cup C \cup D$ is an induced subgraph of G and A, B, C, and D satisfy the assumptions of **4.15**. Hence by **4.15**, G is not {claw, net}-free, a contradiction. □

4.17. *Let G be a 3–connected claw-free graph and $x \in V(G)$. Suppose that $G - x$ is not 3–connected. Then one of the following holds:*

$(a1)$ *$G - x$ is a cycle with four or five vertices,*

$(a2)$ *$G - x$ is both a ring and a 2–string, i.e. $G - x$ is a ring with exactly two non–trivial bricks and either with three bricks (and so $G - x$ is also a 2–string with three bricks, where the middle brick is a triangle) or with four bricks, where two non-trivial bricks are disjoint (and so $G - x$ is also a 2–string with three bricks, where the middle brick is a square); moreover, each non-trivial brick of ring $G - x$ has an inner vertex adjacent to x in G,*

$(a3)$ *$G - x$ is a 2–string which is not a ring; moreover, each end–brick of $G - x$ has an inner vertex adjacent to x and no vertex avoiding the end–bricks is adjacent to x in G.*

Proof (uses **4.16**). Since G is 3–connected and $G - x$ is not 3–connected, $\kappa(G - x) = 2$. By **4.16**, G is either a ring or a 2–string. Suppose that $(a3)$ does not hold. Then $G - x$ is a ring. By definition of a ring $G - x$ has at least two non-trivial bricks. Since G has no induced claw centered at x and since G is 3–connected, ring $G - x$ has the properties described in $(a1)$ and $(a2)$. □

It is easy to prove the following.

4.18. [10] *Let H be a connected graph and Hxz a {claw, net}-free graph. Let X be the set of vertices in H that are on the same distance from x. Then X induces a complete subgraph in H.*

Proof Since $G = Hxz$ is claw-free, $N(x, H) \cup x$ induces a clique Q in G. Let G' be obtained from G by identifying the vertices of Q with a new vertex. The claim follows immediately from the fact that G' is {claw, net}-free. □

4.19. *Let H be a connected graph and $G = zxHx'z'z$, and $x \neq x'$. Suppose that G is {claw, net}-free. Let L_i (L'_i) be the set of vertices in $G - z'$ (respectively, in $G - z$) that are on distance i from z (respectively, from z'), and so $L_0 = z$, $L_1 = x$, $L'_0 = z'$, and $L'_1 = x'$. Let n be the maximum i such that $L_i \neq \emptyset$. Then each L_i and L'_j induce cliques in G and*

$(a1)$ *if $G - zz'$ is {claw, net}-free, then $x' \in L_n$, $L_n - x' \subseteq L'_2 \subseteq L_{n-1} \cup L_n - x'$, and if in addition $n \geq 3$, then $xx' \notin E(G)$ and*

$(a2)$ *if $G - zz'$ is not {claw, net}-free, then $x' \in L_2$, $L_2 - x' \subseteq L'_2 \subseteq (L_2 - x') \cup L_3$, and $xx' \in E(G)$.*

Proof (uses **4.18**). By **4.18**, applied to $G - z' = zxA$ and $G - z = z'x'A$, each L_i and each L'_j induces a clique in G. Obviously, $(a2)$ follows from **4.18**. We prove $(a2)$. Since G is {claw, net}-free and $G - zz'$ is not {claw, net}-free, $G - zz'$ has an induced net with dangling edges xz and $x'z'$. Therefore $xx' \in E(G)$. Claim $(a2)$ follows. □

4.20. *Let A be a connected graph, $G = zxAx'z$, and $x \neq x'$. Suppose that G is {claw, net}-free. Let L_i (L'_i) be the set of vertices in $A - \{x, x'\}$ that are on distance i from x (respectively, from x'). Let n be the maximum i such that $L_i \neq \emptyset$. Let r be the maximum k*

such that x' is adjacent to a vertex in L_k, and so $1 \leq r \leq n$. Then each L_i and L'_j induce cliques in G, $r \in \{1, 2, n\}$, and $L'_1 \subseteq L_{r-1} \cup L_r$. Moreover if $r = 2$ and $L'_1 \subseteq L_2$, then $L'_1 \neq L_2$, $L'_2 = L_1 \cup L_2 \setminus L'_1$, and L'_1 is not adjacent to L_3 (and so $L'_k = L_k$ for $3 \leq k \leq n$).

Proof (uses **4.18**). Let $E(G) = E$. By **4.18** for zxH, each L_i induces a clique in G. Similarly, each L'_i induces a clique in G.

(**p1**) We prove that $r \in \{1, 2, n\}$ and $L'_1 \subseteq L_{r-1} \cup L_r$ (here we put $L_0 = \emptyset$). Since G' has no induced claw centered at x', clearly $L'_1 \subseteq L_{r-1} \cup L_r$.

We claim that if $3 \leq r \leq n-1$, then $N \not\subseteq L_r$, and so $L'_1 \cap L_{r-1} \neq \emptyset$. Suppose, on the contrary, that $L'_1 \subseteq L_r$. Let l_{r-1} be a vertex in L_{r-1} adjacent to L'_1. Since l_{r-1} is not adjacent to L_{r+1}, by **4.18** for $G' = H'x'z'$, L'_1 is not adjacent to L_{r+1}. Since L_r is adjacent to L_{r+1}, there is $l_r \in L_r \setminus L'_1$ such that l_r is adjacent to a vertex l_{r+1} in L_{r+1}. Since $l_{r-1}, l_r \in L'_2$, clearly $l_{r-1}l_r \in E$. If there is $l'_{r-1} \in L_{r-1}$ not adjacent to L'_1, then $l'_{r-1}l_{r+1} \in L'_3$, and so $l'_{r-1}l_{r+1} \in E$, a contradiction. Otherwise, since $r \geq 3$, there is a vertex l_{r-2} in L_{r-2} adjacent to l_{r-1}. Then $l_{r-2}, l_{r+1} \in L'_3$, and so $l_{r-2}l_{r+1} \in E$, a contradiction.

Next we claim that if $3 \leq r \leq n-1$, hen $L_r \subseteq L'_1$. Suppose not, i.e. there is $l_r \in L_r \setminus L'_1$. Since, as we showed above, $L'_1 \not\subseteq L_r$, there is $l_{r-1} \in L'_1 \cap L_{r-1}$. Let l_{r-2} be a vertex in L_{r-2} adjacent to l_{r-1}. Then $l_{r-2}, l_r \in L'_2$, and so $l_{r-2}l_r \in E(G)$, a contradiction. Similarly if $r \leq n-1$, then since $L'_1 \not\subseteq L_{r-1}$, clearly $L_{r-1} \subseteq L'_1$.

Now we claim that $r \in \{1, 2, n\}$. Suppose not, i.e. $3 \leq r \leq m-1$. Then from the above two claims, $L'_1 = L_{r-1} \cup L_r$. Therefore there are vertices l_{r-2} in L_{r-2} and l_{r+1} in L_{r+1} adjacent to L'_1. Then $l_{r-2}l_{r+1} \in L'_2$, and so $l_{r-2}l_{r+1} \in E$, a contradiction.

(**p2**) Now we prove that if $r = 2$ and $L'_1 \subseteq L_2$, then $L'_1 \neq L_2$, $L'_2 = L_1 \cup (L_2 \setminus L'_1)$, and L'_1 is not adjacent to L_3. Indeed, let l_1 be a vertex in L_1 adjacent to L'_1. If there is $l_3 \in L_3$ adjacent to L'_1, then $l_1, l_3 \in L'_2$, and so $l_1l_3 \in E$, a contradiction. Therefore L'_1 is not adjacent to L_3 and, in particular, $L'_1 \neq L_2$. Since L_2 induces a clique in G, clearly $L_2 \setminus L'_1 \subseteq L'_2$. Since $L'_1 \subset L_2$, there is $l_2 \in L_2 \setminus L'_1$ adjacent to $l'_3 \in L_3$. Since $l_1, l_2 \in L'_2$, by **4.18** for $zx'H'$, $l_1l_2 \in E$. If there is $l'_1 \in L_1 \setminus L'_2$, then $l'_1, l'_2 \in L'_3$, and so $l'_1l'_3 \in E$, a contradiction. Therefore $L_1 \subseteq L'_2$. Thus $L'_2 = L_1 \cup (L_2 \setminus L'_1) \cup L_2$. □

Given a {claw, net}-free graph G, an edge e of G is called *removable*, if $G - e$ is also {claw, net}-free.

4.21. *Let G be a {claw, net}-free graph.*

($c1$) Let G be a circle $a_2a_1A_1b_1Bb_2A_2a_2$, where B and each A_i are connected subgraphs of G. If $V(B) = \{b_1b_2\}$ or $d(b_i, A_i) \geq 2$ for some $i \in \{1, 2\}$, then a_1a_2 is a removable edge of G.

($c2$) Let G be a circle $a_2a_1A_1xA_2a_2$, where each A_i is a connected subgraph of G. If $d(b_i, A_i) \geq 2$ for every $i \in \{1, 2\}$, then a_1a_2 is a removable edge of G.

Proof If a_1a_2 belongs to no triangle and no square in G, then obviously a_1a_2 is removable. So we assume that a_1a_2 belongs to either a triangle or a square in G.

We first prove ($c1$). Suppose, on the contrary, that a_1a_2 is not removable, i.e. $G - a_1a_2$ is not {claw, net}-free. Since G is a circle with four bricks and a_1a_2 is one of them, a_1a_2 belongs to no triangle in G. Therefore $G - a_1a_2$ has an induced net, say N. Then $V(B) \neq \{b_1b_2\}$ and $a_1b_1b_2a_2$ is a path in N. Since $d(b_i, A_i) \geq 2$ for some $i \in \{1, 2\}$, there is $b_ix_i \in E(A_i)$. Then $(N - a_i) \cup b_ix_i$ forms an induced net in G. Therefore G is not {claw, net}-free, a contradiction.

We now prove $(c2)$. Suppose, on the contrary, that a_1a_2 is not removable, i.e. $G - a_1a_2$ is not {claw, net}-free. Then one of the following holds:

$(h1)$ $G - a_1a_2$ has an induced claw C containing a path a_1xa_2 and avoiding $A_i - \{a_i, x\}$, say for $i = 1$,

$(h2)$ $G - a_1a_2$ has an induced net N containing a_ix for some $i \in \{1, 2\}$, say for $i = 1$.

Since $d(b_i, A_1) \geq 2$, there is $xx_i \in E(A_i)$. Now if $Y \in \{C, N\}$, then $(Y - a_i) \cup xx_i$ forms an induced subgraph in G isomorphic to Y, a contradiction. □

¿From **4.21** we have, in particular:

4.22. *Let G be a {claw, net}-free ring. If an edge e of G forms a brick in G, then e is removable.*

5 On Hamiltonian cycles and s-traces containing a given edge in {claw, net}-free graphs

As we mentioned above (see **3.9**), in [8]) we proved the following.

5.1. *Let G be a 2–connected {claw, net}-free graph and $uv \in E(G)$. Then uv belongs to a track of G if and only if $G - \{u, v\}$ is connected.*

¿From **5.1** we have:

5.2. *Let G be a {claw, net}-free ring having n bricks, and so $n \geq 3$. Let $\ddot{B}_i$ be a brick of G, $\beta_i \in E(B_i)$, and $U = \{\beta_i : i = 1, \dots, n\}$. Suppose that β_i is incident to an inner vertex in B_i if $v(B_i) \geq 3$. Then G has a track containing U.*

Given a graph G, $s \in V(G)$, and $e \in E(G)$, we call $(G; s, e)$ *traceable* if G has an s–trace containing e. Our next result provides a characterization of traceable triples $(G; s, e)$ for 2–connected {claw, net}-free graphs G.

We define three sets $\mathcal{F}_i$, $i \in \{1, 2, 3\}$, of triples $(G; s, e)$, where G is a connected graph, $s \in V(G)$, and $e = x_1x_2 \in E(G)$, as follows:

$(f1)$ $(G; s, e) \in \mathcal{F}_1$ if and only if $G - \{x_1, x_2\}$ is not connected and $s \in \{x_1, x_2\}$,

$(f2)$ $(G; s, e) \in \mathcal{F}_2$ if and only if G has three connected subgraphs S_1, S_2, and X with at least three vertices each such that $G = sS_1x_1Xx_2S_2s$, and

$(f3)$ $(G; s, e) \in \mathcal{F}_3$ if and only if G has four connected subgraphs S, S_1, S_2, and X with at least three vertices each such that $s \in V(S) - \{s_1, s_2\}$, and $G = s_2Ss_1S_1x_1Xx_2S_2s_2$.

Let $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3$.

Obviously, if G has an s–trace containing e, then $(G; s, e) \notin \mathcal{F}$, i.e. $\mathcal{F}$ is a set of non-traceable triples $(G; s, e)$. It turns out that for 2–connected {claw, net}-free graphs, converse is also true.

5.3. *Let G be a 2–connected {claw, net}-free graph, $s \in V(G)$ and $e \in E(G)$. Then the following are equivalent:*

$(a1)$ *G has an s–trace containing e and*

$(a2)$ *$(G; s, e) \notin \mathcal{F}$.*

Proof (uses **3.7**, **3.6**, **3.9**, **4.1**, and **4.2**). Obviously $(a1) \Rightarrow (a2)$. We prove $(a2) \Rightarrow (a1)$ by induction $v(G)$. If $v(G) = 3$, then our claim is obviously true. So let $v(G) \geq 4$. Let $e = uv$ and $G' = G - s$. Since G is 2–connected, $|N(s, G)| \geq 2$.

(**p1**) Suppose that $s \in \{u, v\}$, say $s = u$.

Suppose that G' is 2–connected. By **3.9**, G' has a v–trace P. Then svP is an s–trace in G containing uv.

Now suppose that G' is not 2–connected. Then by **4.2**, G' is a chain and s is adjacent to an inner vertex of each end–block in G'. If v belongs to no end–block of G', then G has an induced claw centered at s, a contradiction. Therefore let v be a vertex of an end–block B of G'. If v is the boundary vertex of B, then $(G; s, e) \in \mathcal{F}$. This contradicts $(a2)$. Therefore v is an inner vertex of B. By **3.6**, G' has a v–trace P. Then svP is an s–trace in G containing $e = uv$.

(**p2**) Now suppose that $s \notin \{u, v\}$.

(**p2.1**) Suppose that G' is 2–connected. If there is $y \in N(s, G)$ such that $(G'; y, e) \notin \mathcal{F}$, then by the induction hypothesis there is an s'–trace P containing uv. Then $ss'P$ is an s–trace in G containing uv. Therefore we assume that $(G'; y, e) \in \mathcal{F}$ for every $y \in N(s, G)$. Then $G' - \{u, v\}$ is not connected. Since G is {claw, net}-free, by **4.1**, there is $z \in N(s, G) \setminus \{u, v\}$, and so $(G'; z, e) \in \mathcal{F}$. Then $G' = x_1 B_1 x_2 \ldots x_{k-1} B_k x_1$ is a ring with $k \geq 3$ and e is an edge of some B_i, say of B_1. Since $(G'; z, e) \in \mathcal{F}$, clearly $\{u, v\} = \{x_1, x_2\}$, and so $e = x_1 x_2$ and B_1 is either a triangle or 3-connected. Since $(G'; y, e) \in \mathcal{F}$ for every $y \in N(s, G)$, clearly $N(s, G) \cap V(B_k x_1 B_1 x_2 B_2 - \{x_1, x_2, x_3, x_{k-1}\}) = \emptyset$. Moreover, for the same reason if $e(B_2) = 1$ ($e(B_k) = 1$), then $x_3 \notin N(s, G)$ (respectively, $x_{k-1} \notin N(s, G)$). If $x_1 \in N(s, G)$ ($x_2 \in N(s, G)$), then G has an induced claw centered at x_1 (at x_2, respectively), a contradiction. Therefore $\{x_1, x_2\} \cap N(s, G) = \emptyset$. t follows that $(G; s, e) \in \mathcal{F}$. This contradicts $(a2)$.

(**p2.2**) Now suppose that G' is not 2–connected. Then G' is a chain and since G is 2–connected, s is adjacent to inner vertices z and z' of the end–blocks B and B' of the chain, respectively.

Suppose that e belongs to an end–block B of G' with the boundary vertex b. If B is 2–connected, then there is a vertex b' in $B - b$ such that $(G; b, b', e) \in \mathcal{L}$. Therefore by **3.7**, B has a $b'b$–trace P containing e. By **3.6**, $G' - (B - b)$ has a $z'b$–trace P'. Then $sz'P'bPb'$ is an s–trace in G containing e. If B is not 2–connected, then $E(B) = uv = e$ and $b \in \{u, v\}$, say $b = v$. By **3.6**, G' has a us'–trace $uvPs'$. Then $uvPs's$ is a us–trace in G containing $e = uv$.

Now suppose that e belongs to an inner block d_1Dd_2 of G'. If $e = d_1d_2$, then $(G; s, e) \in \mathcal{F}$, which contradicts $(a2)$. Therefore $e \neq d_1d_2$, i.e. e is an inner edge of D. By **3.6**, G' has a zz'–trace P containing e. Then szP is an s–trace of G containing e. □

¿From **5.3** we have:

5.4. *Let G be a 2–connected {claw, net}-free graph. Then every edge belongs to a trace in G.*

6 On st-traces in {claw, net}-free graphs

Given a graph G, $\{s, t\} \subseteq V(G)$, and $s \neq t$, we call $(G; s, t)$ *traceable* if G has an st–trace.

In what follows, we give a characterization of traceable triples $(G; s, t)$ for {claw, net}-free graphs G of connectivity two.

We will need the following notions of the *brick–length of a path* and the *brick–distance* between two vertices in a graph which is either a ring or a string. Let G be a ring and P an xy–path in G. We say that P *passes through* a brick uBv of R if P contains u, v, and at least one edge of uBv. We define the *β–length $\beta(P)$ of path P* as follows: $\beta(P) = 2n + t$, where n and t are the numbers of non-trivial and trivial bricks in ring G, which P passes through. A *β–shortest xy*–path in G is an xy–path with the smallest β–length among all xy–paths in G. Given $x, y \in V(G)$, the *β–distance $\beta(x, y)$ between x and y in G* is the β–length of a β–shortest xy–path in G. The corresponding notions for a string are defined similarly.

6.1. *Let G be a {claw, net}-free graph and $\kappa(G) = 2$. Let $u, v \in V(G)$ and $u \neq v$. Then the following are equivalent:*

$(a1)$ *G has a uv–trace and*

$(a2)$ *$(G; u, v)$ satisfies exactly one of the following conditions:*

$(a2.1)$ *G is a ring and $\beta(u, v) \leq 1$,*

$(a2.2)$ *G is a 2–string and $G - \{u, v\}$ is connected.*

Proof (uses **3.2**, **3.7**, **3.12**, **3.13**, **3.14**, **4.2**, **4.5**, **4.8**, **4.9**, **4.16**, and **4.22**). Obviously $(a1) \Rightarrow (a2)$. We prove $(a2) \Rightarrow (a1)$ by induction on $v(G)$. If G is a cycle or a diamond, then our claim is obviously true. Therefore let $v(G) \geq 5$ and G be not a cycle. Since G is a {claw, net}-free graph and $\kappa(G) = 2$, by **4.16**, G is either a ring or a 2–string.

(**p1**) Suppose that G is a ring. Obviously $(G; u, v)$ satisfies $(a2.1)$ if and only if exactly one of the following holds:

$(h1)$ there are two adjacent bricks $\ddot{A} = aAx$ and $\ddot{B} = xBb$ of G such that either $u, v \in V(A \cup B - \{a, b\})$ or $v(A) = 2$, $u = a$, and $v \in V(B - b)$,

$(h2)$ there are two bricks $\ddot{A} = aAa'$ and $\ddot{A} = b'Bb$ of G connected by an edge $a'b'$ and such that $u \in V(A - \{a, a'\})$, $v \in V(B - \{b, b'\})$.

We can assume that $v \in V(B - b)$.

(**p1**.**1**) Suppose that $(h1)$ holds.

(**p1**.**1**.**1**) Suppose that $u \in A - \{a, x\}$ and $v \in B - \{b, x\}$. Then $v(A) \geq 3$ and $v(B) \geq 3$. Let $G_x = G - x$, $A_x = A - x$, $B_x = B - x$, and $G' = G - (A \cup B - \{a, b\})$. Then $u \in A_x - a$ and $v \in B_x - b$. Since G is claw-free, $\dot{N}(x, A)$ and $\dot{N}(x, B)$ are cliques. Since aAx is a brick of a ring and $v(A) \geq 3$, clearly either A is a triangle or $\bar{A}$ 3–connected, and so A_x is either of one edge or 2–connected. Similarly B_x is either of one edge or 2–connected. Obviously G_x is {claw, net}-free and $\kappa(G_x) = 1$. Therefore by **4.5**, G_x is a string and, clearly, aA_x and bB_x are the end–blocks of G_x.

Since aAx and bBx are non-trivial bricks of a ring, G has distinct vertices $a_1 \in V(A_x)$, $b_1 \in V(B_x)$, $\{a_2, b_2\} \subseteq V(G' - \{a, b\})$ such that $a_1aG'bb_1$, a_2aA_x, and b_2bB_x are induced subgraphs of G. Since G is {claw, net}-free, clearly $a_1aG'bb_1$, a_2aA_x, and b_2bB_x are also {claw, net}-free. By **3.1**, $a_1aG'bb_1$ has an a_1b_1–trace a_1aPbb_1.

Suppose that both A and B are triangles, and so $V(A) = \{a, x, u\}$, $V(B) = \{b, x, v\}$. Then $uaPbxv$ is a uv–trace in G.

Now suppose that one of A, B, say A, is not a triangle. By **3.13**, B_x has a vb–trace R and A_x has a ua–trace L containing an edge st from $\dot{N}(x, A_x)$, and so $L = uLstLa$. Then $uLsxtLaPbRv$ is a uv–trace in G.

(**p1**.**1**.**2**) Now suppose that $u \in V(B - b)$ and $v \in V(B - \{b, x\})$ (the case $u \in A - x$ and $v = x$ is similar). Then $v(B) \geq 3$, and so either B is a triangle or $\bar{B} = B \cup bx$ is 3–connected. Let $G_v = G - v$ and $B_v = B - v$. Then G_v is 2–connected, and so since G is a ring, $\kappa(G_v) = 2$. Since G is {claw, net}-free, both G_v and B_v are also {claw, net}-free. Thus by **4.16**, G_v is either a ring or a 2–string.

If there is a vertex z adjacent to v such that G_v has a uz–trace P, then $uPzv$ is a uv–trace in G. So it is sufficient to show that there is a vertex z with the above property.

Suppose that B is a triangle. Then $E(B_v) = bx$ and $u \in \{b, x\}$, say $u = x$. By **3.2**, G_v has a track C and obviously C contains bx. Then $(C - bx) \cup bv$ is a uv–trace in G.

Now suppose that $\bar{B}$ is 3–connected. Then $\kappa(B_v) \geq 1$. If $\kappa(B_v) = 1$, then since B_v is {claw, net}-free, by **4.5**, B_v is a string $D_1d_1 \dots d_{s-1}D_s$, where $s \geq 2$ and each D_i is either of one edge or 2–connected. Clearly $x \in D_1 - d_1$ and $b \in D_s - d_{s-1}$. Now $\kappa(G_v) = 2$ because $\kappa(G) = 2$, $\bar{B}$ is 3–connected, and $v \in V(B - \{b, x\})$. Therefore G_v has a circle representation R and $\ddot{T}$ is a brick of R if and only if $\ddot{T}$ is either a brick of G distinct from $\ddot{B}$ or a block of B_v. If, in particular, B_v is 2–connected, then clearly B_v form a brick of R.

(**p1**.**1**.**2**.**1**) Suppose that G_v is a ring. Since B is 2–connected, $N(v, G) \cap V(D_1 - d_1) \neq \emptyset$ and $N(v, G) \cap V(D_s - d_{s-1}) \neq \emptyset$. Since G has no induced claw centered at v, clearly $N(v, G) \subseteq V(D_1) \cup V(D_s)$. Since $\bar{B} = B \cup bx$ is 3–connected, either B_v is 2–connected or $B_v = xD_1d_1D_2b$ or $B_v = xD_1d_1d_2D_3b$. Since $u \in V(B - b)$ and $N(v, G) \subseteq V(D_1) \cup V(D_2)$, there is $z \in N(v, G)$ such that pair $\{u, v'\}$ satisfies $(h1)$ or $(h2)$ with respect to the blocks of B_v containing them. By the induction hypothesis, G_v has a uz–trace uPz. Then $uPzv$ is a uz–trace of G.

(**p1.1.2.2**) Now suppose that G_v is not a ring, and so G_v is a 2–string. Since G is a ring, by **4.8**, one of the following holds:

($c1$) $G = xaCcbBx$, where bBx is a non-trivial brick of R, and so $E(A) = xa$,

($c2$) $G = aAxBba$, where both aAx and xBb are non-trivial bricks of R.

Since G_v is not a ring, $B_v = pDqFp$, where $\{b, x\} \subseteq V(F)$. Since G_v is {claw, net}-free, both $\dot{N}(x, B_v)$ and $\dot{N}(b, B_v)$ are cliques. Therefore $\{b, x\} \subseteq V(F - \{p, q\}$. Since $\bar{B}$ is 3–connected and G has no induced claw centered at v, $D \cup pq$ and $F \cup \{bx, pq\}$ are 3–connected and there are $d \in V(D - \{p, q\}) \cap N(v, G)$ and $f \in V(F - \{p, q\}) \cap N(v, G)$. Then 2–string G_v has exactly four bricks and $G_v - \{d, f\}$ is connected. Hence there is $z \in \{d, f\}$ such that $G_v - \{u, z\}$ is connected, and so $(G_v; u, z)$ satisfies ($a2.2$). By the induction hypothesis, G_v has a uz–trace P. Then $uPzv$ is a uv–trace in G.

(**p1.2**) Now suppose that ($h2$) holds. By **4.22**, $H = G - a'b'$ is {claw, net}-free. Since G is a ring, H is a string. By **3.7**, H has a uv–trace, which is also a trace of G.

(**p2**) Now suppose that G is a 2–string. Let $G_u = G - u$. Clearly G_u is {claw, net}-free and $v(G_u) < v(G)$.

Since G is a 2–string, we have: $G = B_1X_1B_2X_2 \ldots X_{k-1}B_k$, where $E(B_i) \cap E(B_j) = \emptyset$ for every $i \neq j$, $V(B_i) \cap V(B_{i+1}) = X_i = \{x_i, x'_i\} \subseteq V(G)$, all X_i's are different, and $\bar{B}_1 = B_1 \cup x_1x'_1$, $\bar{B}_k = B_k \cup x_{k-1}x'_{k-1}$, and $\bar{B}_i = B_i \cup \{x_ix'_i, x_{i+1}x'_{i+1}\}$ for $i \not\in \{1, k\}$ are either triangles or squares or 3–connected graphs. Obviously every vertex 2–cut of G is either a vertex 2-cut of a square brick of G or one of X_i's.

(**p2.1**) Suppose that u belongs to a vertex 2–cut of G. Then $\kappa(G_u) = 1$, and so by **4.5**, G_u is a string. Hence G_u satisfies one of the following conditions:

($c1$) $G_u = A_1xA_2$, where A_1 and A_2 are the end–blocks of G_u (we put $x = a_1 = a_2$),

($c2$) $G_u = A_1a_1a_2A_2$, where A_1 and A_2 are the end–blocks, $a_1 \in V(A_1)$, and $a_2 \in V(A_2)$.

Since by ($a2.2$), $\{u, v\}$ is not a 2–cut of G, we can assume that $v \in V(A_2 - a_2)$. Since G is 2–connected, there is $a \in N(u, G) \cap V(A_1 - a_1)$. By **3.7**, G_u has an av–trace aPv. Then $uaPv$ is a uv–trace in G.

(**p2.2**) Now suppose that both u and v belong to no 2–cut of G, and so both u and v are inner vertices of some bricks of G. Then each brick meeting $\{u, v\}$ is either 3–connected or a triangle end–block of G. If u is an inner vertex of B_i, then $N(u, G) = N(u, B_i)$. Hence G_u is 2–connected. By **4.16**, G_u is either a ring or a 2–string or 3–connected.

(**p2.2.1**) Suppose that G_u is a ring. Then 2–string G has at most three bricks.

(**p2.2.1.1**) Suppose that 2–string G has three bricks.

Suppose that $u \in V(B_1 - X_1)$. If $\bar{B}_2$ is 3–connected, then $\bar{B}_2 \cup B_3$ is not 3–connected but forms a brick of ring G_u, a contradiction. Therefore $\bar{B}_2$ is either a triangle or a square. But then G is not a 2–string, a contradiction.

Now suppose that $u \in V(B_2 - (X_1 \cup X_2))$, and so $\bar{B}_2$ is 3–connected. Since G_u is a ring,

$G_u = x_1B_1x_1'D'x_2'B_3x_2Dx_1$, where D and D' are connected graphs and we can assume that $x_1 \neq x_2$.

Assume that u is adjacent to an inner vertex b of B_2, say $b \in V(D - \{x_1, x_2\}$. Since $\bar{B}_2$ is 3–connected and both $\bar{B}_1$ and $\bar{B}_3$ are 2–connected, $G_u - \{x_1, x_2\}$ is connected. Obviously $G_u - \{x_1, x_2\}$ has at least three end–blocks. This contradicts **4.2**.

Now we can assume that $V(B_2) = Bnd(B_2) \cup u$, and so $N(u, G) \subseteq Bnd(B_2)$. Since $\bar{B}_2$ is 3–connected, $N(u, G) = Bnd(B_2)$. Since v is an inner vertex of a brick of G and $v \neq u$, clearly v is an inner vertex of an end–brick of G, say $v \in V(B_1 - X_1)$. Then $x_1 \subseteq N(u, G)$ and $(G_u; v, x_1)$ satisfies $(h1)$. By the induction hypothesis, G_u has a vx_1–trace P. Then ux_1Pv is a uv–trace in G.

(**p2.2.1.2**) Now suppose that 2–string G has exactly two bricks, i.e. $G = B_1XB_2$, where $X = \{x, x'\} \subseteq V(G)$ and each $\bar{B}_i$ is either a triangle or 3–connected. We can assume that $u \in V(B_1 - X)$, and so $\bar{B}_1$ is 3–connected. Then $B_1 - u$ is a string $D_1d_1 \dots d_{s-1}D_s$, $s \geq 2$, and G_u is the ring $xD_1d_1 \dots d_{s-1}D_sx'B_2x$. Since $\bar{B}_1$ is 3–connected, each non-trivial brick D_i of ring G_u has an inner vertex adjacent to u in G. Since G has no induced claw centered at u, at most two of D_i's are non-trivial bricks of ring G_u and $s \leq 4$. Now it is easy to check that there is $z \in V(G_u - v) \cap N(u, G)$ such that $(G_u; v, z)$ satisfies $(h1)$. By the induction hypothesis, G_u has a vz–trace P. Then $uzPv$ is a uv–trace in G.

(**p2.2.2**) Suppose that G_u is a 2–string. We remind that $u \in V(B_r)$, both u and v belong to no vertex 2–cut of G, and so both u and v are inner vertices of some bricks of 2–string G. If there is $z \in N(u, G - v)$ such that $(G_u; v, z)$ satisfies $(a2.2)$ (i.e. $G_u - \{v, z\}$ is connected), then by the induction hypothesis, G_u has a vz–trace P and then $uzPv$ is a uv–trace in G. So our goal is to show that there exists a vertex $z \in N(u, G - v)$ such that $G_u - \{v, z\}$ is connected.

If $\bar{B}_r - u$ is 3–connected, then obviously there is $z \in N(u, G - v)$ such that $G_u - \{v, z\}$ is connected. So we assume that $\bar{B}_r - u$ is not 3–connected.

Suppose that $\bar{B}_r$ is 3–connected. Since $\bar{B}_r - u$ is not 3–connected and G_u is a 2–string, $\bar{B}_r - u = D_1YD_2$, where $Y = \{y, y'\}$, $x_{r-1}x_{r-1}' \in E(D_1)$ and $x_rx_r' \in E(D_2)$ if B_r forms an inner brick of G, and $x_1x_1' \in E(D_2)$ if $B_r = B_1$. Since $\bar{B}_r$ is 3–connected and G has no induced claw centered at u, each D_i has an inner vertex in $N(u, G)$. Therefore there is $z \in N(u, G)$ such that $G_u - \{v, z\}$ is connected.

Now suppose that $\bar{B}_r$ is not 3–connected. Then by **4.9**, $\bar{B}_r$ is either a triangle or a square. Every vertex of a square brick of 2–string G belongs to a vertex 2–cut of G. Thus $\bar{B}_r$ forms a triangle end-brick of G, say $\bar{B}_r = \bar{B}_1$, because u belongs to no vertex 2-cut of G, u is an inner vertex of B_r, and a triangle inner brick of a 2-string has no inner vertices. Therefore $V(B_1) = \{u, x_1x_1'\}$ and $N(u, G - v) = N(u, G) = \{x_1x_1'\}$. Then $G_u - \{v, z\}$ is connected for $z \in \{x_1x_1'\}$.

(**p2.2.3**) Now suppose that G_u is 3–connected. Then B_r forms an end–brick of G and B_r is a triangle. Thus $G = x_1ux_2Hx_1$, where $v \in H - \{x_1, x_2\}$ and $H = G_u$ is 3–connected. Then by **3.14**, G has a uv–trace. □

The following is an analog of **6.1** for {claw, net}-free graphs of connectivity one.

6.2. *Let G be a {claw, net}-free graph, G is a string $B_1x_1 \dots x_{k-1}B_k$, and p_1, p_2, q_1, q_2 different vertices in G. Suppose that $p_1 \in In(B_1)$, $q_1 \in In(B_k)$, $p_2 \in B_i$, $q_2 \in B_j$, and $1 < i \le j < k$. Then the following are equivalent:*

$(c1)$ *G has two disjoint paths p_1Pp_2 and q_1Qq_2 such that $V(P \cup Q) = V(G)$ and*

$(c2)$ *$\beta_G(p_2, qq_2) \le 1$, i.e, one of the following holds:*

$(c2.1)$ *$j = i$ and $\{p_2, q_2\} \cap In(B_i) \neq \emptyset$,*

$(c2.2)$ *$j = i + 1$ and $p_2 \in In(B_i)$, $q_2 \in In(B_j)$,*

$(c2.3)$ *$j = i + 2$, B_{i+1} is a one edge block of G, and $p_2 \in In(B_i)$, $q_2 \in In(B_j)$.*

Proof (uses **3.7** and **6.1**). Let $G_1 = B_1x_1yzx_{k-1}B_k$ and $G_2 = yx_1B_2x_2 \dots B_{k-1}x_{k-1}zy$, where y and z are new vertices. Since G {claw, net}-free, both G_1 and G_2 are {claw, net}-free. Also G is a string implies that G_1 is a string and G_2 is a ring. By **3.7**, G_1 has a p_1q_1–trace C_1. By **6.1**, has a p_2q_2–trace C_2 if and only if $(c2)$ holds. Obviously path x_1yzx_{k-1} belongs to both C_1 and C_2. Let $C = (C_1 \cup C_2) - \{y, z\}$. Then $C = P \cup Q$, where P is a p_1p_2–path, Q is a q_1q_2–path, $V(P) \cap V(Q) = \emptyset$, and $V(P \cup Q) = V(G)$. □

Our next result is a characterization of traceable triples $(G; s, t)$ for 2–connected {claw, net}-free graphs G.

We define the sets $\mathcal{P}_i$, $i \in \{1.2.3\}$, of triples $(G; s, t)$, where G is a graph, $\{s, t\} \subseteq V(G)$, and $s \neq t$, as follows:

$(p1)$ $(G; s, t) \in \mathcal{P}_1$ if and only if $G - \{s, t\}$ is not connected,

$(p2)$ $(G; s, t) \in \mathcal{P}_2$ if and only if there are connected subgraphs Z_1, Z_2, and Z_3 of G such that $G = z_0Z_1z_1Z_2z_2Z_3z_0$, each $v(Z_i) \ge 3$, $s \in V(Z_2 - \{z_1, z_2\})$, and $t = z_0$,

$(p3)$ $(G; s, t) \in \mathcal{P}_3$ if and only if there are connected subgraphs Z_1, Z_2, Z_3, and Z_4 of G such that $G = z_0Z_1z_1Z_2z_2Z_3z_0$, each $v(Z_i) \ge 3$, $xs \in V(Z_1 - \{z_0, z_1\})$, and $t \in V(Z_3 - \{z_2, z_3\})$.

Let $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2 \cup \mathcal{P}_3$.

We will also use sometimes sets $\mathcal{P}_i(G)$ defined as follows: $\{s, t\} \in \mathcal{P}_i(G)$ if and only if $(G; s, t) \in \mathcal{P}_i$. Accordingly let $\mathcal{P}(G) = \mathcal{P}_1(G) \cup \mathcal{P}_2(G) \cup \mathcal{P}_3(G)$.

It is easy to see the following.

6.3. *Let G be a connected graph and $x, y \in V(G)$. Then $(G; x, y) \in \mathcal{P}$ if and only if there is a circle representation $x_1B_1 \dots x_{k-1}B_kx_1$ of G such that the β–distance between x and y in this circle is at least two.*

6.4. *Let G be a {claw, net}-free graph. Then*

$(a1)$ *if G is 3–connected, then $\mathcal{P}(G) = \emptyset$,*

$(a2)$ *if G is a 2–string, then $\mathcal{P}(G) = \mathcal{P}_1(G)$, and*

($a3$) *if G is 3–connected and $x \in V(G)$, then $\mathcal{P}(G-x) = \mathcal{P}_1(G-x)$.*

Proof Claims ($a1$) and ($a2$) are obviously true. Claim ($a3$) follows from the properties of $G-x$ given in **4.17**. □

Obviously

6.5. *Let G be a connected graph. If G has an st–trace, then $(G; s, t) \notin \mathcal{P}$.*

By **6.3**, Theorem **6.1** can be reformulated using $\mathcal{P}$. This reformulation shows that the converse of **6.5** is true for {claw, net}-free graphs of connectivity two.

6.6. *Let G be a {claw, net}-free graph and $\kappa(G) = 2$. Let $s, t \in V(G)$ and $s \neq t$. Then the following are equivalent:*

($a1$) *G has a st–trace and*

($a2$) *$(G; s, t) \notin \mathcal{P}$, namely:*

($a2.1$) *if G is a ring, then $\{G; s, t\} \notin \mathcal{P}$ and*

($a2.2$) *if G is a 2–string, then $(G; s, t) \notin \mathcal{P}_1$.*

It turns out that the converse of **6.5** is true for every 2–connected {claw, net}-free graph.

6.7. *Let G be a 2–connected {claw, net}-free graph and $x, y \in V(G)$, $x \neq y$. Then G has an xy–trace if and only if $(G; x, y) \notin \mathcal{P}$.*

Proof (uses **4.16**, **6.1**, **6.4**, and **6.5**). By **6.5**, if G has an xy–trace, then $(G; x, y) \notin \mathcal{P}$. We prove by induction on $v(G)$ that if $(G; x, y) \notin \mathcal{P}$, then G has an xy–trace. If $\kappa(G) = 2$, then by **6.1**, our claim is true. So we assume that $v(G) \geq 4$ and G is 3–connected, and so $|N(x, G)| \geq 3$. Then $|N(x, G-y)| \geq 2$. Let $G_x = G - x$. Since G is 3–connected and {claw, net}-free, G_x is 2–connected and {claw, net}-free, respectively. Obviously if P is a yz–trace for some vertex $z \in N(x, G-y)$, then $xzPy$ is a xy–trace in G. Thus we assume that G_x has no yz–trace for every $z \in N(x, G-y)$. Then by the induction hypothesis, $(G_x; y, z) \in \mathcal{P}$ for every $z \in N(x, G-y)$. Hence $\kappa(G_x) = 2$. By **4.16**, G_x is either a ring or a 2–string.

(**p1**) Suppose that G_x is a ring, and so G_x has at least three bricks. By **6.4**, $\mathcal{P}(G_x) = \mathcal{P}_1(G_x)$. It follows that there is $z \in N(x, G-y)$ such that $(G_x; y, z) \notin \mathcal{P}$ (i.e $G_x - \{y, z\}$ is connected), a contradiction.

(**p2**) Now suppose that G_x is a 2–string. Then again **6.4**, $\mathcal{P}(G_x) = \mathcal{P}_1(G_x)$. Since G is 3–connected, every end-brick of G_x has an inner vertex in $N(x, G)$. It follows that there is $z \in N(x, G-y)$ such that $(G_x; y, z) \notin \mathcal{P}$ (i.e $G_x - \{y, z\}$ is connected), a contradiction. □

6.8. *Let G be a 3–connected {claw, net}-free graph and L a 3–vertex path in G. Then the following are equivalent:*

($a1$) *G has a track containing L and*

($a2$) *$G - L$ is connected.*

Proof (uses **4.16**, **6.4**, and **6.7**). Obviously $(a1) \Rightarrow (a2)$. We prove $(a2) \Rightarrow (a1)$. By $(a2)$, $G - L$ is connected. Let $L = xyz$ and $G_y = G - y$. Since G is 3–connected, obviously G_y is 2–connected.

If G' has a xz–trace P, then $xPzyx$ is a track of G containing L. So we assume that G_y has no xz–trace. Then by **6.7**, $\{x, z\} \in \mathcal{P}(G')$, and so $\kappa(G) = 2$. By **4.16**, G' is either a ring or a 2–string.

Suppose that G_y is a ring, and so G_y has at least three bricks. By **6.4**, $\mathcal{P}(G_x) = \mathcal{P}_1(G_x)$. Then $(G'; x_1, x_3) \in \mathcal{P}_1$, and so $G - L$ is not connected, a contradiction.

Now suppose that G' is a 2–string. Then again $(G'; x, z) \in \mathcal{P}_1$, and so $G - L$ is not connected, a contradiction. □

¿From **6.8** we can obtain the following.

6.9. *Let G be a 3–connected {claw, net}-free graph and L a 3–path in G. Then G has a trace containing L.*

Proof (uses **4.16** and **6.8**). Let $L = x_1x_3x_2$ be a 3–path in G and $G' = G - x_3$. Since G is 3–connected, obviously G' is 2–connected. If P belongs to a track in G, then our claim is obviously true. So suppose that L belongs to no track in G. Then by **6.8**, $G' - \{x_1, x_2\}$ is not connected. Therefore $G' - x_i$ is a string and x_j is a vertex cut of $G' - x_i$, where $\{i, j\} = \{1, 2\}$. By **4.16**, $G' - x_i$ has a trace $P_i = z_iP_ix_iP_iz'_i$, where $z_i \in G_i - \{x_1, x_2\}$, $i \in \{1, 2\}$. Then $z_1P_1x_1x_3x_2P_2x_2z_2$ is a trace of G containing L. □

7 On *st*-traces containing a given edge in {claw, net}-free graphs

Given a graph G, $\{s, t\} \subseteq V(G)$, $s \neq t$, and $e \in E(G)$, we call $(G; s, t, e)$ *traceable* if G has an st–trace containing e. In this subsection we will discuss the problem of recognizing traceable 4–tuples $(G; s, t, e)$ for {claw, net}-free graphs G.

In **3.6** we gave, in particular, a characterization of traceable 4–tuples $(G; s, t, e)$ for {claw, net}-free strings G.

We will discuss the problem of recognizing traceable 4–tuples $(G; s, t, e)$ for {claw, net}-free rings G. Obviously, if $(G; s, t)$ is not traceable, then $(G; s, t, e)$ is also not traceable. Therefore we will assume that $(G; s, t)$ is traceable.

7.1. *Let $G = x_1B_1x_2 \dots x_kB_kx_1$ be a {claw, net}-free ring, and so $k \geq 3$. Let $s, t \in V(G)$ and $s \neq t$. Let $\beta_i = b_ib'_i \in E(B_i)$. Let $U = \{\beta_i : i = 1, \dots, k\}$.*

$(a1)$ Let $s \in V(B_1 - x_1)$, $t \in V(B_3 - x_4)$, and B_2 forms a one edge brick β_2. Then G has an st–trace containing $U - \beta_2$ if and only if $(B_1; s, x_1, \beta_1) \notin \mathcal{L}$, $(B_3; t, x_4, \beta_3) \notin \mathcal{L}$, and each β_i, $i \geq 4$, is incident to an inner vertex in B_i unless $v(B_i) = 2$.

$(a2)$ Let $s \in V(B_1 - \{x_1, x_2\})$, $t \in V(B_2 - x_3)$, and $\beta_1 \in E(B_1 - x_2)$. Then G has an st–trace containing U if and only if $(B_1; s, x_1, \beta_1) \notin \mathcal{L}$, $(B_2; t, x_3, \beta_2) \notin \mathcal{L}$, and each β_i, $i \geq 3$,

is incident to an inner vertex in B_i *unless* $v(B_i) = 2$.

$(a3)$ *Let* $s, t \in V(B_1)$ *and* $\{s,t\} \neq \{x_1, x_2\}$ *and* β_2 *and* β_k *are not incident to* $\{x_1, x_2\}$. *Then* G *has an* st*–trace containing* U *if and only if each* β_i, $i \geq 2$, *is incident to an inner vertex in* B_i *unless* $v(B_i) = 2$ *and none of the following holds for* $(B_1; s, t, x_1, x_2, \beta_1)$, *where* $\beta_1 = uv$:

$(b1)$ $\{s,t\} \cap \{u,v\} \cap \{x_1,x_2\} \neq \emptyset$ *or* $\{s,t\} \cap \{u,v\} \neq \emptyset$, *say* $t = u$, *and* $(B_1 \cup x_1x_2) - \{s,u,v\}$ *is not connected,*

$(b3)$ *there is* $\{a,b\} \subseteq V(B_1 - \{t, x_1, x_2\})$ *such that* x_1 *and* x_2 *belong to different components* C_1 *and* C_2 *of* $(B_1 - x_1x_2) - \{a,b\}$, *respectively,* $\{a,b\} \cap \{u,v\} \neq \emptyset$, *and for some* $\{i,j\} = \{1,2\}$, $s = x_i$, $v(C_i) \geq 2$, *and either* $\{a,b\} = \{u,v\}$ *and* $t \notin C_i$ *or* $\{a,b\} \neq \{u,v\}$, *say* $v \notin \{a,b\}$, *and* $v \in V(C_j)$,

$(b4)$ $\{s,t\} \cap \{u,v\} \neq \emptyset$, *say* $t = u$, *and there is* $z \in V(B_1 - \{t, x_1, x_2\})$ *such that* x_1 *and* x_2 *belong to different components* C_1 *and* C_2 *of* $(B_1 - x_1x_2) - \{t,z\}$, *respectively, and for some* $i \in \{1,2\}$, $x_i = v$, $v(C_i) \geq 2$ *and* $s \notin V(C_i)$,

$(b5)$ $\{s,t\} \cap \{u,v\} \neq \emptyset$, *say* $t = u$, *and there is* $z \in V(B_1 - \{t, x_1, x_2\})$ *such that* x_1 *and* x_2 *belong to different components* C_1 *and* C_2 *of* $(B_1 - x_1x_2) - \{t,z\}$, *respectively, and for some* $\{i,j\} = \{1,2\}$, $s = x_i$, $v(C_i) \geq 2$, *and* $v \notin V(C_j)$.

Proof (uses **3.6**, **3.8**, **4.12**, **4.19**, **4.20** **4.22**, and **5.2**). The necessity of the condition in each claim (ai) for the existence of the corresponding st–trace is obvious. We prove that the condition in each (ai) is also sufficient. Let us define (G_1, G_2, e) as follows:

$(a'1)$ in case $(a1)$ $G_1 = x_1B_1x_2x_3B_3x_4x_1$, $G_2 = (G \cup x_1x_4) - ((B_1 \cup B_3) - \{x_1, x_4\})$, and $e = x_1x_4$,

$(a'2)$ in case $(a2)$ $G_1 = x_1B_1x_2B_2x_3x_2x_1$, $G_2 = (G \cup x_1x_3) - ((B_1 \cup B_2) - \{x_1, x_3\})$, and $e = x_1x_3$, and

$(a'3)$ in case $(a3)$ $G_1 = B_1 \cup x_1x_2$, $G_2 = (G \cup x_1x_2) - (B_1 - \{x_1, x_2\})$, and $e = x_1x_2$.

Since G is a {claw, net}-free ring, by **4.12**, both G_1 and G_2 are also {claw, net}-free rings. By **5.2**, G_2 has a track H containing all β_i's in G_2.

Clearly in cases $(a1)$ and $(a2)$ G has a required st–trace in G if and only if G_1 has an st–trace P containing β_1 and β_2, and $(H \cup P) - e$ is one of them. In case $(a3)$ G has a required st–trace in G if and only if G_1 has an st–trace P containing uv and x_1x_2.

We now prove that in each case G_1 has an st–trace P if the assumptions of the case are satisfied.

Consider case $(a1)$. Since G_1 is a ring, $G'_1 = G_1 - \beta_2$ is a string with the end–bricks $\ddot{B}_1$ and $\ddot{B}_3$. Since G_1 is {claw, net}-free, by **4.22**, G'_1 is also {claw, net}-free. Now by **3.6**, G'_1 (and therefore also G_1) has an st–trace P containing β_1 and β_2 if and only if the assumptions in $(a1)$ are satisfied.

Consider case $(a2)$. By **3.8**, $B_1 - x_2$ has an sx_1–trace P'_1 containing β_1 if and only if $(B_1 - x_2; s, x_1, \beta_1) \notin \mathcal{L}$. Since G is claw–free, $N(x_2, B_1)$ induces a clique in B_1. Therefore $(B_1 - x_2; s, x_1, \beta_1) \notin \mathcal{L}$ if and only if $(B_1; s, x_1, \beta_1) \notin \mathcal{L}$. By **3.8**, B_2 has a tx_2–trace P_2 containing β_2 if and only if $(B_2; s, x_2, \beta_2) \notin \mathcal{L}$. Thus G_1 has an st–trace $sP'_1x_1x_2P_2t$ if and only if assumptions of $(a2)$ are satisfied.

Now consider case $(a3)$. Since $x_1B_1x_2$ is a brick of a ring, $\bar{B}_1$ is either a triangle or 3–connected. ¿From the assumption of $(c3)$ it follows that $\bar{B}_1$ is not a triangle. Thus $\bar{B}_1$ is 3–connected. Now from **4.19** and **4.20** it follows that $B_1 \cup x_1x_2$ has no st–trace containing uv and x_1x_2 if and only if condition (bk) holds for some $k \in \{1, 2, 3, 4\}$. □

Next we will describe some results on the problem of recognizing traceable 4–tuples $(G; s, t, e)$ for {claw, net}-free 2–strings G.

7.2. *Let $G = B_1X_1 \dots X_{n-1}B_n$ be a {claw, net}-free 2–string, where $X_i = \{x_i, x'_i\}$. Let $uv \in E(B_1)$, $s, t \in V(G)$, $s \neq t$, and $\{s, t\} \cap \{u, v\} = \emptyset$. Then the following are equivalent:*

$(a1)$ G has an st–trace containing uv and

$(a2)$ the following holds:

$(a2.1)$ $\{s, t\} \neq \{u, v\}$ and

$(a2.2)$ $G - \{s, t\}$ and $G - \{u, v\}$ are connected.

Proof (uses **4.9**, **4.19**, **4.20**, and **6.6**). Obviously $(a2) \Rightarrow (a1)$. We prove $(a1) \Rightarrow (a2)$. Since G is a 2–string, by **4.9**, each $\bar{B}_i$ is either 3–connected or a triangle or a square.

Since $(G; s, t, uv)$ satisfies $(a2)$, by **6.6**, G has an st–trace P. If $uv \in E(P)$, which is the case if $\bar{B}_1$ is a triangle or a square, then we are done. So we assume that $uv \notin E(P)$ and $\bar{B}_1$ is 3–connected. Let $P_1 = P \cup B_1$, and so $uv \notin E(P_1)$.

Suppose that $\{s, t\} \cap V(B_1 - X_1) = \emptyset$. Then P_1 is an $x_1x'_1$–trace of B_1. By $(a2.2)$, $\{u, v\} \neq X_1$. Since $n \geq 2$, there are $y_1, y'_1 \in V(B_2 - X_1)$ such that $x_1y_1, x'_1y'_1 \in E(G)$.

Since $\bar{B}_1$ is 3–connected, from **4.19** applied to $y_1x_1B_1x'_1y'_1$ if $y_1 \neq y'_1$, or from **4.20** applied to $y_1x_1B_1x'_1y'_1$ if $y_1 = y'_1$ it follows that B_1 has an $x_1x'_1$–trace T containing uv. Then $(P - (P_1 - X_1)) \cup T$ is an st–trace containing uv.

Suppose that $s \in V(B_1 - X_1)$ and $t \notin V(B_1)$. Then P_1 is an sz–trace of either B_1 or $B_1 - z'$, where $\{z, z'\} = X_1$. As above, since $\bar{B}_1$ is 3–connected, it follows from **4.19** or, accordingly, from **4.20** that B_1 has an sz–trace T and $B_1 - z'$ has an sz–trace T'. If P_1 is an sz–trace B_1, then $(P - (P_1 - X_1)) \cup T$ is an st–trace of G containing uv. If P_1 is an sz–trace $B_1 - z'$, then $(P - (P_1 - X_1)) \cup T'$ is an st–trace of G containing uv.

Now suppose that $s, t \in V(B_1)$. Then $P_1 \cup x_1x'1$ is an st–trace of $\bar{B}_1$. Since $\bar{B}_1$ is 3–connected, it follows from **4.19** or, accordingly, from **4.20** that $\bar{B}_1$ has an st–trace T containing uv and $x_1x'_1$. Then $(P - (P_1 - X_1)) \cup (T - x_1x'_1)$ is an st–trace of G containing uv. □

We consider 4-tuples $(G; s, t, uv)$, where G is a graph, $uv \in E(G)$, $\{s, t\} \subseteq V(G)$, $s \neq t$, and $s \notin \{u, v\}$.

We define the set $\mathcal{T}_1 = \mathcal{T}_1^1 \cup \mathcal{T}_1^2 \cup \mathcal{T}_1^3$ of 4-tuples $(G; s, t, uv)$ as follows:

(t_1^1) $(G; s, t, uv) \in \mathcal{T}_1^1$ if and only if $t \in \{u, v\}$ and $G - \{u, v\}$ is not connected,

(t_1^2) $(G; s, t, uv) \in \mathcal{T}_1^2$ if and only if $t \in \{u, v\}$, say $t = u$, there is $x \in V(G - \{u, v\})$ such that $G - \{t, x\}$ is not connected, and $\{s, v\}$ avoids one of the components of $G - \{t, x\}$,

(t_1^3) $(G; s, t, uv) \in \mathcal{T}_1^3$ if and only if $t \in \{u, v\}$, say $t = u$, $G - \{s, v\}$ is not connected, and the component of $G - \{s, v\}$ containing t has at least two vertices.

It is easy to see that by defined of $\mathcal{L}$ in Section 3, $(G; s, t, uv) \in \mathcal{L}$ if and only if either $\{s, t\} = \{u, v\}$ and $v(G) \geq 3$ or $\{s, t\}$ belongs to one of the components of $G - \{u, v\}$ or $(G; s, t, uv) \in \mathcal{T}_1^0 \cup \mathcal{T}_1^1 \cup \mathcal{T}_1^2 \cup \mathcal{T}_1^3$.

We define the set $\mathcal{T}_2 = \mathcal{T}_2^1 \cup \mathcal{T}_2^2 \cup \mathcal{T}_2^3$ of 4-tuples $(G; s, t, uv)$ as follows: $(G; s, t, uv\} \in \mathcal{T}_2$ if and only if $t \in \{u, v\}$ and there are two connected subgraphs A and B in G such that $G = AYB$, $Y = \{y_1, y_2\} \subseteq V(G)$, $y_1 \neq y_2$, $s \in V(A - Y)$, $uv \in E(B)$, $(B - y_1y_2) - \{u, v\}$ has exactly two components C_1 and C_2, $y_1 \in C_1$, $y_2 \in C_2$, and one of the following holds:

(t_2^1) $V(C_i - x_i) \neq \emptyset$ for every $i \in \{1, 2\}$; put $(G; s, t, uv) \in \mathcal{T}_2^1$,

(t_2^2) $y_1Ay_2 = y_1A_1yA_2y_2$, $s \in V(A_i$, $y_1y_2 \notin E(B)$, and $V(C_i - x_i) \neq \emptyset$ for some $i \in \{1, 2\}$; put $(G; s, t, uv) \in \mathcal{T}_2^2$,

(t_2^3) $y_1Ay_2 = y_1sy_2$; put $(G; s, t, uv) \in \mathcal{T}_2^3$.

Let $\mathcal{T}_3$ denote denote the set of 4-tuples $(G; s, t, uv)$ such that there are two connected subgraphs A and B in G such that $G = sAxBzs$, $uv \in E(B - z)$, $V(A - \{s, x\}) \neq \emptyset$, $x \in \{u, v\}$, say $x = u$, $t \neq v$, and $B - \{t, v\}$ has exactly two components C_x and C_z such that $x \in C_x$, $z \in C_z$, and $V(C_x - x) \neq \emptyset$.

Let $\mathcal{T} = \mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}_3$.

Obviously, if G has an st–trace containing uv, then $(G; s, t, uv) \notin \mathcal{T}$. In other words, $\mathcal{T}$ is a set of non-traceable 4-tuples $(G; s, t, uv)$.

7.3. *Let $G = B_1X_1 \dots X_{n-1}B_n$ be a {claw, net}-free 2–string, where $X_i = \{x_i, x'_i\}$. Suppose that $(G; s, t, uv)$ satisfies the following condition:*

(h) $uv \in E(B_r)$, $\{u, v\} \neq \{x_r, x'_r\}$, $s \in V(B_p - X_{r-1})$, and $t \in V(B_rX_r \dots X_{n-1}B_n - X_{r-1})$, where $p < r$, and so $s \notin \{u, v\}$.

Then the following conditions $(a1)$ and $(a2)$ are equivalent:

$(a1)$ exactly one of the following holds:

$(a1.1)$ G has an st–trace containing uv provided either $p < r - 1$ or $p = r - 1$ and either $\bar{B}_p$ is 3–connected or $\bar{B}_p$ forms a triangle end–brick of G,

$(a1.2)$ G has a zt–trace containing uv for some $z \in X_{r-2}$ provided $p = r - 1$ and either $\bar{B}_p$ is a square (and so $s \in X_{r-2}$) or $\bar{B}_p$ forms a triangle inner brick of G and

$(a2)$ $(G; s, t, uv) \notin \mathcal{T}'$.

Proof (uses **3.6**, **3.7**, **4.2**, **4.5**, **4.12**, **4.13**, **4.14**, **4.16**, and **6.7**).

Obviously $(a1) \Rightarrow (a2)$. We will prove $(a2) \Rightarrow (a1)$ by induction on $v(G)$, namely, we will show that either $(a1)$ or $(a2)$ holds. If G is a diamond, then our claim is obviously true. Therefore let $v(G) \geq 5$. Let $G^* = (G; s, t, uv)$.

(**p1**) Suppose that $p = 1$, $r = 2$, $\bar{B}_2$ is 3–connected, and $\bar{B}_1 = B_1 \cup x_1x'_1$ is a triangle, and so $V(B_1) = \{s, x_1, x'_1\}$.

Let $\{i, j\} = \{1, 2\}$, $x_1 = z_1$, and $x'_1 = z_2$, and so $X_1 = \{z_1, z_2\}$. Let $G_i = G - z_i$, $H = G - s$, and $H_i = H - z_i$ for $i \in \{1, 2\}$. Since $\bar{B}_2$ is 3–connected, each H_i is 2–connected.

Since G is {claw, net}-free, $G-z$, H, each G_i and each H_i are also {claw, net}-free. Obviously $G_i = sz_jH_i$ and $N(z_j, H_i) = N(z_j, B_2 - z_i)$. Therefore since G_i is claw-free, $\dot{N}(z_j, H_i)$ is a clique.

Suppose that $\{u, v\} = \{z_1, z_2\}$. If $t \notin \{u, v\}$, then by **3.7**, G_i has an st–trace sz_jP_jt for some $i \in \{1, 2\}$, and so $sz_iz_jP_jt$ is an st–trace of G containing $z_1z_2 = uv$. If $t \in \{u, v\}$, then $G^* \in \mathcal{T}_1^1$.

Now suppose that $\{u, v\} \neq \{z_1, z_2\}$. Let $G' = G - t$.

Suppose that $t \in \{u, v\}$, say $t = u$, and so $q = r$. If $\ddot{B}_2$ is not an end–brick of G and $t \in \{x_2, x_2'\}$, then $G - \{t, t'\}$, where $\{x_1, x_1'\} = \{t, t'\}$, has two components and $\{s, v\}$ avoids one of the components, and therefore $G^* \in \mathcal{T}_1^2$. If $\ddot{B}_2$ is an end–brick of G, z_1 and z_2 belong to different components of $B - \{u, v\}$, and each of these components has at least two vertices, then $G^*) \in \mathcal{T}_2^1$. Otherwise $G' = G - t$ is 2–connected and $(G'; s, v) \notin \mathcal{P}$. By **6.7**, G' has an sv–trace P. Then $sPvu = sPvt$ is an st–trace in G containing uv. Notice that if $v = z_i$, then there is no sv–trace of G' containing sz_i.

Now suppose that $t \notin \{u, v\}$ or equivalently, $\{s, t\} \cap \{u, v\} = \emptyset$. Then $uv \in G'$. By (h), $t \notin X_1$, and so $X_1 \subseteq G'$ and X_1 is a vertex 2–cut of G'. Since $\kappa(G) = 2$, clearly G' is connected. Since G' is {claw, net}-free, by **4.5** and **4.16**, G' is either a string or a ring or a 2–string.

(**p1**.**1**) Suppose that $z_1z_2 \in E(G)$. Since $G_i = sz_jH_i$ is {claw, net}-free, $\dot{N}(z_j, H_i)$ is a clique. Let $H' = (H - z_1z_2) - \{u, v\}$.

Suppose that H' is not connected. Since $\bar{B}_2$ is 3–connected, H' has exactly two components, say C_1 and C_2, and each $z_i \in V(C_i)$. Obviously, one of the components of H' is a subgraph of B_2 and, if $\ddot{B}_2$ is not an end–brick of G, then the other component is not a subgraph of B_2 and contains a vertex from X_2. Let $D_i = C_i - z_i$. We can assume that $t \in V(D_i)$. Since $\dot{N}(z_i, B_2 - z_j)$ is a clique and $z_i \in V(C_i)$, clearly $\dot{N}(z_i, C_i) = \dot{N}(z_i, B_2 - z_j)$, and so D_i is connected. Then D_i and C_j are the two components of $H' - z_i = H_i - \{u, v\}$. Therefore since $\{s, t\} \cap \{u, v\} = \emptyset$, $(H_i; t, z_j, uv) \notin \mathcal{L}$. By **3.7**, H_i has a z_jt–trace P_j containing uv. Then $sz_iz_jP_it$ is an st–trace of G containing uv, sz_i, and z_1z_2.

Now suppose that H' is connected. Since $\dot{N}(z_i, B_2 - z_j)$ is a clique, $H' - z_i = H_i - \{u, v\}$ is connected. Therefore again $(H_i; t, z_j, uv) \notin \mathcal{L}$. By **3.7**, H_i has a z_jt–trace P_i containing uv. Then $sz_iz_jP_it$ is an st–trace of G containing both uv and z_1z_2.

(**p1**.**2**) Suppose that $z_1z_2 \notin E(G)$. Let, as above, $G' = G - t$, and so G' is either a string or a ring or a 2–string.

(**p1**.**2**.**1**.) Suppose that G' is a ring. Then each sz_i forms a one-edge brick of ring G', and so $H = G' - s$ is a string. Therefore $\ddot{B}_2$ is an end-brick of 2–string G, and so $G = B_2 \cup \{sz_1, sz_2\}$ and z_1, z_2 are inner vertices of different end–bricks in string H. Since $\{s, t\} \cap \{u, v\} = \emptyset$, clearly $uv \in E(H)$. Since $\bar{B}_2 = B_2 \cup z_1z_2$ is 3–connected, each non-trivial brick of H and each end-brick of H has an inner vertex adjacent to t in G. Since G has no induced claw centered at t, one of the following holds:

$(h1)$ H is a z_1z_2-path P with three or four edges and $N(t, G) = V(P)$,

$(h2)$ H is either $H_1(h_1 = h_2)H_2$ or $H_1h_1h_2H_2$, where for $i \in \{1, 2\}$, H_i is a block of H,

$z_i \in In(H_i)$, $H_i \cup h_i z_i$ is either an edge or a triangle or 3–conneced, and there is $t_i \in In(H_i)$ such that $tt_i \in E(G)$.

It suffices to show that for some $\{i, j\} = \{1, 2\}$, H has an $t_i z_j$–trace T_{ij} containing uv. By **3.7** $(c1)$, H has a $t_1 z_2$–trace T. If uv is a trivial block of H, then T is $t_1 z_2$–trace T_{12} in H containing uv. So we can assume that uv is not a trivial block of H, say $uv \in E(H_1)$ and H_1 is a non-trivial block. Obviously there is $h'_2 \in V(H_2)$ such that $sz_1 H_1 h_1 h'_2$ is an induced subgraph in G'. Then if $\{u, v\} \neq \{h_1, z_1\}$, then by **3.6**, H_1 has a $h_1 z_1$–trace T_1 containing uv. By **3.7** $(c1)$, $H - (H_1 - h_1)$ has an $(h_1 t_2$–trace T_2. Then $z_1 T_1 h_1 T_2 t_2$ is a required trace T_{21}.

(**p1**.**2**.**2**) Suppose that G' is a string. Let $\ddot{D}_1$ and $\ddot{D}_2$ be the end-bricks of string G'. Since $\bar{B}_2$ is 3–connected and G' is not 2–connected, either $t \in X_2$ or $t \notin V(B_1 \cup B_2)$. Therefore $B_1 \cup B_2 - t \subseteq D_1$ and D_1 is 2–connected. Let d_i be the boundary vertex of the end-brick $\ddot{D}_i$. Since G is 2–connected, there is an inner vertex t' of D_2 such that $tt' \in E(G)$. By **3.7**, $G' - (D_1 - d_1)$ has a $d_1 t'$–trace P_2. Since $\dot{N}(d_1, D_1)$ is a clique, $\{s, t\} \cap \{u, v\} = \emptyset$, and $\bar{B}_2$ is 3–connected, we have: $(D_1; s, d_1, uv) \notin \mathcal{T}_1^0 \cup \mathcal{T}_1^1 \cup \mathcal{T}_1^2 \cup \mathcal{T}_1^3$. Therefore since $\{s, d_1\} \neq \{u, v\}$, clearly $(D_1; s, d_1, uv) \notin \mathcal{L}$. Then by **3.7**, D_1 has an sd_1–trace P_1 containing uv. Hence $sP_1 d_1 P_2 t$ is an st–trace in G containing uv.

(**p1**.**2**.**3**) Now suppose that G' is a 2–string. Then $N = N(t, G - X_1) \neq \emptyset$ and $(G'; s, t', uv)$ satisfies $(h1)$ for every $t' \in N$. If there is $t' \in N$ such that $(G'; s, t', uv)$ satisfies $(a2)$, then by the induction hypothesis, G' has an st'–trace P containing uv, and so $sPt't$ is an st–trace in G containing uv. Otherwise $(G'; s, t', uv) \in \mathcal{T}'$. If $t' \in N \setminus \{u, v\}$, then $(G'; s, t', uv) \notin \mathcal{T}_1 \cup \mathcal{T}_2$. So we assume that either $t' \in N \setminus \{u, v\}$ and $(G'; s, t', uv) \in \mathcal{T}_3^1$ or $t' \in N \subseteq \{u, v\}$, and so $N(t, G) \subseteq \{z_1, z_2, u, v\}$.

We claim that in this case $G' - s$ has a $z_1 z_2$–trace L containing uv. Indeed, we can assume that $z_1 \notin \{u, v\}$. Since G' is claw-free, $N(z_1, G' - \{s, z_2\})$ induces a clique in G'. Let $z'_1 \in N(z_1, G' - \{s, z_2\})$. By (h), $\{s, t\}$ meets every component of $G - \{u, v\}$. It follows that $(G' - \{s, z_1\}; z'_1, z_2, uv) \notin \mathcal{L}$. Therefore by **3.7**, $G' - \{s, z_1\}$ has a $z'_1 z_2$–trace L' containing uv. Then $L = z_1 z'_1 L' z_2$ is a $z_1 z_2$–trace in $G' - s$ containing uv.

If $t' \in N \setminus \{u, v\}$ and $(G'; s, t', uv) \notin \mathcal{T}_1$, then $N(t, G - \{u, v\}) \cap \{z_1, z_2\} \neq \emptyset$ because $\bar{B}_2$ is 3–connected and $d(s, G') = 2$. If $N \subseteq \{u, v\}$, then also $N(t, G - \{u, v\}) \cap \{z_1, z_2\} \neq \emptyset$ because $\bar{B}_2$ is 3–connected. Let $z_1 \in N(t, G - \{u, v\})$.

If $t' \in N \subseteq \{u, v\}$ and $(G'; s, t', uv) \in \mathcal{T}_1^1$, then $\{s, t\}$ avoids one of the components of $G - \{u, v\}$; this contradicts (h). Otherwise, since $z_1 \in N(t, G)$, clearly $sz_2 L z_1 t$ is an st–trace containing uv.

(**p2**) Now suppose that the assumption of (**p1**) does not hold. We will use (**p1**) and the decomposition of 2–strings in **4.12** to prove our claim. Let $1 \leq k \leq l \leq n$, $X_k = \{f_1, f_2\}$, $X_{l-1} = \{d_1, d_2\}$, and let $F^k = B_1 X_1 \ldots X_{k-1} B_k$, $D^l = B_l X_l \ldots X_{n-1} B_n$, $F_k = F^k \cup \{ff_1, ff_2\}$, and $D_l = \{dd_1, dd_2\} \cup D^l$, where f and d are new vertices. Let $s \in F^k$ and $uv \in D^l$.

Since G is {claw, net}-free, by **4.12**, both F_k and D_l are also {claw, net}-free. In what follows, F_k and D_l will be 2–strings. Obviously $(F_k; s, f, ff_i)$ satisfies $(a2)$, i.e. $(F_k; s, f, ff_i) \notin \mathcal{T}'$. Hence if F_k is a 2–string and $s \notin \{f_1, f_2\}$, then by the induction hypothesis, F^k has

sf_i–trace sP_if_i for every $i \in \{1,2\}$. If F_k is a 2–string, then, F^k has an f_1f_2–trace P. If $s \notin \{f_1, f_2\}$ and $\bar{B}_k$ is 3–connected, then F_k is a 2–string, $F^k - f_j$ is 2–connected, and $F_k - f_j = F_i^k f_i f$, where $\{i,j\} = \{1,2\}$. Then by **3.7**, $F^k - f_j$ has an sf_i–trace P_i'.

Notice that if $t \in \{u,v\}$ and $G - t$ is not 2–connected, then $(G; s, t, uv) \in \mathcal{T}_1$. Therefore we assume that if $t \in \{u,v\}$, then $G - t$ is 2–connected.

(**p2.1**) Suppose that $\bar{B}_r$ is either a triangle or a square. Let $k = r-1$ and $l = r+1$. Since G is a 2–string, by **4.14**, both F_k and D_l are also 2–strings. We can assume that if $\bar{B}_r$ is a triangle, then $f_1d_1 \in E(G)$, and $f_2 = d_2$, and if $\bar{B}_r$ is a square, then $f_1d_1 \in E(G)$ and $f_2d_2 \in E(G)$.

(**p2.1.1**) Suppose that $\{u,v\} \neq X_{r-1}$.

Suppose that B_r forms an end–brick of 2–string G, i.e. $r = n$. Then using sf_i–trace P_i in F^k for $i \in \{1,2\}$ in case $s \notin \{f_1, f_2\}$, and sf_j–trace P_j of F^k for $\{i,j\} = \{1,2\}$ in case $s = f_i$, it is easy to obtain an st–trace of G containing uv.

Now suppose that B_r does not form an end–brick of G, i.e. $r < n$. We can assume that $uv = f_1d_1$. If $\bar{B}_r$ is a square, then F^k has an sf_1–trace P_1 and D^l has an td_1–trace Q_1, and therefore $sP_1f_1d_1Q_1t$ is an st–trace contatining $f_1d_1 = uv$. If $\bar{B}_r$ be a triangle, then by **4.13**, $\bar{B}_{r-1}$ is either a triangle or 3–connected. Hence $F^k - f_2$ has an sf_1–trace P_1'. Thus $sP_1')f_1d_1Q_1t$ is an st–trace containing $f_1d_1 = uv$.

(**p2.1.2**) Now suppose that $\{u,v\} = X_{r-1}$.
Then by **3.7**, D^l has a td_2–trace Q_2. By **4.13**, $\bar{B}_{r-1}$ is either a triangle or 3–connected, and if $\bar{B}_r$ is a square, then $\bar{B}_{r-1}$ is 3–connected. Therefore by **3.7**, $F^k - f_2$ has an sf_1–trace P_1'. Now if $\bar{B}_r$ is a triangle, then $sP_1'f_1(f_2 = d_2)Q_2t$ is an st–trace of G containing $f_1f_2 = uv$. If $\bar{B}_r$ is a square, then $sP_1'f_1f_2d_2Q_2t$ is an st–trace of G containing $f_1f_2 = uv$.

(**p2.2**) Now suppose that $\bar{B}_r$ is 3–connected. Let $l = r$. Since G is a 2–string, by **4.14**, D_r is also a 2–string.

If $(D_r; d, t, uv)$ satisfies $(a2)$, then by (**p1**), D^r has a d_it–trace Q_i containing uv for some $i \in \{1,2\}$.

If $(D_r; d, t, uv)$ does not satisfy $(a2)$, then $(D_r; d, t, uv) \in \mathcal{T}'$. Since d has degree two in D_r, clearly $(D_r; d, t, uv) \notin \mathcal{T}_1^3 \cup \mathcal{T}_3^1$. If $(D_r; d, t, uv) \in \mathcal{T}_1$, then also $(G; d, t, uv) \in \mathcal{T}_1$. If $(D_r; d, t, uv) \in \mathcal{T}_2^1$, then also $(G; d, t, uv) \in \mathcal{T}_2^1$. If $(D_r; d, t, uv) \in \mathcal{T}_2^2$, then also $(G; d, t, uv) \in \mathcal{T}_2^2$. Therefore $(D_r; d, t, uv) \in \mathcal{T}_2^3$, i.e. $t \in \{u,v\}$, $D^r - \{u,v\}$ has exactly two components C_1 and C_2, $d_i \in C_i$, and $D^r = B_r$. If $V(C_i - d_i) \neq \emptyset$ for every $i \in \{1,2\}$, then $(G; s, t, uv) \in \mathcal{T}_2^1$. Therefore we assume that in this case $V(C_i) = d_i$ for some $i \in \{1,2\}$.

(**p2.2.1**) Suppose that $\bar{B}_{r-1}$ is 3–connected. Let $k = r-1$ and, as above, $l = r$. Then $\{f_1, f_2\} = X_k = \{d_1, d_2\}$, say $f_1 = d_1$ and $f_2 = d_2$. By (h), $s \notin \{f_1, f_2\}$. Since G is a 2–string, by **4.14**, F_k is also a 2–string. Therefore, as above, F^k has an sf_i–trace P_i and $F^k - f_j$ has an sf_i–trace P_i' for every $\{i,j\} = \{1,2\}$.

If $(D_r; d, t, uv)$ satisfies $(a2)$, then $sP_i'(f_i = d_i)Q_it$ is an st–trace in G containing uv. So we assume that $(D_r; d, t, uv)$ does not satisfy $(a2)$. Then, as it is shown at the beginning of **(p2.2)**, $t \in \{u,v\}$, $D^r - \{u,v\}$ has exactly two components C_1 and C_2, $d_j \in V(C_j)$,

$D^r = B_r$, and $V(C_i) = d_i$ for some $i \in \{1,2\}$, say for $i = 2$. Since $\bar{B}_r$ is 3–connected, clearly $D' = D^r - d_2$ is 2–connected, $D' - \{u,v\}$ is connected, and $D_r - d_2 = dd_1D'$. Since $\{d_1, t\} \neq \{u, v\}$, $D' - \{u, v\} = C_1$ is connected, G has no induced claw centered at t, clearly $(D'; t, d_1, uv) \notin \mathcal{L}$. By **3.7**, D' has an td_1–trace Q containing uv. Then sP_1d_1Qt is an st–trace in G containing uv.

(**p2.2.2**) Now suppose that $\bar{B}_{r-1}$ is either a triangle or a square. Let $k = r - 2$ and $l = r$. We can assume that if $\bar{B}_{r-1}$ is a triangle, then $f_1d_1 \in E(G)$ and $f_2 = d_2 \neq s$, and if $\bar{B}_k$ is a square, then $f_1d_1 \in E(G)$ and $f_2d_2 \in E(G)$. Since G is a 2–string, by **4.14**, F_k is also a 2–string. Therefore, as above, for every $\{i,j\} = \{1,2\}$ if $s \neq f_i$, then F^k has an sf_i–trace P_i and if $s \notin \{f_1, f_2\}$, then $F^k - f_j$ has an sf_i–trace P'_i.

(**p2.2.2.1**) Suppose that $s \notin \{f_1, f_2\}$.

Suppose $(D_r; d, t, uv)$ satisfies $(a2)$. Then by (**p1**), D^r has a td_i–trace Q_i containing uv for some $i \in \{1,2\}$. If $\bar{B}_{r-1}$ is a square, then $sP_if_id_iQ_it$ is an st–trace in G containing uv. If $\bar{B}_{r-1}$ is a triangle and $i = 1$, then $sP'_1f_1d_1Q_1t$ is an st–trace contatining uv. If $\bar{B}_{r-1}$ is a triangle and $i = 2$, then $sP_2(f_2 = d_2)Q_2t$ is an st–trace contatining uv.

Now suppose that $(D_r; d, t, uv)$ does not satisfy $(a2)$. As it is shown at the beginning of **(p2.2)**, $t \in \{u,v\}$, $D^r - \{u,v\}$ has exactly two components C_1 and C_2, $d_j \in C_j$, $D^r = B_r$, and $V(C_i) = d_i$ for some $i \in \{1,2\}$. If $V(C_1) = d_1$, then, as in **(p2.2.1)**, G has an st–trace in G containing uv. If $V(C_2) = d_2$, then $(G^* \in \mathcal{T}_2^2$.

(**p2.2.2.2**) Now suppose that $s \in \{f_1, f_2\}$. We can assume that $s = f_1$.

Suppose that D^r has a td_2–trace Q_2 containing uv. If $\bar{B}_{r-1}$ is a triangle, then $sP_2(f_2 = d_2)Q_2t$ is an st–trace contatining uv. If $\bar{B}_{r-1}$ is a square, then $sP_2f_2d_2Q_2t$ is an st–trace in G containing uv.

Now suppose that D^r has no td_2–trace containing uv. Then by **3.7** applied to $G - f_1$, $(D^r; d_2, t, uv) \in \mathcal{L}$, i.e. either $\{d_2, t\} \neq \{u, v\}$ or $(D^r; d_2, t, uv) \in \mathcal{T}_1^0 \cup \mathcal{T}_1^1 \cup \mathcal{T}_1^2 \cup \mathcal{T}_1^3$. If $(D^r; d_2, t, uv) \in \mathcal{T}_1^0$, then $G^* \in \mathcal{T}_1^0$, a contradiction. If $(D^r; d_2, t, uv) \in \mathcal{T}_1^i$, then either G has an induced claw centered at d_2 or $G^* \in \mathcal{T}_1^i$ for $i \in \{1,2,3\}$, a contradiction. Therefore $\{d_2, t\} = \{u, v\}$, say $d_2 = u$ and $t = v$. Then $G - \{s, u\}$ is not connected and the component containing v has at least two vertices. Therefore $G^* \in \mathcal{T}_1^3$. □

Next we will discuss the problem of recognizing traceable 4–tuples $(G; s, t, e)$ for 3–connected {claw, net}-free graphs G.

We start by giving a short proof of the following strengthening of **3.3**.

7.4. *Let G be a 3-connected {claw, net}-free graph, $xz \in E(G)$, and $y \in V(G - \{x,z\}$. Then G has an $\{x,y\}$–trace containing xz if and only if $G - \{x,y,z\}$ is connected.*

Proof (uses **3.3**, **6.1**, and **6.4**). Obviously if G has a xy–trace containing xz, then $\{x,y,z\}$ is not a 3–cut of G. We prove by induction on $v(G)$ that if $G - \{x,y,z\}$ is connected then G has an $\{x,y\}$–trace containing xz. Obviously our claim is true for K_4. So we assume that $v(G) \geq 5$. Let $G_x = G - x$. Then $y, z \in V(G_x)$ and $y \neq z$. Since G is 3–connected and {claw, net}-free, clearly G_x is 2–connected and {claw, net}-free. If G_x has a yz–trace P, then $xzPy$ is a xy–trace in G containing xz. Therefore it is sufficient to show that G_x has a yz–trace.

If G_x is 3–connected, then by **3.3**, G_x has a yz–trace. Therefore we assume that G_x is not 3–connected, and so $\kappa(G_x) = 2$. Since $\{x, y, z\}$ is not a 3–cut of G, clearly $G_x - \{y, z\}$ is connected, and so $(G_x; y, z) \in \mathcal{P}_1$. By **6.4**, $\mathcal{P}(G_x) = \mathcal{P}_1(G_x)$. Therefore by **6.1**, G_x has a yz–trace. □

Now we are ready to give a characterization of traceable 4–tuples $(G; s, t, e)$ for 3–connected {claw, net}-free graphs G.

7.5. *Let G be a 3–connected {claw, net}-free graph, $uv \in E(G)$, and $s, t \in V(G)$, $s \neq t$. Then the following are equivalent:*

$(a1)$ G has an st–trace containing $e = uv$ and

$(a2)$ $\{s, t\} \neq \{u, v\}$ and either $\{s, t\} \cap \{u, v\} = \emptyset$ or $\{s, t\} \setminus \{u, v\} = z$ and $G - \{z, u, v\}$ is connected.

Proof (uses **3.7**, **3.13**, **4.16**, **4.17**, **6.1**, **6.7**, **7.3**, and **7.4**). Obviously, $(a1) \Rightarrow (a2)$. We will prove $(a2) \Rightarrow (a1)$ by induction on $v(G)$, namely, we will show that either $(a1)$ or $(a2)$ holds. If $v(G) \leq 4$, then our claim is obviously true. So we assume that $v(G) \geq 5$. If $\{s, t\} \cap \{u, v\} = z$ and $G - \{z, u, v\}$ is connected, then by **7.4**, $(a1)$ holds, i.e. G has an st–trace containing uv. So let $\{s, t\} \cap \{u, v\} = \emptyset$. Let $G' = G - s$. Since G is {claw, net}-free, G' is also {claw, net}-free. It is sufficient to show that there is a vertex s' adjacent to s in G such that G' has an $s't$–trace containing uv. By **4.16**, G' is either 3–connected or a ring or a 2–string. Let $E(G) = E$.

(**p1**) Suppose that $G' = B_1 X_1 \ldots X_{k-1} B_k$ is a 2–string and $uv \in E(B_r)$. We can assume that $t \in V(B_r X_1 \ldots X_{k-1} B_k - X_{r-1})$ for $r \geq 2$ (otherwise we could exchange the role of s and t). Since G is 3–connected, s is adjacent in G to an inner vertex s_i of end–brick $\ddot{B}_i$ of 2–string G', $i \in \{1, k\}$, and if $\bar{B}_i$ is a square, then s is adjacent to both vertices of degree two in B_i. By **7.3**, G' has an inner vertex s_1 of B_1 such that $ss_1 \in E$ and G' has an s_1t–trace P containing uv if and only if $(G'; s_1, t, uv) \in \mathcal{T}'$.

Since s_1 is an inner vertex of an end–brick of G, clearly $(G'; s_1, t, uv) \notin \mathcal{T}_1^3 \cup \mathcal{T}_3^1$. If $(G'; s_1, t, uv) \in \mathcal{T}_1^1$, then $G - \{s, u, v\}$ is not connected. If $(G'; s_1, t, uv) \in \mathcal{T}_1^2$, then $t \in X_r$. Then $(G'; s_k, t, uv) \notin \mathcal{T}$ and $(G'; s_k, t, uv)$ satisfies assumption (h) of **7.3**. By **7.3**, G' has an $s_k t$–trace containing uv. Thus let $(G'; s_1, t, uv) \in \mathcal{T}_2$. Put $x_{k-1} = x$ and $x'_{k-1} = x'$. Then $t \in \{u, v\}$, $B_r = B_k$, $\bar{B}_r$ is 3–connected, and x, x' belong to different components C, C' of $(B_r - xx') - \{u, v\}$, say $x \in C$ and $x' \in C'$. Let $\{t, t'\} = \{u, v\}$.

(**p1.1**) Suppose that $N(s, G) \cap V(B_r) \subseteq \{x, x', u, v\}$. Then $s_k \in \{u, v\}$.

Suppose that $V(C - x) \neq \emptyset$ and $V(C' - x') \neq \emptyset$. Then there are $c \in V(C - x)$ and $c' \in V(C' - x')$ such that $uc, uc' \in E$. Hence $\{u; c, c', s\}$ induces a claw in G centered at u, a contradiction.

Suppose that $V(C - x) = \emptyset$ and $V(C' - x') \neq \emptyset$. Since $\bar{B}_r$ is 3–connected, there is $c' \in V(C' - x')$ such that $c'u \in E$. If $sx \notin E$, then $\{u, c', s, x\}$ induces in G a claw centered at u, a contradiction. So let $sx \in E$. Then by **6.7**, $G - \{s,t\}$ has an xt'–trace P. Then $sxPt't$ is an st–trace in G containing $tt' = uv$.

Now suppose that $V(C-x) = \emptyset$ and $V(C'-x') = \emptyset$. If $xx' \notin E$ and $sz \notin E$ for every $z \in \{x, x'\}$, then $\{u; s, x, x'\}$ induces a claw in G centered at u, a contradiction. If $sz \in E$ for some $z \in \{x, x'\}$, then as above, G has an st–trace containing uv. So let $xx' \in E$. Let $H = G' - (B_r - x)$. Then $G' - (B_r - \{t', x\}) = Hxt'$ is a string and s_1 is an inner vertex of the end-brick of Hxt' avoiding u. By **3.7**, H' has an s_1x-trace P. Then $ss_1Pxx't't$ is an st–trace of G containing $tt' = uv$.

(**p1.2**) Now suppose that there is $z \in N(s,G) \cap V(B_r - \{x, x', u, v\})$. We can assume that $z \in V(C-x)$. Then $S = (B_r - xx') - t$ is a string, x and x' are inner vertices of different end–bricks of S, and t' is a cut vertex of S. Since $\bar{B}_r$ is 3–connected and G has no induced claw centered at t, S has at most three bricks, at most two of them are non-trivial, and each non-trivial brick is an end-brick of S. Since $z \neq x$ and t' is a cut vertex of S, by **6.7**, $G - \{s,t\}$ has a zt'–trace P. Then $szPt't$ is an st–trace in G containing $tt' = uv$.

(**p2**) Suppose that G' is a ring. Then by **4.17**, each non-trivial brick B_i of G' has an inner vertex b_i in $N(s,G)$ and one of the following holds:
($d1$) G' is a cycle with at most five vertices,
($d2$) $G' = x_1B_1xB_2x_2x_1$, where B_1 and B_2 form non-trivial bricks,
($d3$) $G' = x_1B_1x_2x_3x_4x_1$, where B_1 forms a non-trivial brick,
($d4$) $G' = x_1B_1x_2x_3B_2x_4x_1$, where B_1 and B_2 form non-trivial bricks.

If ($d1$) holds, then obviously G has an st–trace containing uv.

(**p2.1**) Suppose that ($d2$) holds. We can assume that $t \in V(B_1)$. If a, b, c are different vertices and e is an edge in B_i, then let bP_ic denote a bc–trace in B_i, $bP_i(a)c$ a bc–trace in $B_i - a$, and $bP_i[e]c$ a bc-trace in B_i containing e. By **3.13**, B_i has a bc–trace $bP_i(a)c$ for $\{a,c\} = \{x, x_i\}$ and $b \notin \{x, x_i\}$. By **3.7**, B_i has a bc–trace bP_ic for $c \in \{x, x_i\}$ and $b \neq c$. If $e = uv$, $c \in \{x, x_i\}$, $b \neq c$, and $(B_i; b, c, uv) \notin \mathcal{L}$, then by **3.7**, B_i has a bc–trace $bP_i[e]c$.

(**p2.1.1**) Suppose that $\{u,v\} = \{x_1, x_2\}$. If $t \in \{x_1, x_2\}$, say $t = x_1$, then $sb_1P_1(x_1)xP_2x_2t$ is an st–trace containing uv. If $t \notin \{x_1, x_2\}$, say $t \in V(B_1 - x_1)$, then $sb_2P_2(x)x_2x_1P_1t$ is an st–trace containing uv.

(**p2.1.2**) Suppose that $uv \in E(B_1)$ and $\{u,v\} = \{x, x_1\}$. Then $G - \{u,v\}$ is not connected, and so if $t \in \{u,v\}$, then (G, uv, s, t) does not satisfy ($a2$), a contradiction. Therefore $t \notin \{u,v\}$. Then $sb_2P_2(x)x_2x_1xP_1(x_1)t$ is an st–trace containing uv.

(**p2.1.3**) Suppose that $uv \in E(B_1)$ and $\{u,v\} \neq \{x, x_1\}$.

(**p2.1.3.1**) Suppose that $B_1 - \{u,v\}$ is connected (which is the case if $\{u,v\} \cap \{x, x_1\} \neq \emptyset$ because $\bar{B}_1$ is 3–connected). If $x_1 \notin \{u,v\}$, then $sb_2P_2xP_1[e]t$ is an st–trace containing uv. If $x \notin \{u,v\}$, then $sb_2P_2(x)x_2x_1P_1[e]t$ is an st–trace containing uv.

(**p2.1.3.2**) Suppose that $B_1 - \{u,v\}$ is not connected. Since $\bar{B}_1$ is 3–connected, $B_1 - \{u,v\}$ has exactly two components C and C_1 with $x \in C$ and $x_1 \in C_1$.

Suppose that $t \in C$. Then t and x_1 are in different components of $B_1 - \{u, v\}$ and $(B_1; t, x_1, e) \notin \mathcal{L}$. Hence $sb_2P_2(x)x_2x_1P_1[e]t$ is an st–trace containing uv.

Suppose that $t \in C_1$. Then t and x are in different components of $B_1 - \{u, v\}$ and $(B_1; t, x, e) \notin \mathcal{L}$. Hence $sb_2P_2xP_1[e]t$ is an st–trace containing uv.

Suppose that $t \in \{u, v\}$. Then $G' - t$ is a ring. Let $N_1 = N(s, G) \cap V(B_1)$.

Suppose that $N_1 \subseteq \{u, v, x, x_1\}$. Then $b_1 \in \{u, v\}$. If either $x, x_1 \notin N_1$ or $x_1 \notin N_1$ and $C \neq x$ or $x \notin N_1$ and $C_1 \neq x_1$ or $C \neq x$ and $C_1 \neq x_1$, then b_1 is the center of an induced claw in G, a contradiction. If $C = x$, then $t'x$ forms a brick of $G' - t$. By **6.1**, $G' - t$ has a $t'b_2$–trace P. Then sb_2Pvt is an st–trace containing uv. If $C \neq x$, then by the above argumemts, $C_1 = x_1$ and $s \in N_1$. Then $t'x_1$ forms a brick of $G' - t$. By **6.1**, $G' - t$ has a vx_1–trace P_1. Then $sx_1P_1t't$ is an st–trace containing uv.

Now suppose that there is $b_1 \in N_1 \setminus \{u, v, x, x_1\}$. Then as in **(1.2)**, $S = B_1 - t$ is a string, x and x_1 are inner vertices of different end–bricks of string S, and t' is a cut vertex of S. Since $\bar{B}_1$ is 3–connected, every non-trivial brick and every end–brick of S has an inner vertex adjacent to t. Since G has no induced claw centered at t, S has at most three bricks, at most two of them are non-trivial, and each non-trivial brick is an end-brick of S. Therefore by **6.1**, ring $G' - t$ has a b_1t'–trace P. Then $sb_1Pt't$ is an st–trace in G containing $tt' = uv$.

(**p2.1.4**) Now suppose that $uv \in E(B_2)$ and $t \neq x$. This case can be reduced to the case (**p2.1.3**) by putting $t := b_2$ and $b_2 := t$.

(**p2.2**) Now suppose that either ($d3$) or ($d4$) holds, and so G' has four or five bricks. These cases are similar to case (**p2.1**).

(**p3**) Now suppose that G' is 3–connected. If there is a vertex $s' \in N(s, G)$ such that (G', s', t, uv) satisfies ($a2$), then by the induction hypothesis, G' has an $s't$–trace P containing uv, and so $ss'Pt$ is an st–trace in G containing uv. Therefore it is sufficient to show that there is a vertex $s' \in N(s, G)$ such that (G', s', t, uv) satisfies ($a2$).

Since G is 3–connected, $N(s, G) - \{u, v\} \neq \emptyset$. If there is $y \in N(s, G) - \{u, v, t\}$ such that $(G'; y, t, uv)$ satisfies ($a2$), then we are done. So suppose not, i.e. $(G'; y, t, uv) \in \mathcal{T}$ for every $y \in N(s, G) - \{u, v, t\}$. Since G is 3–connected, there is $y \in N(s, G) - \{u, v, t\}$ such that $(G'; y, t, uv) \notin \mathcal{T}_1 \cup \mathcal{T}_3$. Therefore $(G'; y, t, uv) \notin \mathcal{T}_1 \cup \mathcal{T}_3$ for every $y \in N(s, G) - \{u, v, t\}$. Then $t \in \{u, v\}$ and $G' - \{u, v, y\}$ is not connected for every $y \in N(s, G) - \{u, v, t\}$.

Since G' is claw-free and $\{u, v, y\}$ is a minimal vertex cut of G', by **4.1**, $G' - \{u, v, y\}$ has exactly two components, say A_1 and A_2. If $N(s, G) = \{u, v, y\}$, then $G - \{u, v, y\}$ has at least three components. Then by **4.1**, G has an induced claw, a contradiction. Therefore there is $a_i \in V(A_i) \cap (N(s, G) \setminus \{u, v, y\})$ for some $i \in \{1, 2\}$, say for $i = 1$, and so $a_1 \neq t$. Since $(G'; a_1, t, uv)$ does not satisfy ($a2$), $G' - \{a_1, u, v\}$ is not connected. Let B_1 be the component of $G' - \{a_1, u, v\}$ avoiding y. Since $a_1 \in A_1$, obviously B_1 is an induced subgraph of A_1.

We claim that $G' - \{a_1, y, z\}$ is connected for every $z \in \{u, v\}$, which implies that $a_1y \in E$ and $A_1 - a_1 = B_1$. Indeed, suppose that this is not the case for some $z \in \{u, v\}$, say for $z = u$. Let C be the component of $G' - \{a_1, y, u\}$ avoiding v. Then u is adjacent to vertices a, b, c in A_2, B_1, and C, respectively. Hence $\{u; a, b, c\}$ induces a claw in G centered at u, a contradiction.

We also claim that $N(s,G) \cap V(A_1) = a_1$. Suppose not, i.e. there is $b \in N(s,G) \cap V(B_1)$. Since $(G'; b, t, uv)$ does not satisfy $(a2)$, $G' - \{b, u, v\}$ is not connected. Then $G' - \{b, u, y\}$ is also not connected, and so, as above, u is the center of an induced claw in G, a contradiction.

Our next claim is that there is $a_2 \in N(s,G) \cap V(A_2)$. Suppose not, i.e. $N(s,G) \cap V(A_2) = \emptyset$. Since $d(s,G) \geq 3$, there is $z \in N(s,G) \cap \{u,v\}$. Since $\{a_1, u, v\}$ is a minimal cut of G', there is $z_1 \in N(z,G) \cap V(B_1)$. Since $\{u, v, y\}$ is a minimal vertex cut in G', there is $z_2 \in N(z,G) \cap V(A_2)$. Then $\{z; z_1, z_2, s\}$ induces a claw in G centered at z, a contradiction.

Now by the above arguments for (A_1, a_1), applied to (A_2, a_2), we obtain $a_2 y \in E$, $N(s,G) \cap V(A_2) = a_2$, and $B_2 = A_2 - a_2$ is a component of $G' - \{a_2, u, v\}$. Since G is 3–connected, $xy \in E(G)$ for some $x \in \{u, v\}$, say for $x = u$. Since $\{a_i, u, v\}$ is a minimal cut of G', there is $x_i \in N(z,G) \cap V(B_i)$ for $i \in \{1,2\}$. Then $\{x; x_1, x_2, y\}$ induces a claw in G centered at z, a contradiction. $\square$

8 More on Hamiltonicity of {claw, net}-free graphs

In this section we give generalizations or variations of some of the above results.

Using **3.9** and **7.4**, it is easy to prove the following generalization of **6.8**.

8.1. *Let G be a k–connected {claw, net}-free graph, $k \geq 2$, L be a path in G, and $v(L) \leq k$. Then the following are equivalent:*

$(a1)$ G has a track containing L and

$(a2)$ $G - L$ is connected.

8.2. *Let G be a connected {claw, net}-free-free graph and L a path in G. Suppose that $V(L)$ is a minimal vertex cut of G. Then G has a trace containing L.*

Proof (uses **3.7** and **4.1**). Let x_1 and x_2 be the end–vertices of L. Since G is {claw, net}-free, by **4.1**, $G-L$ has exactly two components, say C_1 and C_2. Let $G_i = G-(L-x_j)$, where $\{i, j\} = \{1, 2\}$. Since $V(L)$ is a minimal vertex cut of G, clearly each G_i is connected and x_i is a cut vertex of G_i. By **3.7**, G_i has a trace $c_1 P_i c_2$, where $c_j \in C_j$. Then $c_1 P_1 x_1 L x_2 P_2 c_2$ is a trace in G containing L. $\square$

Now we can obtain the following generalization of **6.9** for k–connected graphs.

8.3. *Let G be a k–connected {claw, net}-free graph, $k \geq 1$, L a path in G, and $v(L) \leq k$. Then G has a trace containing L.*

Proof (uses **3.1**, **8.1**, and **8.2**). If $v(L) = 1$, then our claim follows from **3.1**. So let $k \geq 2$ and $v(L) \geq 2$.

Suppose that $G - L$ is not connected. Since G is k–connected, clearly $v(L) = k$, and so $V(L)$ is a minimal vertex cut of G. Then by **8.2**, G has a trace containing L.

Now suppose that $G - L$ is connected. Then by **8.1**, L belongs to a track of G, and therefore also to a trace of G. □

There are infinitely many examples showing that the claim of **8.3** is not true if $v(L) = k + 1$.

The following is a strengthening of **8.1**.

8.4. *Let G be a k–connected {claw, net}-free graph, $k \geq 2$, $e = uv \in E(G)$, and L a path in G with $v(L) \leq k - 1$. Then the following are equivalent:*

($c1$) G has a track containing e and L and

($c2$) the subgraph L' in G, induced by $E(L) \cup e$, is either a path or not connected (i.e. has two components L and uev) and if L' is a path, then $G - L'$ is connected.

Proof (uses **7.5** and **8.1**). Obviously ($c1$) ⇒ ($c2$). We prove ($c2$) ⇒ ($c1$). If $k = 2$ or $v(L) = 1$, then our claim follows from **8.1**. Therefore let $k \geq 3$. Let s and t be the end–vertices of L and $G' = G - V(L - \{s, t\})$. Since G is {claw, net}-free, G' is also {claw, net}-free. Since $v(L) \leq k - 1$, clearly $|V(L) - \{s, t\}| \leq k - 3$. Since G is k–connected, G' is 3–connected. Obviously ($c2$) implies that $(G'; s, t, uv)$ satisfies ($a2$) in **7.5**. Therefore by **7.5**, G' has an st–trace P containing uv. Then $P \cup L$ is a track in G containing both e and L. □

Using **8.4**, it is easy to prove the following.

8.5. *Let G be a k–connected {claw, net}-free graph, $k \geq 5$, C_1 and C_2 disjoint cycle in G, and $v(C_1 \cup C_2) \leq k + 1$. Then for every $e_1 \in E(C_1)$ and $e_2 \in E(C_2)$ there exists a track in G containing $C_1 - e_1$ and $C_2 - e_2$, and therefore avoiding $\{e_1, e_2\}$.*

8.6. (see **1.7**) *Let G be a k–connected {claw, net}-free graph, $k \geq 2$, $e = uv$ an edge and L an xy–path in G, $v(L) \leq k - 1$, and $\{u, v\} \cap V(L) \subset \{x, y\}$. Then G has a trace containing e and L.*

Proof Suppose that $V(L) \cap \{u, v\} \neq \emptyset$, i.e. $L \cup uv$ induces a path, say xPv in G. Then $v(P) \leq k$. Let G' be the graph obtained from G by adding a new vertex c and each edge cz, $z \in V(G)$. Since G is {claw, net}-free, G' is also {claw, net}-free. Since G is k–connected, G' is $(k + 1)$–connected. By **8.4**, G' has a track, say C, containing P. Then $C - c$ is a trace in G containing $L \cup e$.

Now suppose that $V(L) \cap \{u, v\} = \emptyset$. Then by **8.4**, G has a track, say D, containing $L \cup e$. Then $D - d$ is a trace of G containing $L \cup e$ for every $d \in E(D - e) \setminus E(L)$. □

There are infinitely many $(k + 1)$–connected {claw, net}-free graphs G with two disjoint paths S and L such that $v(S) + v(L) = k + 1$, $v(P) \geq 3$, $v(L) \geq 3$, and G has no track containing $S \cup L$. Therefore the claims of **8.4** and **8.6** are not true if condition "*$e \in E(G)$ and L is a path in G with $v(L) \leq k - 1$*" is replaced by condition "*S and L are disjoint paths in G such that $v(S) + v(L) = k + 1$*".

8.7. *Let G be a k–connected {claw, net}-free graph, $k \geq 3$, C be a cycle in G, $v(C) \leq k-1$, $e = uv \in E(C)$, and $s, t \in V(G)$, $s \neq t$. Then the following are equivalent:*

$(c1)$ G has an st–trace containing $C - e$, and therefore avoiding e, and

$(c2)$ (G, C, s, t, uv) satisfies exactly one of the following conditions:

$(c2.1)$ $\{s,t\} \cap V(C) = \emptyset$,

$(c2.2)$ $V(C) \cap \{s,t\} = z \in \{u,v\}$ and $G - (C \cup (\{s,t\} - z))$ is connected.

Proof (uses **7.5**). Obviously $(c1) \Rightarrow (c2)$. We prove $(c2) \Rightarrow (c1)$. Let $G' = G-(C-\{u,v\})$. Obviously G' is {claw, net}-free and G has an st–trace containing $C - e$ if and only if G' has an st–trace containing $e = uv$. By **7.5**, G' has an st–trace containing uv if and only if (G', s, t, uv) satisfies $(a2)$ in **7.5**. Obviously (G', s, t, uv) satisfies $(a2)$ if and only if (G, C, s, t, uv) satisfies $(c2)$. □

It is easy to see that the following is true for every connected graph.

8.8. *Let G be a connected graph, L_1 and L_2 two disjoint paths in G, and x_i is an end–vertex of L_i, $i \in \{1,2\}$. If $G - (L_1 \cup L_2)$ is not connected, then G has no x_1x_2–trace containing $L_1 \cup L_2$.*

The following is a generalization of **7.4** for k–connected {claw, net}-free graphs that shows, in particular, that the converse of **8.8** is also true if $v(L_1) + v(L_2) \leq k$.

8.9. *Let G be a k–connected {claw, net}-free graph, $k \geq 3$, L_1 and L_2 be two disjoint paths in G, $v(L_1) + v(L_2) \leq k$, and x_i be an end–vertex of L_i, $i \in \{1,2\}$. Then the following are equivalent:*

$(a1)$ G has an x_1x_2–trace containing $L_1 \cup L_2$,

$(a2)$ G has a z_1z_2–trace containing $L_1 \cup L_2$ for every end–vertex z_i of L_i, $i \in \{1,2\}$, and

$(a3)$ $G - (L_1 \cup L_2)$ is connected.

Proof (uses **7.4**). Obviously $(a1) \Leftrightarrow (a3)$ implies $(a1) \Leftrightarrow (a2)$. As we mentioned above, $(a1) \Rightarrow (a3)$. We prove $(a3) \Rightarrow (a1)$ by induction on k. By **7.4**, our claim is true for $k = 3$. So we assume that $k \geq 4$ and, therefore, that $v(L_1) \geq 2$. Let $G' = G - x_1$ and $x_1x_1' \in E(L_1)$. Since G is k–connected and {claw, net}-free, clearly G' is $(k-1)$–connected and {claw, net}-free. By $(a2)$, $V(L_1 \cup L_2)$ is not a vertex cut of G. Therefore $V(L_1 \cup L_2) \setminus x_1$ is not a vertex cut of G'. By the induction hypothesis, G' has an $x_1'x_2$–trace P containing $(L_1 \cup L_2) - x_1$. Then $x_1x_1'Px_2$ is an x_1x_2–trace P containing $L_1 \cup L_2$. □

8.10. *Let G be a graph, $uv \in E(G)$, L an xt–path in G, $V(L) \cap \{u,v\} = \emptyset$, and $s \in V(G-L)$. Suppose that G is k–connected and {claw, net}-free, and $k \geq v(L) + 2$. Then the following are equivalent:*

$(a1)$ G has an st–trace containing L and uv and

$(a2)$ $G - (L \cup \{u,v\})$ is connected.

Proof (uses **7.5**). Obviously $(a1) \Leftrightarrow (a2)$. We prove $(a2) \Rightarrow (a1)$. If $s \in \{u, v\}$, then by **8.9**, our claim is true. So let $s \notin \{u, v\}$. If $v(L) = 1$, then by **7.5**, our claim is true. Now our claim can be easily proved by induction on $v(L)$. □

¿From **8.10** we have, in particular:

8.11. *Let G be a graph, x_1x_2 and y_1y_2 two disjoint edges in G, s and t two vertices of G, and $|\{s,t\} \cap \{x_1, x_2, y_1, y_2\}| \le 1$. Let ρ be the distance between $\{s,t\}$ and $\{x_1, x_2, y_1, y_2\}$ in G. Suppose that G is k–connected and {claw, net}-free and $k \ge \rho + 4$. Then G has an st–trace containing x_1x_2 and y_1y_2. In particular, if $|\{s,t\} \cap \{x_1, x_2, y_1, y_2\}| = 1$ and G is a 4–connected {claw, net}-free graph, then G has an st–trace containing x_1x_2 and y_1y_2.*

As we mentioned before, our proofs provide polynomial–time algorithm for solving the corresponding Hamiltonicity problems for {claw, net}-free graphs.

Given a graph G, two different vertices s, t and two disjoint edges e_1 and e_2 in G, we call $(G; s, t, e_1, e_2)$ *traceable*, if G has an st–trace containing e_1 and e_2. Theorems **8.9** and **8.11** suggest the following questions.

8.12. Problem. *Is there a polynomial–time algorithm to recognize traceable 5–tuples $(G; s, t, e_1, e_2)$, where G is a {claw, net}-free 4–connected graph (or {claw, net}-free k–connected graph for some fixed k) and $\{s, t\}$ is not incident to $\{e_1, e_2\}$?*

8.13. Problem. *Is there a positive integer k such that $(G; s, t, e_1, e_2)$ is traceable if G is a k–connected {claw, net}-free graph and $\{s, t\}$ is not incident to $\{e_1, e_2\}$?*

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