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APPROXIMATIONS FOR AND CONVEXITY OF PROBABILISTICALLY CONSTRAINED PROBLEMS WITH RANDOM RIGHT-HAND SIDES

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Abstract. We consider probabilistically constrained problems, in which the multivariate random variables are located in the right-hand sides. The objective function is linear, and its optimization is subject to a set of linear constraints as well as a joint probabilistic constraint enforcing that the joint fulfillment of a system of linear inequalities with random right-hand side variables be above a prescribed probability level p. To deal with such complex problems, we describe a solving method based on the p-efficiency concept for discretely distributed random variables, and also propose some alternative formulations applicable to both discrete and continuous probability distributions, and involving the substitution of the joint probabilistic constraint by a set of individual constraints, the Boole's inequality, the binomial moment bounding scheme, and Slepian's inequality, respectively. The common advantage of these formulations is that they involve the computation of joint probabilistic constraints of lower dimension than this of the joint probabilistic constraint included in the original formulation. We analyze their computational tractability, and evaluate their constraining power relying on three datasets, in which random variables have a normal distribution. We then prove that the function $E[\varepsilon - T_i x | \varepsilon - T_i x > 0]$ enforcing a reliability level d_i is concave except for very large (small) values of $T_i x$ (d_i). We study the relationship between the service levels p_i and d_i defined by the functions and $P(T_i x \geq \varepsilon_i) \geq p_i$ and $E[\varepsilon_i - T_i x | \varepsilon_i - T_i x > 0] \leq d_i$, and use the STABIL problem to provide a power management interpretation of the results.

1 Objectives

The two following programs

$$\begin{aligned} &\min\ c^T x \\ &s.to\ \ Tx = \varepsilon \\ &\quad Ax \geq b \\ &\quad x \geq 0 \end{aligned},$$

and

$$\begin{aligned} &\min\ c^T x \\ &s.to\ \ Tx \geq \varepsilon \\ &\quad Ax \geq b \\ &\quad x \geq 0 \end{aligned}, \tag{1}$$

where A in an *[m x n]*-dimensional matrix, T in an *[r x n]*-dimensional matrix, c and x are n-dimensional vectors, b and ε are m- and r-dimensional vectors, respectively, are the underlying deterministic programs for static stochastic programming models, whose objective function is linear, and is subject, in addition to a set of linear and non-negativity constraints, to probabilistic constraints, the random variables of which being in the right-hand sides of the constraints.

Denoting by $F(z) = P(\varepsilon \leq z)$ the known probability distribution of the multivariate random variable ε, the stochastic programming problem associated with (1) is the probabilistically constrained problem expressed as follows:

$$\begin{aligned} &\min\ c^T x \\ &s.to\ \ \mathrm{P}(Tx \geq \varepsilon) \geq p \\ &\quad Ax \geq b \\ &\quad x \geq 0 \end{aligned}$$

that can alternatively be written (Prékopa, 1988, Sen, 1992) as

$$\begin{aligned} &\min\ c^T x \\ &s.to\ \ Tx - \varepsilon \geq 0 \\ &\quad F(z) \geq p \\ &\quad Ax \geq b \\ &\quad x \geq 0 \end{aligned},$$

in which p is a prescribed probability level, usually close to 1.

Programming under probabilistic constraints was introduced by Charnes et al. [1958], who considered a set ($i = 1,\ldots,r$) of individual probabilistic constraints imposed on each particular random event:

$$\begin{aligned} &\min c^T x \\ &s.to\ \mathbb{P}(T_i x \geq \varepsilon_i) \geq p_i, i = 1,\ldots,r \\ &\quad Ax \geq b \\ &\quad x \geq 0 \end{aligned}$$

The use of multiple individual probabilistic constraints $i = 1,\ldots,r$ is somewhat relatively easy to handle. The robustness of such formulations is however questionable; individual probabilistic constraints are only appropriate in case of a system composed of different components that do not affect each other. However, in most situations, probabilistic constraints, taken independently, do not provide an accurate representation of the considered system.

Subsequently, Miller and Wagner [1965] proposed a formulation for joint probabilistic constraints:

$$\begin{aligned} &\min c^T x \\ &s.to\ \mathbb{P}(T_i x \geq \varepsilon_i, i = 1, \dots, r) \geq p \\ &\quad Ax \geq b \\ &\quad x \geq 0 \end{aligned} \tag{2}$$

with the restriction that each of the random variables are independent of each another:

$$\rho_{i,j} = 0, \forall i, j, i \neq j .$$

The program (2) can alternatively be rewritten as:

$$\begin{aligned} &\min c^T x \\ &s.to \prod_{\mathrm{i}=1}^{\mathrm{r}} \mathbb{P}(T_i x \geq \varepsilon_i) \geq p \\ &\quad Ax \geq b \\ &\quad x \geq 0 \end{aligned} \tag{3}$$

By taking the logarithm of both sides of the constraint

$$\prod_{i=1}^{r} \mathbb{P}(T_i x \geq \varepsilon_i) \geq p \ ,$$

we obtain the program below

$$\begin{aligned} &\min c^T x \\ &s.to \sum_{\mathrm{i}=1}^{\mathrm{r}} \ln F_i(T_i x) \geq \ln p \\ &\quad Ax \geq b \\ &\quad x \geq 0 \end{aligned} , \tag{4}$$

since the elements *i, i=1,…,r* of the vector ε are independent random variables with probability distribution F_i.

The general case, that does not require the random right-hand side variables be independent of each another, i.e., $\rho_{i,j} \neq 0, i \neq j$, was introduced by Prékopa [1970, 1973]. The formulation is similar to (2) but does not allow the transformation of (2) into (3), nor (4). The solving of the general case proposed by Prékopa is very challenging, and involves the complex task of dealing with possibly non-convex, non-linear programs: they contain non-linear constraints entailing the computation of joint probabilities for multivariate random variables.

First, multivariate probability distributions are very difficult to compute, or even to approximate. Asserting that the handling of nonlinear constraints involving multivariate probability distributions is more difficult that of a nonlinear objective function, Komarómi [1986], for example, proposed a dual method for handling probabilistic constrained problems, in which their dual problems characterized by linear constraints and a concave objective function is solved.

Second, multivariate normal distributions are not always concave. For example, if the standard univariate normal distribution is well known to be concave, the standard bivariate normal distribution $\Phi(\varepsilon_1, \varepsilon_2; \rho)$ is concave [Prékopa, 1970] in ε_1 and ε_2 if:

- $p \geq 0.5$ and $\rho \geq 0$ or $\rho = -1$;
- $p \geq 0.355$ and $-1 < \rho \leq 0$;
- $p \geq 0.638$ regardless of the value of the correlation coefficient ρ.

More generally, the standard *n*-variate normal distribution $\Phi(z_1,...,z_n)$ is concave in the set $\{z \,|\, z_i \geq \sqrt{n-1}, i = 1,...,n\}$ [Prékopa, 2001]. However, the concavity of other continuous and discrete distributions has yet to be studied in greater detail.

In the next section, we focus on the probabilistic program (2), in which the objective function is linear and the multivariate random variables are located in the right-hand sides of the probabilistic constraints. This program enforces that the probability of the joint fulfillment of a system of linear inequalities with random right-hand sides be above a prescribed probability level *p*. We present some alternative formulations for the program (2), analyze the *p*-efficiency concept for approximating the probabilistic constraints of program (2) in which the random variables are discretely distributed, and describe a discrete moment bounding scheme [Prékopa, 1999]. The introduction of the bounding schemes into the stochastic problems result in the computation of the joint probability distribution of several lower dimensional vectors instead of this of a single higher dimensional random vector.

In Section 3, we evaluate the computational tractability and the constraining power of some alternative formulations applicable to multivariate random variables, and investigate the usefulness of the binomial moment bounding schemes for solving probabilistic programs. We report the results on three data sets well known in the stochastic programming literature. In Section 4, we analyze the convexity of non-linear constraints enforcing a certain type of reliability level (Prékopa, 1995), and interpret them in the context of power management. Section 5 contains some concluding remarks.

2 Formulation

In this section, we introduce alternative formulations for the probabilistic program (6). The first one, relying on the p-efficiency concept, is applicable to multivariate random having a discrete probability distribution, while the other formulations can be applied to variables regardless of the type (discrete or normal) of probability distribution.

2.1 *p*-efficiency approximation for discretely distributed random variables

In the case of random variables discretely distributed, the program (2) can be transformed into a disjunctive program using the concept of *p*-efficiency [Prékopa, 1990]. Let $p \in [0,1]$, and F be the probability distribution function of a *s*-dimensional integer random variable $\varepsilon \in \mathbb{Z}_+^s$, a point *v*, $v \in R^s$, is called a *p*-efficient point of the probability distribution function F if:

$$F(v) \geq p, \qquad \text{and}$$

$$\text{there is no } v' \leq v, v' \neq v \text{ such that } F(v') \geq p$$

The inequality "≤" for vector must be understood coordinate-wise. It has been shown [Dentcheva et al., 2000] that for any probability distribution, the set of *p*-efficiency points, $v^i,...,v^I$, for an integer random vector is non-empty and finite. Consequently, the program (2) can be substituted by the following disjunctive program:

$$\begin{aligned} &\min c^T x \\ &s.to \;\; Tx \geq v^i \\ &\qquad Ax \geq b \\ &\qquad v^i \in \mathrm{K}^p \\ &\qquad x \geq 0 \end{aligned} \tag{5}$$

in which $\mathrm{K}^p = \bigcup_{i=1}^{I} (v^i + R_+^s)$, and the disjunctive constraint $Tx \geq v^i$ that must hold for at least one i replaces the joint probabilistic constraint $\mathbb{P}(T_i x \geq \varepsilon_i, i = 1,...,n) \geq p$. The *p*-efficient points are multi-dimensional vectors that must be found prior to the optimization of (5) with respect to x.

For small dimensional problems, the easiest way is to enumerate all *p*-efficient points v^i and to process the associated problems. However, for larger problems, the brute force approach that consists of enumerating all *p*-efficient points can be overly time- and resource-consuming; it is than better suited to convexify the problem (5) as follows:

$$\begin{aligned} \min \;\; & c^T x \\ \text{s.to} \;\; & Ax \geq b \\ & Tx \geq \sum_{i=1}^{I} \lambda_i v^i \\ & \sum_{i=1}^{I} \lambda_i = 1 \\ & \lambda_1 \geq 0, i = 1,...,I \\ & x \geq 0 \end{aligned} \quad , \tag{6}$$

that imposes *Tx* be greater than or equal to a convex combination of *p*-efficient points.

The optimal value of (6) is the lower bound (for a minimization problem) for the optimal value of (5). Since all *p*-efficient points are not know, the solving of (6) is carried out using a column generation method. For continuous random variables, the cone generation method (Dentcheva et al., 2000, 2001, 2002, Beraldi and Ruszczyński, 2002a, Prékopa, 2003) is shown to be very efficient.

2.2 Bounding scheme for multivariate random variables

The approximation schemes presented in this section are applicable regardless of the type (i.e., discrete or continuous) of the probability distribution of the random variables. In the case of random variables having a continuous probability distribution, the computation of a joint probability is extremely difficult: the "highest" joint normal distribution for which a computation is provided [see, *inter alia*, Daley, 1974, Szántai, 1988, Drezner, 1994] is the trivariate one.

2.2.1 Boole's lower bound on the intersection of events

An alternative formulation for (2) can be obtained using Boole's lower bound relative to the intersection of r events. Denoting by

$$F_{i\ldots j}(z_i,\ldots,z_j)=\mathbb{P}(\varepsilon_i\le z_i,\ldots,\varepsilon_j\le z_j),1\le i<j\le r$$

the joint probability associated with the components i to j of the r-dimensional random vector, by A_i the event $\varepsilon_i\le z_i, i=1,\ldots,r,$ and setting

$$S_k=\sum_{i=1}^{r-k}F_{i\ldots k}(z_i,\ldots,z_j), j=i+k, 1\le k\le r-1 \quad ,$$

the following inequality follows from the application of Boole's lower bound for the intersection of r events:

$$\mathbb{P}(A_i\cap\ldots\cap A_r)\ge S_1-(r-1) \tag{7}$$

Using (7) for replacing the joint probabilistic constraint in (2), the following program results:

$$\begin{aligned}&\min c^T x\\ &s.to \sum_{i=1}^{r}F_i(T_i x)-(r-1)\ge p\\ &\qquad Ax\ge b\\ &\qquad x\ge 0\end{aligned} \tag{8}$$

2.2.2 Formulation with individual probabilistic constraints

A formulation containing a finite set of individual probabilistic constraints instead of the joint probabilistic constraint, and enforcing the requirements in (2) is given below:

$$\begin{aligned}&\min c^T x\\ &s.to\ \ T_i x\ge F_i^{-1}(p_i), i=1,\ldots,r\\ &\qquad \sum_{i=1}^{r}(1-p_i)\le 1-p\\ &\qquad Ax\ge b\\ &\qquad x\ge 0\end{aligned} \tag{9}$$

in which $p_i,\ldots,p_r$ are random variables constrained to be at least equal to p, which is a fixed parameter, and F_i^{-1} refers to the inverse probability distribution of the random variable ε_i.

The formulation (9) is valid, since constraints

$$T_i x\ge F_i^{-1}(p_i), i=1,\ldots,r, \quad \text{and}$$

$$\sum_{i=1}^{r}(1-p_i)\le 1-p$$

imply that

$$
\begin{aligned}
&T_i x \geq F_i^{-1}(p_i), i = 1,...,r,\\
&= \mathbb{P}(T_i x \geq \varepsilon_i, i = 1,...,r)\\
&= 1 - \mathbb{P}((T_1 x < \varepsilon_1) \cup ... \cup (T_r x < \varepsilon_r))\\
&\geq 1 - \mathbb{P}\sum_{i=1}^{r}(T_i x < \varepsilon_i)\\
&= 1 - \sum_{i=1}^{r}(1 - \mathbb{P}(T_i x < \varepsilon_i))\\
&= 1 - \sum_{i=1}^{r}(1 - p_i)\\
&\leq p
\end{aligned}
$$

2.2.3 Binomial moments bounding scheme

The formulation presented in this section is based on the binomial moment problem (Prékopa, 1999, Dentcheva et al., 2002) formulated for a finite number of events $A_1,...,A_r$, defined in a specified probability space. We use sharp lower and upper bounds for the probability of the following functions of events: $A_1 \cap ... \cap A_r$, $A_1 \cup ... \cup A_r$. Denoting by ε the number of events out of $A_1,...,A_r$, Prékopa [1995] shows that

$$E\left[\binom{\varepsilon}{k}\right] = S_k, k = 1,...,r,$$

where

$$S_k = \sum_{i=1}^{r-k} P(A_i \cap ... \cap A_j), j = i + k, 1 \leq k \leq r - 1$$

is he k^{th} moment of the random variable ε, and $S_0 = 1$.

The binomial moment problem is defined as the following linear programming problem:

$$
\begin{aligned}
&\min(\max) \sum_{i=0}^{r} f_i v_i\\
&s.\ to \quad \sum_{i=0}^{r}\binom{i}{k} v_i = S_k, k = 0,...,m \qquad (10)\\
&\qquad\qquad v_i \geq 0, i = 0,...,r
\end{aligned}
$$

where $f_0,...,f_r$ are some constants and $m < r$.
If

$$f_i = \begin{cases} 1, \text{ if } i = r,\\ 0, \text{ if } i < r, \end{cases} \qquad (11)$$

then the optimal values of (10) provide lower (upper) bounds for $\mathbb{P}(A_1 \cup ... \cup A_r)$.

If

$$f_i = \begin{cases} 1, \text{ if } i \geq 1, \\ 0, \text{ if } i = 0, \end{cases} \tag{12}$$

then the optimal values of (10) provide lower (upper) bounds for $\mathbb{P}(A_1 \cap ... \cap A_r)$.
Since

$$\mathbb{P}(A_1 \cap ... \cap A_r) = 1 - \mathbb{P}(\overline{A_1} \cup ... \cup \overline{A_r}) \ ,$$

with $\overline{A_i}$ being the complementary event of A_i, Prékopa [1999] establishes that the sharp lower (upper) bound for $\mathbb{P}(A_1 \cap ... \cap A_r)$ using the minimization (maximization) problem (10) and the function (11) is the same as 1 - the sharp upper (lower) bound for $\mathbb{P}(\overline{A_1} \cup ... \cup \overline{A_r})$ using the maximization (minimization) problem (10) and the function (12).

Using the results above, we formulate the following linear programs

$$\begin{aligned}
\min(\max) \quad & v_r \\
s.to \quad & v_0 + v_1 + v_2 + ... + v_r = 1 \\
& v_1 + 2v_2 + 3v_3 + ... + rv_r = \sum_{i=1}^{r} F_i(T_i x) \\
& v_2 + \binom{3}{2} v_3 + ... + \binom{r}{2} v_r = \sum_{i=1}^{r-1} F_{ij}(T_i x, T_j x), \ j = i+1 \\
& ... \\
& v_m + \binom{m+1}{m} v_{m+1} + ... + \binom{r}{m} v_r = \sum_{i=1}^{r-m+1} F_{ij}(T_i x, ..., T_j x), \ j = i+m-1 \\
& v_0, v_1, ..., v_r \geq 0
\end{aligned}$$

which optimal values provide lower and upper bounds for $F(z_i,...,z_j)$. Using this bounding scheme on $F(z_i,...,z_j)$ and introducing it in the formulation of (2) for replacing the joint probabilistic constraints, we obtain the two following approximations for (2):

$$\begin{aligned}
\min \quad & c^T x + \alpha v_r \\
s.to \quad & Ax \geq b \\
& v_0 + v_1 + v_2 + ... + v_r = 1 \\
& v_1 + 2v_2 + 3v_3 + ... + rv_r = \sum_{i=1}^{r} F_i(T_i x) \\
& v_2 + \binom{3}{2} v_3 + ... + \binom{r}{2} v_r = \sum_{i=1}^{r-1} F_{ij}(T_i x, T_j x), \ j = i+1 \\
& ... \\
& v_m + \binom{m+1}{m} v_{m+1} + ... + \binom{r}{m} v_r = \sum_{i=1}^{r-m+1} F_{ij}(T_i x, ..., T_j x), \ j = i+m-1 \\
& v_0, v_1, ..., v_{r-1} \geq 0 \\
& v_r \geq p \\
& x \geq 0
\end{aligned} \quad , \tag{13}$$

and

$$
\begin{aligned}
\max \quad & -c^T x + \alpha v_r \\
s.to \quad & Ax \geq b \\
& v_0 + v_1 + v_2 + \ldots + v_r = 1 \\
& v_1 + 2v_2 + 3v_3 + \ldots + rv_r = \sum_{i=1}^{r} F_i(T_i x) \\
& v_2 + \binom{3}{2} v_3 + \ldots + \binom{r}{2} v_r = \sum_{i=1}^{r-1} F_{ij}(T_i x, T_j x), j = i+1 \\
& \ldots \\
& v_m + \binom{m+1}{m} v_{m+1} + \ldots + \binom{r}{m} v_r = \sum_{i=1}^{r-m+1} F_{ij}(T_i x, \ldots, T_j x), j = i+m-1 \\
& v_0, v_1, \ldots, v_{r-1} \geq 0 \\
& v_r \geq p \\
& x \geq 0
\end{aligned}
\tag{14}
$$

where α is an arbitrary non-negative number.

2.2.4 Bound using Slepian's inequality

The Slepian' inequality can be stated as follows: if $\varepsilon_i, i = 1,\ldots,r$ has a standard multivariate normal probability distribution, and *R* and *R'* are two possible correlation matrices which respective elements s_{ij} and s'_{ij} are such that

$$s_{ij} \geq s'_{ij}, i \neq j, i, j = 1,\ldots,r \quad , \tag{15}$$

then it can shown that [Gupta, 1965, Bawa, 1973]

$$\mathbb{P}(\varepsilon_i \leq T_i x, i = 1,\ldots,r; R) \geq \mathbb{P}(\varepsilon_i \leq T_i x, i = 1,\ldots,r; R') \tag{16}$$

for a same value taken by the decision variable $T_i x$.

Consequently, when (15) holds, the solving of (2) for the correlation matrix *R'* provides an upper found on the objective value of (2) for the correlation matrix *R*. Denoting by $z'^* = c^T x'^*$ ($z^* = c^T x^*$) the optimal value of (2) associated with *R' (R)*, and by $x_i'^*$ (x_i^*) the optimal values of the decision variables, the following cutting plane

$$z^* \leq z'^* \tag{17}$$

and disjunctive cuts

$$
\begin{aligned}
& x_i^* \leq x_i'^* + \delta_i M, i = 1,\ldots,r \\
& \sum_{i=1}^{r} \delta_i \leq r-1, \delta_i \in \{0,1\}, i = 1,\ldots,r
\end{aligned}
\quad , \tag{18}
$$

where *M* is a large positive number, and δ is a *r*-dimensional binary vector, can be introduced in the program (2). In particular, if *R* is such that $s_{ij} \geq 0$, for all *i,j*, and *R'* is such that $s'_{ij} = 0$, for all $i \neq j$, then (16) can be rewritten as:

$$\mathbb{P}(\varepsilon_i \leq T_i x, i = 1,\ldots,r; R) \geq \sum_{i=1}^{r} \ln F_i(T_i x : R') \tag{19}$$

In the next section, we shall further evaluate the respective computational tractability and constraining power of the formulations presented in this section.

3 Computational results

The solving of the stochastic program (2) containing joint probabilistic constraints with dependent random variables is very challenging; the complexity of the task increases with the dimension of the random variable. In the preceding section, we give some alternative formulations for (2).

In this section, we shall comment on their respective computational tractability. We note that the program (9) is convex. Indeed, if a random variable ε has a log-concave probability density function, i.e., the normal distribution is log-concave, then the distribution of ε and its marginal distributions are log-concave, which implies that the constraint in (9) is convex. Therefore, $F_i(T_i x)$ is concave for $T_i x \geq F^{-1}(p_i), i = 1,...,r$. It has also been shown that the log-concavity property does not carry over for sums, which means that the constraint

$$\sum_{i=1}^{r} F_i(T_i x) - (r-1) \geq p$$

in (8) is not guaranteed to be concave, which in turn implies that the program (8) is not necessarily convex. The same remark applies for the programs (13) and (14). Moreover, the programs (13) and (14) contain equality constraints involving nonlinear functions.

In the remaining part of this section, we shall evaluate the constraining power of the formulation (2), and of its approximations (8), (9), (13) and (14). Computational results are derived from the application of the programs listed above to three data sets, for which different parameter settings are considered. In each case handled, the random variables are assumed to have a joint normal distribution.

3.1.1 Reservoir Management

The first data set is related to reservoir management. The objective is to design a reservoir system, in which reservoirs are used to hedge against the possibility of flooding that may occur as a result of random stream of water. The probabilistic constraints impose limits on the probability that the water rises above reservoir capacities. The capacities x_j of the reservoirs *j* considered, are the decision variables, which respective capacities are limited from above by V_j, and are to be designed in such a way that they are able to retain the flood of two different water sources coming in random quantities ε_j, preventing it from continuing to downstream locations. The parameter c_j refers to the cost per unit of capacity for reservoir *j*. The reader is referred to Prékopa and Szántai [1978] and Prékopa [1995] for more detailed explanations.

The associate program for a reservoir system containing two reservoirs takes the following form:

- $$\begin{aligned} &\min\ c_1 x_1 + c_2 x_2 \\ &s.to\ \mathbb{P}\begin{pmatrix} x_1 + x_2 \geq \varepsilon_1 + \varepsilon_2 \\ x_2 \geq \varepsilon_2 \end{pmatrix} \geq p, \\ &0 \leq x_1 \leq V_1 \\ &0 \leq x_2 \leq V_2 \end{aligned}$$

  when considering formulation (2);

- $$\begin{aligned} &\min\ c_1x_1 + c_2x_2 \\ &s.to\ F(x_1) + F(x_1 + x_2) - (2-1) \geq p \\ &\quad 0 \leq x_1 \leq V_1 \\ &\quad 0 \leq x_2 \leq V_2 \end{aligned}$$

  when considering formulation (8);

- $$\begin{aligned} &\min\ c_1x_1 + c_2x_2 \\ &s.to\ x_1 + x_2 \geq F^{-1}(p_1) \\ &\quad x_2 \geq F^{-1}(p_2) \\ &\quad (1 - p_1) + (1 - p_2) \leq 1 - p \\ &\quad 0 \leq x_1 \leq V_1 \\ &\quad 0 \leq x_2 \leq V_2 \end{aligned}$$

  when considering formulation (9);

- $$\begin{aligned} &\min\ c_1x_1 + c_2x_2 \\ &\text{s.to}\ \ v_0 + v_1 + v_2 = 1 \\ &\quad v_1 + 2v_2 = F(x_1) + F(x_1 + x_2) \\ &\quad v_2 = F(x_1, x_1 + x_2) \\ &\quad 0 \leq x_1 \leq V_1 \\ &\quad 0 \leq x_2 \leq V_2 \\ &\quad v_2 \geq p \end{aligned}$$

  when considering the formulation (13), with $\alpha = 0$.

The random variables ε_1 and ε_2 are normally distributed, $\varepsilon_1 \sim N(1, 0.1)$, $\varepsilon_2 \sim N(2, 0.2)$, and the covariance ($\text{cov}[\varepsilon_1, \varepsilon_2]$) between ε_1 and ε_2 is known. In the second probabilistic constraint, the random variable ε_2 is univariate: (for all the parameters settings), while, in the joint probabilistic constraint, the random variable $\theta_1 = \varepsilon_1 + \varepsilon_2$ is bivariate with expected value:

$$E[\theta_1] = E[\varepsilon_1 + \varepsilon_2] = E[\varepsilon_1] + E[\varepsilon_2] = \mu_1 + \mu_2 \ ,$$

and variance

$$\begin{aligned} Var[\theta_1] &= Var[\varepsilon_1 + \varepsilon_2] \\ &= E[(\varepsilon_1 + \varepsilon_2 - \mu_1 - \mu_2)^2] \\ &= E[(\varepsilon_1 - \mu_1)^2 + (\varepsilon_2 - \mu_2)^2 + 2(\varepsilon_1 - \mu_1)(\varepsilon_2 - \mu_2)] \\ &= Var[\varepsilon_1] + Var[\varepsilon_2] + 2\,\text{cov}[\varepsilon_1, \varepsilon_2] \end{aligned} \ ,$$

with

$$\text{cov}[\varepsilon_1, \varepsilon_2] = \rho_{1,2} \cdot \sigma_1 \cdot \sigma_2 ,$$

where $\rho_{1,2}$ denotes the correlation level between ε_1 and ε_2 and σ_1 and σ_2 are given, referring to the standard deviation of ε_1 and ε_2.

Table 1 displays the different problem instances settings considered, i.e., the different values taken by the coefficients of the objective function, by the correlation levels between random variables, by the reliability level *p*, and by the maximal quantity that the reservoirs can contain, while Table 2 reports the value of the objective function and these of the decision variables for the various problem formulations discussed in the preceding sections.

Table 1: Parameter settings

Case #	c_1	c_2	p	ρ	$E[\theta_1]$	cov	$Var[\theta_1]$	V_1	V_2
1	2	1	0.9	0	3	0	0.05	0.8	2.5
2	2	1	0.9	-0.8	3	-0.016	0.018	0.8	2.5
3	2	1	0.9	0.8	3	0.016	0.082	0.8	2.5
4	1	2	0.9	0	3	0	0.05	0.8	2.5
5	1	2	0.9	-0.8	3	-0.016	0.018	0.8	2.5
6	1	2	0.9	0.8	3	0.016	0.082	0.8	2.5
7	1	2	0.99	0	3	0	0.05	2	5
8	1	2	0.99	-0.8	3	-0.016	0.018	2	5
9	1	2	0.99	0.8	3	0.016	0.082	2	5
10	1	2	0.99	0	3	0	0.05	0.8	3
11	1	2	0.99	-0.8	3	-0.016	0.018	0.8	3
12	1	2	0.99	0.8	3	0.016	0.082	0.8	3

It appears that the four programs (2), (8), (9) and (13) result in the same value of the objective function in almost all the parameter settings considered for the reservoir problem. It turns out that approximations of form (8), (9) and (13) are equally (very) tight for such problems of moderate size.

Table 2: Optimal solutions for each parameter setting and formulation

	Program (2) with joint probability constraints			Program (8) with Boole's inequality			Program (9) with set of individual constraints			Program (13) with binomial moment bounding scheme		
#	z	x_1	x_2	z	x_1	x_2	z	x_1	x_2	z	x_1	x_2
1	4.088	0.794	2.500	4.089	0.795	2.500	4.089	0.795	2.500	4.088	0.794	2.500
2	3.853	0.677	2.500	3.854	0.677	2.500	3.854	0.677	2.500	3.853	0.677	2.500
3	INFEASIBLE											
4	5.789	0.8000	2.494	5.790	0.800	2.495	5.790	0.800	2.495	5.789	0.8000	2.494
5	5.585	0.8000	2.393	5.586	0.800	2.393	5.586	0.800	2.393	5.585	0.8000	2.393
6	INFEASIBLE											
7	6.090	1.052	2.519	6.091	1.052	2.520	6.091	1.052	2.520	6.090	1.052	2.519
8	5.858	0.856	2.501	5.858	0.856	2.501	5.858	0.856	2.501	5.858	0.856	2.501
9	6.218	1.193	2.513	6.250	1.189	2.530	6.250	1.189	2.530	6.218	1.193	2.513
10	6.243	0.800	2.721	6.243	0.800	2.721	6.243	0.800	2.721	6.243	0.800	2.721
11	5.870	0.800	2.535	5.870	0.800	2.535	5.870	0.800	2.535	5.870	0.800	2.535
12	6.532	0.800	2.866	6.533	0.800	2.866	6.533	0.800	2.866	6.532	0.800	2.866

It appears that the programs (3) and (4) result in the same optimal value of the objective function, and do not present major difference in terms of computational tractability. Their optimal value of the objective function is roughly the same as this of the programs analyzed in Table 2. However, unlike programs (8), (9) and (13) that are widely applicable, programs (3) and (4) can only be used if the random variables are independent of each other.

Table 3: Optimal solutions with (3) and (4) for cases when random variables are independent

	Program (3)					Program (4)				
#	z	x_1	x_2	p_1	p_2	z	x_1	x_2	p_1	p_2
1	4.088	0.794	2.500	0.994	0.906	4.088	0.794	2.500	0.994	0.906
4	5.789	0.800	2.494	0.993	0.906	5.789	0.800	2.494	0.993	0.906
7	6.091	1.052	2.519	0.995	0.995	6.090	1.052	2.519	0.995	0.995
10	6.243	0.800	2.721	1.000	0.990	6.244	0.800	2.722	1.000	0.990

3.1.2 Coffee blending

The second data set is related to coffee blending. The objective function is linear and is minimized subject to a set of linear constraints related to the limited availability of the coffee types, the fulfillment of quality requirements, and a joint probability constraint that imposes that the demand for coffee be satisfied with probability *p,* representing the reliability level of the production system. The reader is referred to Prékopa [1995] and Szántai [1988] for more detailed explanations. Denoting by *D* the feasible set determined by the linear constraints, and setting

$$\varepsilon_1 = \sum_{k=1}^{8} x_{k1}, \varepsilon_2 = \sum_{k=1}^{8} x_{k2}, \text{ and } \varepsilon_3 = \sum_{k=1}^{8} x_{k3},$$

the program is formulated as:

- $$\begin{aligned} &\min c^T x \\ &s.to\ x \in D \\ &\mathbb{P}\begin{pmatrix} \sum_{k=1}^{8} x_{k1} \ge \varepsilon_1 \\ \sum_{k=1}^{8} x_{k2} \ge \varepsilon_2 \\ \sum_{k=1}^{8} x_{k3} \ge \varepsilon_3 \end{pmatrix} \ge p \\ &x_{kl} \ge 0, k = 1,...,8 \end{aligned}$$

  in terms of (2);

- $$\begin{aligned} &\min c^T x \\ &s.to\ x \in D \\ &F(\sum_{k=1}^{8} x_{k1}) + F(\sum_{k=1}^{8} x_{k2}) + F(\sum_{k=1}^{8} x_{k3}) - (3-1) \ge p \\ &x_{kl} \ge 0, k = 1,...,8 \end{aligned}$$

  in terms of (8);

- $$\begin{aligned} &\min \; c^T x \\ &s.to \;\; x \in D \\ &\qquad \sum_{k=1}^{8} x_{k1} \geq F_1^{-1}(p_1) \\ &\qquad \sum_{k=1}^{8} x_{k2} \geq F_2^{-1}(p_2) \\ &\qquad \sum_{k=1}^{8} x_{k3} \geq F_3^{-1}(p_3) \\ &\qquad \sum_{i=1}^{3} (1-p_i) \leq 1-p \\ &\qquad x_{kl} \geq 0, k=1,\ldots,8 \end{aligned}$$

  in terms of (9);

- $$\begin{aligned} &\min \;\; c^T x \\ &s.to \;\; x \in D \\ &\quad v_0 + v_1 + v_2 + v_3 = 1 \\ &\quad v_1 + 2v_2 + 3v_3 = F(\sum_{k=1}^{8} x_{k1}) + F(\sum_{k=1}^{8} x_{k2}) + F(\sum_{k=1}^{8} x_{k3}) \\ &\quad v_2 + 3v_3 = F(\sum_{k=1}^{8} x_{k1}, \sum_{k=1}^{8} x_{k3}) + F(\sum_{k=1}^{8} x_{k1}, \sum_{k=1}^{8} x_{k2}) + F(\sum_{k=1}^{8} x_{k2}, \sum_{k=1}^{8} x_{k3}) \\ &\quad x_{kl} \geq 0, k=1,\ldots,8 \\ &\quad v_0, v_1, v_2 \geq 0 \\ &\quad v_3 \geq p \end{aligned}$$

  in terms of (13) with $\alpha = 0$ and when considering the first two binomial moments.

Assuming that

$$\begin{cases} \xi_1 \sim N(3, 0.5) \\ \xi_2 \sim N(40, 5) \\ \xi_3 \sim N(20, 3) \end{cases},$$

we can compute (*Table 4*) the optimal objective value and the values of the random variables p_1, p_2, p_3 for different values of the reliability level p for the programs (8) and (9) .

Table 4: Optimal solutions for programs involving Boole's inequality and a set of individual probabilistic constraints

p	Program (9) with set of individual constraints				Program (8) with Boole's inequality			
	p_1	p_2	p_3	Objective value	p_1	p_2	p_3	Objective value
0.9	0.992	0.924	0.983	22988	0.992	0.924	0.983	22988
0.95	0.996	0.963	0.991	23910	0.996	0.963	0.991	23910
0.99	0.999	0.993	0.998	25703	0.999	0.993	0.998	25703

It can be seen that programs (8) and (9) respectively involving Boole's inequality and a set of individual constraints result in the same optimal solution.

For programs (2) and (13) involving the joint probabilistic constraint and the use of the binomial moment bounding scheme, the correlation levels between random variables must be specified. We consider the three following correlation matrices R_1, R_2 and R_3:

$$R_1 = \begin{pmatrix} 1 & 0.1 & 0.1 \\ 0.1 & 1 & 0.9 \\ 0.1 & 0.9 & 1 \end{pmatrix}, \quad R_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad R_3 = \begin{pmatrix} 1 & 0.1 & 0.1 \\ 0.1 & 1 & -0.9 \\ 0.1 & -0.9 & 1 \end{pmatrix}$$

Results for the programs (2) and (13) are given in *Table 5*.

Table 5: Optimal solutions for the programs involving a joint probabilistic constraint and the binomial moment bounding scheme

p	Correlation level	Program (2) with joint probability constraints	Program (13) with binomial moment bounding scheme: MIN	Program (3)	Program (4)
0.9	R_1	22564.0	22575.2		
0.9	R_2	22949.4	22958.2	22958.3	22958.4
0.9	R_3	22961.6	22970.1		
0.95	R_1	23603.6	23587.3		
0.95	R_2	23866.6	23896.8	23896.8	23897.1
0.95	R_3	23885.2	23901.3		
0.99	R_1	25500.6	25493.7		
0.99	R_2	25702.0	25700.0	25700.0	25701.1
0.99	R_3	25680.6	25701.5		

Regardless of the reliability level considered, it can be seen that the optimal objective values obtained with programs (2) and (13) for any of the three correlation matrices are lower than the optimal solution obtained with programs (8) and (9). It is logical that the optimal solution of (13) be lower than these of programs (8) and (9), since (13) provides a lower bound on the objective value, while (8) and (9) are approximations of (2), enforcing requirements that at least as demanding as those of (2). The low magnitude of the gap between the optimal solutions of (13) on the one side and those of (8) and (9) on the other side indicates that (8) and (9) are tight approximations of (13).

Considering the correlation matrices R_1, R_2 and R_3, it can be seen that (20) applies

$$s_{ij}^{(1)} \geq s_{ij}^{(2)}, i \neq j, i, j = 1,...,r \quad \text{and} \quad s_{ij}^{(1)} \geq s_{ij}^{(3)}, i \neq j, i, j = 1,...,r \quad ,$$

therefore allowing the reliance on Slepian's inequality
$\mathbb{P}(\varepsilon_i \leq T_i x, i = 1,...,r; R_1) \geq \mathbb{P}(\varepsilon_i \leq T_i x, i = 1,...,r; R_2)$ and $\mathbb{P}(\varepsilon_i \leq T_i x, i = 1,...,r; R_1) \geq \mathbb{P}(\varepsilon_i \leq T_i x, i = 1,...,r; R_3)$
and the introduction of the inequalities (18). For any reliability level enforced, it can be seen in Table 5 that the optimal values obtained when considering the correlation matrix R_1 are lower than obtained when considering correlation matrices R_2 and R_3.

The solving of the stochastic program associated with correlation matrix R_1 can moreover be solved with the introduction of inequality (19). Finally, it can also be seen that (4) is a bit less constraining than (3).

3.1.3 Power management: STABIL problem

The last problem considered is the STABIL problem (Prékopa et al., 1976, 1980) involving the construction of a plan for the Hungarian electrical energy sector in the seventies. It has a linear objective function minimizing the profit function multiplying by –1, while satisfying 106 deterministic constraints (manpower balance, investment features, foreign trade balance, balance of the state budget, finance, and electricity demand satisfaction), as well as four stochastic constraints.

The problem is formulated by:

$$\begin{aligned} & \min c^T x \\ & s.to\ a_i x \geq b_i, i = 1,\dots,106 \\ & \mathbb{P}\begin{pmatrix} -25x_{25} \geq \varepsilon_1 \\ -16.67x_{26} \geq \varepsilon_2 \\ 0.8696x_{24} + x_{40} \geq \varepsilon_3 \\ 0.9(x_1 + x_2 + x_3 + x_4) - 0.115x_{24} \geq \varepsilon_4 \end{pmatrix} \geq p \\ & x \geq 0 \end{aligned} \tag{21}$$

where $c^T x$ is given by x_{35} - x_{36}, x_{35} and x_{36} representing respectively the increase in the wage bill and the enterprise profit before taxation, and $\varepsilon_i, i = 1,\dots,4$ are normally distributed random variables with the following means and variances:

$$\begin{cases} \mu_1 = -48313, & \sigma_1 = 483 \\ \mu_2 = -426\ , & \sigma_2 = 4 \\ \mu_3 = 16000\ , & \sigma_3 = 160 \\ \mu_4 = 14950\ , & \sigma_4 = 190 \end{cases}.$$

The joint distribution of the random variables is normal, and their covariance matrix is given by:

$$C = \begin{pmatrix} 1 & -0.8 & 0.4 & 0.4 \\ -0.8 & 1 & 0.1 & 0.1 \\ 0.4 & 0.1 & 1 & 0.9 \\ 0.4 & 0.1 & 0.9 & 1 \end{pmatrix}$$

The first two components of (21) restrain the planned deficit of foreign trade (in $US and roubles) to be below a certain level, while the last two components express the relationships between the electrical sector and the other sectors of the Hungarian economy. The reader is referred to Prékopa et al. [1976 and 1980] for a more detailed description of the model.

Below, the solving of the programs associated with formulations (2) containing joint probability constraints, (8) based on Boole's inequality, (9) containing a set of individual constraints, and (13) based on the binomial moment bounding scheme are reported and discussed. Two reliability levels (*p = 0.9 , 0.95)* are considered.

Table 6: Optimal solutions for each formulation and p = 0.9

	Program (2) with joint probability constraints	Program (8) with Boole's inequality	Program (9) with set of individual constraints	Program (13) with binomial moment bounding scheme: MIN	Program with (14) binomial moment bounding scheme: MAX
x_1	6432.371	6497.096	6497.683	6305.828	6510.645
x_2	12806.469	12820.380	12820.400	12779.188	12823.766
x_6	8.236	20.485	20.484	0.045	0
x_7	1.417	2.113	2.112	1.362	1.330
x_{24}	18400.000	18400.000	18400.000	18400.000	18400.000
x_{25}	1892.749	1897.938	1897.940	1889.279	1889.260
x_{26}	24.676	25.091	25.091	24.643	24.625
x_{40}	1917.425	1923.028	1923.030	1913.923	1913.885
x_{35}	93.762	93.780	93.780	93.727	93.782
x_{36}	4464.090	4463.128	4463.130	4465.968	4463.226
z	- 4370.328	- 4369.349	- 4369.810	-4372.241	-4369.444

In *Table 7* below, we report the "individual" reliability levels associated with the optimal solution of (9) for the STABIL problem and with an overall service level p equal to 0.9.

Table 7 : "Individual" reliability level with p = 0.9

i	p_i
1	95.06%
2	99.96%
3	100.00%
4	94.97%

Using the binomial moment bounding formulation (13), we obtain a lower bound on the objective function equal to - 4372.241 and an upper bound equal to - 4369.444.

It can be seen that the binomial bounding scheme provides a very narrow bounding scheme, and that the solving of the problems (2), (8) and (9) provides very similar objective values that fall within the bounds determined by (13) and (14).

A similar conclusion can be observed for p set equal to 0.95. Using the binomial moment bounding formulation (13), we obtain a lower and upper bound on the objective function equal to - 4370.710 and - 4368.350, respectively. The optimal objective value obtained with the problem formulations (2) and (8) is the same, equal to - 4368.56, and is also extremely close to the optimal objective value obtained when p is equal to 0.9. The optimal individual reliability levels p_i, $i = 1, 2, 3, 4$ in (8) are equal to 97.91%, 99.18%, 100.00% and 97.91%, respectively.

4 Convexity analysis of constraints enforcing reliability level

4.1 Concavity of reliability functions

In this section, we analyze the reliability level enforcing that the average measure

$$E[\varepsilon_i - t | \varepsilon_i - t \geq 0] \leq d_i, i = 1,...,r \qquad (22)$$

of violations of the constraint $T_i x \geq \varepsilon_i$ be below a certain threshold d_i. Clearly, the inequality (22) enforces the average to be taken in the only cases when $T_i x \geq \varepsilon_i$ is violated.

Below, we analyze the concavity of the function

$$g(t) = E[\varepsilon - t \mid \varepsilon - t \geq 0] \tag{23}$$

when the random variable ε is normally distributed.

The function (23) can be rewritten as

$$\frac{\int_t^\infty (x-t) f(x)\, dx}{1\text{-}F(t)}$$

where *f(x)* denotes the probability density function and *F(x)* denotes the cumulative probability distribution of the random variable or, alternatively, as

$$\frac{\int_t^\infty (1-F(x))\, dx}{1\text{-}F(t)} \quad .$$

It can be seen that

$$g'(t) = \frac{-1(1-F(t))^2 + f(t)\int_t^\infty (1-F(x))\, dx}{(1-F(t))^2}$$

$$g'(t) = -1 + \frac{f(t)}{1-F(t)} g(t) < 0 \ ,$$

where $\frac{f(t)}{1-F(t)}$ is the hazard rate and is decreasing. Assuming that ε_i has a standard normal distribution with mean 0 and standard deviation1, and thus knowing that

$$\begin{cases} f(t) = \dfrac{e^{-t^2/2}}{\sqrt{2\pi}} \\ f'(t) = -t\, f(t) \end{cases} ,$$

we obtain the following developments for the second derivative of *g(t)*:

$$\begin{aligned} g''(t) &= \left(\frac{d}{dt} \frac{f(t)}{1-F(t)} \right) g(t) + \frac{f(t)}{1-F(t)} g'(t) \\ &= \frac{f'(t)(1-F(t)) + f^2(t)}{(1-F(t))^2} g(t) + \frac{f(t)}{1-F(t)} g'(t) \\ &= \frac{-tf(t)(1-F(t)) + f^2(t)}{(1-F(t))^2} g(t) + \frac{f(t)}{1-F(t)} \left(-1 + \frac{f(t)}{1-F(t)} g(t) \right) \\ &= \frac{-tf(t)g(t)}{1-F(t)} + \frac{f^2(t)g(t)}{(1-F(t))^2} - \frac{f(t)}{1-F(t)} + \frac{f^2(t)g(t)}{(1-F(t))^2} \\ &= \frac{2f^2(t)g(t)}{(1-F(t))^2} - \frac{f(t)}{1-F(t)} (tg(t)+1) \\ &= \underbrace{\frac{f(t)}{1-F(t)}}_{\geq 0} \left(\frac{2f(t)g(t)}{1-F(t)} - tg(t) - 1 \right) \end{aligned}$$

To test the concavity of *g(t)*, we must therefore evaluate the sign of

$$2f(t)g(t)-(t\,g(t)-1)(1-F(t)) \quad (24)$$

The solving of

$$\min_t 2f(t)g(t)-(t\,g(t)-1)(1-F(t)) \quad (25)$$

shows that the expression (24)

- is minimized for

$$t = 7.58067$$

for which the objective value of (25) is equal to -0.0047982

- is positive on

$$]-\infty, 7.31499[\quad .$$

The objective value of (25) becomes negative beyond 7.31499, and is equal to

-0.000415477

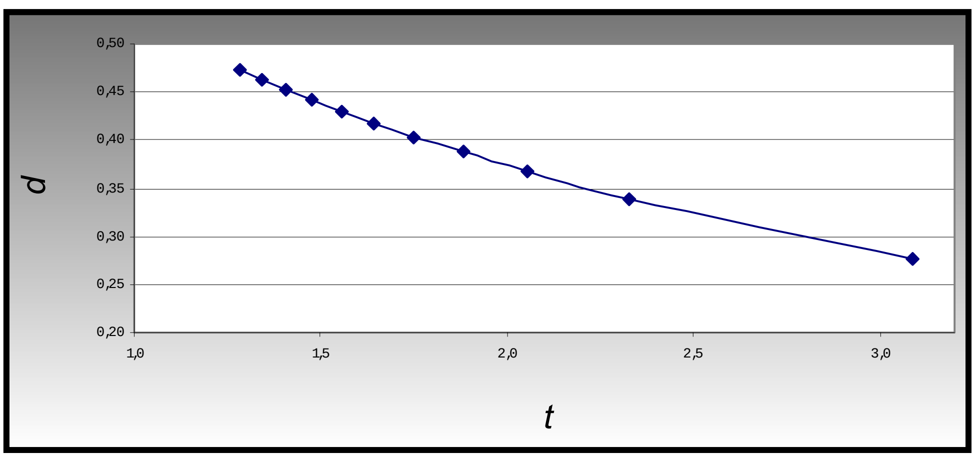
at this point.

Finally, we note that

$$P(\varepsilon \le 7.31499) = 0.99999999999987,$$

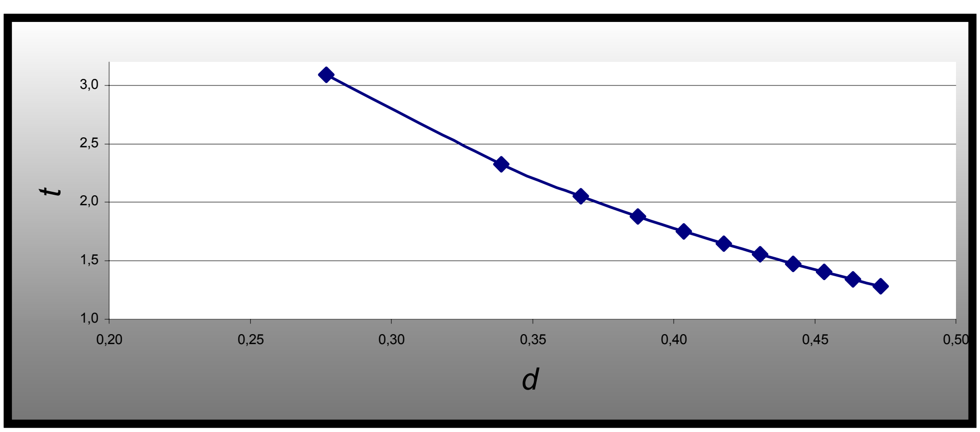
which means that *g(t)* (23) is concave except for extremely unlikely large values of *t*, and that the function (22) will also be concave except for extremely demanding reliability levels d_i characterized by extremely low values for d_i , requiring extremely high values for t for (22) to hold.

Figure 1: Function g(t)



This also implies that the inverse $g^{-1}(t)$ of *g(t)* is convex for most values of *t*.

Figure 2: Function $g^{-1}(t)$



Consequently, the stochastic problem

$$\begin{aligned} &\min c^T x \\ &s.to\ E[\varepsilon_i - T_i x \,|\, \varepsilon_i - T_i x > 0] \le d_i, i = 1,\dots,r \\ &\quad Ax \ge b \\ &\quad x \ge 0 \end{aligned},$$

equivalent to

$$\begin{aligned} &\min c^T x \\ &s.to\ T_i x \ge F^{-1}(d_i), i = 1,\dots,r \\ &\quad Ax \ge b \\ &\quad x \ge 0 \end{aligned}$$

is a non-linear convex problem.

4.1.1 Correspondence between types of reliability levels

In this section, we study the correspondence between the service level p_i enforced by a probabilistic constraint taking the form

$$P(t_i \ge \varepsilon_i) \ge p_i \quad ,$$

which is equivalent to

$$t_i \ge F^{-1}(p_i)$$

and the service level d_i enforced by the constraint (22), alternatively written as

$$t_i \ge g^{-1}(p_i)$$

To determine which value of d_i corresponds to various probabilities (p_i = 0.9 , 0.95, …) of not having a stockout, we solve the following non-linear problem:

$$\begin{aligned} &\min\ d_i \\ &s.to\ \frac{\int_{t^p}^{\infty} (1 - F(x))\, dx}{1\text{-}F(t^p)} \le d_i \\ &\quad d_i \ge 0 \end{aligned} \tag{26}$$

where $t^p = F^{-1}(p_i)$ is given and d_i is the decision variable.

The correspondence between the two types of service levels obtained by solving (26) for different values of p_i are given in *Table 8*.

Table 8 : Correspondence between p_i and d_i

p_i	0.9	0.91	0.92	0.93	0.94	0.95	0.96	0.97	0.98	0.99	0.999
d_i	0.4734	0.4636	0.4533	0.4423	0.4306	0.4179	0.4037	0.3873	0.3672	0.3389	0.2769

4.1.2 Application

After having shown in Section 4.1 that the enforcement of reliability levels of type d_i with normally distributed random variables result in non-linear convex problems, we consider the STABIL problem in which similar reliability levels are prescribed. More precisely, we want to enforce the reliability level d_0 such that

$$\begin{aligned} &\min c^T x \\ &s.to\ E[\varepsilon_i - T_i x \,|\, \varepsilon_i - T_i x > 0] \le d_i, i = 1,...,r \\ &\sum_{i=1}^{r} (d_i\ P(\varepsilon_i - T_i x > 0)) \le d_0 \\ &Ax \ge b \\ &x \ge 0 \\ &d_i \in R, i = 1,...,r \end{aligned} \tag{27}$$

in which $d_i, i = 1, ..., r$ are decision variables.
Substituting $d_i, i = 1, ..., r$ by

$$E[\varepsilon_i - T_i x \,|\, \varepsilon_i - T_i x > 0], i = 1,...,r,$$

the constraint

$$\sum_{i=1}^{r} (d_i\ P(\varepsilon_i - T_i x > 0)) \le d_0$$

becomes

$$\sum_{i=1}^{r} (E[\varepsilon_i - T_i x \,|\, \varepsilon_i - T_i x > 0]\ P(\varepsilon_i - T_i x > 0)) \le d_0 \quad . \tag{28}$$

Since

$$E[\varepsilon_i - T_i x \,|\, \varepsilon_i - T_i x > 0] = \frac{\int_{T_i x}^{\infty} (z_i - T_i x) f(z_i)\ dz_i}{1 - F(T_i x)} \quad ,$$

and

$$P(\varepsilon_i - T_i x > 0) = 1 - F(T_i x) \ ,$$

constraint (28) can be rewritten as

$$\int_{T_i x}^{\infty} (z_i - T_i x) f(z_i)\ dz_i$$

and is thus an "integrated probabilistic constraint" (Klein Haneveld, 1986), in which

$$\eta_i = [z_i - T_i x]_+ , i = 1,...,r$$

is the positive part of the expression $z_i - T_i x, i = 1,...,r$ and can be interpreted, in the STABIL context, as the amount of unserved power. It is thus clear that problem (27) restrains the expected amount of unserved power to be below a certain threshold d.

In *Table 9,* we report the results for a reliability level d equal to 0.4734 , 0.4179 and 0.3389, which respectively correspond to a probability level p of not having a stockout equal to 0.9 , 0.95 and 0.99 (see *Table 8*).

Table 9 : Reliability level d for the STABIL problem

p	0.9	0.95	0.99
d	0.473	0.418	0.339
$\int_{T_1x}^{\infty}(z_1 - T_1x)f(z_1)\,dz_1$	0.158	0.139	0.112
$\int_{T_2x}^{\infty}(z_2 - T_2x)f(z_2)\,dz_2$	0.152	0.135	0.111
$\int_{T_3x}^{\infty}(z_3 - T_3x)f(z_3)\,dz_3$	0	0	0
$\int_{T_4x}^{\infty}(z_4 - T_4x)f(z_4)\,dz_4$	0.163	0.144	0.116
Objective value	-4373.080	-4372.840	-4372.441

5 Concluding remarks

We consider probabilistically constrained problems, in which the multivariate random variables are located in the right-hand sides. The joint probabilistic constraint guarantees the joint fulfillment of a system of linear inequalities with random right-hand side variables to be above a prescribed probability level p. We show the applicability of the p-efficiency concept for approximating joint probabilistic variables with discretely distributed random variables. We then develop approximation bounding schemes applicable to random variables with either a discrete or a continuous probability distribution. The bounding schemes involve the substitution of the joint probabilistic constraint by a set of individual constraints, the Boole's inequality, the binomial moment bounding scheme, and Slepian's inequality, respectively. The advantage of such formulations is that they require the computation of joint probabilistic constraints of lower dimension than this of the joint probabilistic constraint included in the original formulation. We analyze their computational tractability, and show their relative tightness.

Finally, we study a particular reliability level d_i, and show that the function of form $E[\varepsilon - T_i x | \varepsilon - T_i x > 0]$ enforcing it is concave except for very large (small) values of $T_i x$ *(d_i)*. We study the relationship between the service levels p_i and d_i defined by the functions $P(T_i x \geq \varepsilon_i) \geq p_i$ and $E[\varepsilon_i - T_i x | \varepsilon_i - T_i x > 0] \leq d_i$, respectively. We illustrate and interpret such a service level d_i in a power management context.

References

Bawa V. S. 1973. On Chance Constrained Programming Problems with Joint Constraints. *Management Science* **10** (11), 1326-1331.

Beraldi P. and Ruszczyński A. 2002a. The Probabilistic Set Covering Problem. *Operations Research* **50**, 956-967.

Beraldi P. and Ruszczyński A. 2002b. A Branch and Bound Method for Stochastic Integer Problems under Probabilistic Constraints. *Optimization Methods and Software* **17** (3), 359-382.

Boole G. 1854. *Laws of Thought* (American Reprint of 1854 Edition). Dover, NY.

Boole G. 1868. Of Propositions Numerically Definite. In: *Transactions of the Cambridge Philosophical Society*. Part II. XI.

Charnes A. and Cooper W.W. 1959. Chance-Constrained Programming. *Management Science* **5**, 903-923.

Charnes A. and Cooper W.W. 1963. Deterministic Equivalents for Optimizing and Satisficing under Chance Constraints. *Operations Research* **11**, 18-39.

Charnes A., Cooper W.W. and Symonds G.H. 1958. Cost Horizons and Certainty Equivalents: An Approach to Stochastic Programming of Heating Oil. *Management Science* **4**, 235-263.

Daley D.J. 1974. Computation of Bi- and Tri-Variate Normal Integrals. *Applied Statistics* **23** (3), 435-438.

Dentcheva D., Prékopa A. and Ruszczyński A. 2000. Concavity and Efficient Points of Discrete Distributions in Probabilistic Programming. *Mathematical Programming* **89**, 55-77.

Dentcheva D., Prékopa A. and Ruszczyński A. 2001. On Convex Probabilistic Programming with Discrete Distributions. *Nonlinear Analysis* **47**, 1997-2009.

Dentcheva D., Prékopa A. and Ruszczyński A. 2002. Bounds for Probabilistic Integer Programming Models. *Discrete Applied Mathematics* **124**, 55-65.

Drezner Z. 1978. Computation of the Bivariate Normal Integral. *Mathematics of Computation* **32** (141), 277-279.

Drezner Z. 1994. Computation of the Trivariate Normal Integral. *Mathematics of Computation* **62,** 289-294.

Gassmann H.I. 2000. Rectangle Probabilities of Trivariate Normal Distributions. Unpublished preprint available from website http://www.mgmt.dal.ca/sba/profs/hgassmann/

Gupta S.S. 1965. Probability Integrals of Multivariate Normal and Multivariate *t*. *Mathematical Statistics* **34**, 792-828.

Klein Haneveld W.K. 1986. Duality in Stochastic Linear and Dynamic Programming. *Lecture Notes in Economics and Mathematical Systems* 274. Springer-Verlag. New York, NY.

Komáromi É. 1986. A Dual Method for Probabilistic Constrained Problems. *Mathematical Programming Study* **28**, 94-112.

Prékopa A. 1970. On Probabilistic Constrained Programming. *Proceedings of the Princeton Symposium on Mathematical Programming*. Princeton University Press, 113-138.

Prékopa A. 1973. Contributions to the Theory of Stochastic Programming. *Mathematical Programming* **4**, 202-221.

Prékopa A. 1988. Numerical Solution of Probabilistic Constrained Programming Problems. in

Numerical Techniques for Stochastic Optimization. Edited by Ermoliev Y. and Wets R.J.-B. Springer-Verlag. Berlin. 123-139.

Prékopa A. 1990. Dual Method for a One-Stage Stochastic Programming with Random *rhs* Obeying a Discrete Probability Distribution. *Zeitschrift of Operations Research* **34**, 441-461.

Prékopa A. 1995. *Stochastic Programming*. Kluwer. Dordrecht-Boston.

Prékopa A. 1999. The Use of Discrete Moment Bounds in Probabilistic Constrained Stochastic Programming Models. *Annals of Operations Research* **85**, 21-38.

Prékopa A. 2001. On the Concavity of Multivariate Probability Distribution Functions. *Operations Research Letters* **29**, 1-4.

Prékopa A. 2003. Probabilistic Programming Models. Chapter 5 in *Stochastic Programming: Handbook in Operations Research and Management Science 10.* Edited by Ruszczyński A. and Shapiro A. Elsevier Science Ltd, 267-351.

Prékopa A. and Szántai T. 1978. Flood Control Reservoir System Design Using Stochastic Programming. *Mathematical Programming Study* **9**, 138-151.

Prékopa A., Ganczer S., Deák I. and Patyi K. 1975. The STABIL Stochastic Programming Model and its Experimental Application to the Electrical Energy Industry of the Hungarian Economy. *Alkalmazott mat Lapok* **1**, 3-22.

Prékopa A., Ganczer S., Deák I. and Patyi K. 1980. The STABIL Stochastic Programming Model and its Experimental Application to the Electrical Energy Sector of the Hungarian Economy. In *Stochastic Programming.* Edited by Dempster M.A.H. Academic Press. London, 369-385.

Ruszczyński A. and Shapiro A. 2003. *Stochastic Programming: Handbook in Operations Research and Management Science* 10. Elsevier Science Ltd.

Schrage L. 1997. *Optimization Modeling with LINDO, 5th Edition*. Duxbury Press. Pacific Grove, CA.

Sen S. 1992. Relaxations for Probabilistically Constrained Programs with Discrete Random Variables. *Operations Research Letters* **11**, 81-86.

Szántai T. 1988. A Computer Code for Solution of Probabilistic-Constrained Stochastic Programming Problems. In: *Numerical Techniques for Stochastic Optimization.* Edited by Ermoliev Y. and Wets R.J.-B. Springer-Verlag. New York, 229-235.

Szántai T. 2000. Improved Bounds and Simulation Procedures on the Value of the Multivariate Normal Probability Distribution Function. *Annals of Operations Research* **100**, 85-101.

Wolfram S. 2003. *The Mathematica Book, 5th Edition.* Wolfram Media Inc. Champaign, IL.