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Gábor Rudolf[a] Nilay Noyan[b] Farid Alizadeh[c]

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RUTCOR
Rutgers Center for
Operations Research
Rutgers University
640 Bartholomew Road
Piscataway, New Jersey
08854-8003
Telephone: 732-445-3804
Telefax: 732-445-5472
Email: rrr@rutcor.rutgers.edu
http://rutcor.rutgers.edu/~rrr

[a]RUTCOR Rutgers, the State University of New Jersey 640 Bartholomew Road Piscataway, NJ 08854-8003 USA. E-mail: grudolf@rutcor.rutgers.edu
[b]RUTCOR Rutgers, the State University of New Jersey 640 Bartholomew Road Piscataway, NJ 08854-8003 USA. E-mail: noyan@rutcor.rutgers.edu
[c]RUTCOR Rutgers, the State University of New Jersey 640 Bartholomew Road Piscataway, NJ 08854-8003, USA. E-mail: alizadeh@rutcor.rutgers.edu

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Optimality Constraints For the Cone of Positive Polynomials

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Abstract. For a proper cone $\mathcal{K} \subset \mathbb{R}^n$ and its dual cone $\mathcal{K}^*$ the complementary slackness condition $\mathbf{x}^\mathsf{T}\mathbf{s} = 0$ defines an n-dimensional manifold $C(\mathcal{K})$ in the space $\{ (\mathbf{x}, \mathbf{s}) \mid \mathbf{x} \in \mathcal{K},\ \mathbf{s} \in \mathcal{K}^* \}$. When $\mathcal{K}$ is a symmetric cone, this manifold can be described by a set of n bilinear equalities. This fact proves to be very useful when optimizing over such cones, therefore it is natural to look for similar optimality constraints for non-symmetric cones. In this paper we examine the cone of positive polynomials $\mathcal{P}_{2n+1}$ and its dual, the closure of the moment cone $\mathcal{M}_{2n+1}$. We show that there are exactly 4 linearly independent bilinear identities which hold for all $(\mathbf{x}, \mathbf{s}) \in C(\mathcal{K})$, regardless of the dimension of the cones. Since these are not sufficient to describe $C(\mathcal{P}_{2n+1})$ we then look for more complicated constraints and present a set of $2n + 3$ valid cubic conditions. We then establish similar results for the cone of positive polynomials over a finite interval and the cone of positive trigonometric polynomials. In an Appendix we give some examples of cones where our approach can be used to show that no non-trivial bilinear optimality constraints exist.

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Introduction

In this paper we study the complementarity conditions for the cone of positive polynomials and its dual, the closure of the moment cone, over the real line. We also extend some of our results to the positive polynomials and moment vectors over a finite interval of $\mathbb{R}$, and also over trigonometric polynomials. Cones of positive polynomials are a family of non-symmetric cones with many practical applications, such as shape-constrained regression and the approximation of nonnegative functions (see for example [1]).

It is well-known that positive polynomials over the real line are precisely those polynomials that can be written as the sum of squares of other polynomials. This property directly leads to expression of the cone of positive polynomials as a linear map of the cone of positive semidefinite matrices, see for example Nesterov [7]. For instance optimization over the cone of positive polynomials of degree n can be expressed as a semidefinite program over $n \times n$ Hankel matrices. However, this approach may significantly increase the size of the problem and introduce degeneracy, therefore in many cases it is not practical. This motivates us to look for solution methods and optimality conditions which directly apply to the cone of positive polynomials.

A pair of vectors $(\mathbf{x}, \mathbf{s})$, $\mathbf{x} \in \mathcal{K}$, $\mathbf{s} \in \mathcal{K}^*$ is said to satisfy the complementary slackness condition if $\mathbf{x}^\mathsf{T}\mathbf{s} = 0$. In this case it is known ([5]) that if $\mathcal{K}$ is a proper cone (convex, closed, pointed and having a non-empty interior), such pairs constitute an n-dimensional manifold

$$\mathsf{C}(\mathcal{K}) = \left\{(\mathbf{x}, \mathbf{s}) : \mathbf{x} \in \mathcal{K},\ \mathbf{s} \in \mathcal{K}^*,\ \mathbf{x}^\mathsf{T}\mathbf{s} = 0\right\} \subset \mathbb{R}^n \times \mathbb{R}^n.$$

One of the key properties of symmetric cones, i.e. proper self-dual cones whose automorphism group acts transitively on their interior, is that the corresponding manifold $\mathsf{C}(\mathsf{K})$ can be expressed by a set of bilinear relations. For instance in the cone of positive semidefinite matrices $< \mathsf{X}, \mathsf{S} >= \text{Trace}(\mathsf{XS}) = 0$ implies $\mathsf{XS} + \mathsf{SX} = 0$. The latter matrix equality corresponds to a set of scalar bilinear equalities. In turn the existence of bilinear complementary relations has an impact on the design of efficient and numerically stable primal-dual interior point algorithms. Furthermore, the bilinear relationship among vectors indicates existence of an algebra underlying the optimization problem. For instance for symmetric cones the underlying algebras are the Euclidean Jordan algebras which can be exploited effectively to derive primal-dual interior point algorithms, see for example Schmieta and Alizadeh [2] and Faybusovich [4] for further details.

An interesting question is therefore this: for which cones K the manifold $\mathsf{C}(\mathsf{K})$ can be characterized by a set of bilinear relations? In this paper we show that for the cone of positive polynomials $\mathsf{C}(\mathsf{K})$ cannot be entirely represented by such bilinear relations, in fact in general there are only four bilinear complementary relations. We present similar results for nonnegative trigonometric polynomials and nonnegative polynomials on an interval. As in all these cases bilinear constraints prove insufficient to completely describe $\mathsf{C}(\mathcal{K})$, we also look for more complicated conditions. The methods we apply rely on results allowing the parametrization of the boundaries of the cones in question based on the theory of Tchebycheff systems ([6]). In the Appendix we present a version of our approach which can be applied in the absence of such a parametrization, along with some negative results.

The paper is structured as follows: we first present necessary background information about the cone of positive polynomials $\mathcal{P}_{2n+1}$ and its dual, the closure of the moment cone $\mathcal{M}_{2n+1}$. The next section contains our main results concerning bilinear optimality constraints. Section 3 deals with cubic constraints while sections 4 and 5 extend our results for polynomials over finite intervals and trigonometric polynomials. In the Appendix we give some negative results for specific low-dimensional cones.

Notation

For a polynomial represented by the vector of its coefficients $\mathbf{p} = (p_0, \ldots, p_n)$ the corresponding polynomial function is denoted by $p(t) = p_0 + p_1 t + p_2 t^2 + \ldots + p_n t^n$.

Throughout the paper we adopt the following convention: if for a range of indices the lower bound is greater than the upper bound, the range is considered to be empty.

For a matrix $(A_{ij})_{i,j=1,\ldots,n} \in \mathbb{R}^{n\times n}$ let $\mathrm{vec}(A) = (A_{11}, A_{12}, \ldots, A_{nn}) \in \mathbb{R}^{n^2}$ and let $\mathrm{Null}(A)$ denote the nullspace of A.

The convex hull of a set $S \subset \mathbb{R}^n$ is denoted by $\mathrm{conv}(S)$, the closure of S is denoted by $\bar{S}$. The Minkowski sum of sets $S, T \subset \mathbb{R}^n$ is $S + T = \{\mathbf{s} + \mathbf{t} \mid s \in S,\ t \in T\}$.

The ray spanned by a vector $\boldsymbol{v} \in \mathbb{R}^n$ is denoted by $\mathbb{R}_+\boldsymbol{v} = \{\lambda\boldsymbol{v} \mid \lambda \geq 0\}$. The linear space spanned by a set of vectors $V = \{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k\}$ is denoted by $\langle V\rangle = \langle \boldsymbol{v}_1, \ldots, \boldsymbol{v}_k\rangle$.

If $\mathcal{A} = a_1, a_2, \ldots, a_n$ and $\mathcal{B} = b_1, b_2, \ldots, b_m$ are two sequences, $\mathcal{A}, \mathcal{B}$ denotes their concatenation, $a_1, \ldots, a_n, b_1, \ldots, b_m$.

The reverse of a vector $\boldsymbol{v} = (v_1, v_2, \ldots, v_{n-1}, v_n)$ is denoted by $\overleftarrow{\boldsymbol{v}} = (v_n, v_{n-1}, \ldots, v_2, v_1)$.

For the dual of a cone $\mathcal{K} \subset \mathbb{R}^n$ we use the notation $\mathcal{K}^* = \left\{\mathbf{s} \in \mathbb{R}^n \mid \mathbf{x}^T\mathbf{s} \geq 0\ \forall \mathbf{x} \in \mathcal{K}\right\}$.

The parity of an integer m is denoted by $m \pmod 2 = \begin{cases} 0 \text{ if } m \text{ is even} \\ 1 \text{ if } m \text{ is odd} \end{cases}$.

1 Positive Polynomials over $\mathbb{R}$ and the Moment Cone

Let us first introduce the cone of positive polynomials and the moment cone:

Definition 1

$$\mathcal{P}_{2n+1} = \left\{(p_0, \ldots, p_{2n}) \in \mathbb{R}^{2n+1} \mid p(t) = p_0 + p_1 t + p_2 t^2 + \ldots + p_{2n} t^{2n} \geq 0 \quad \forall t \in \mathbb{R}\right\}$$

denotes the cone of positive polynomials of degree $2n$.

The moment cone of dimension $2n+1$ *is defined as*

$$\mathcal{M}_{2n+1} = \mathrm{conv}\{\mathbf{c}_{2n+1}(t) \mid t \in \mathbb{R}\}, \text{ where } \mathbf{c}_{2n+1}(t) \stackrel{\text{def}}{=} (1, t, t^2, \ldots, t^{2n})^T.$$

Remark 1 *This is not the traditional definition of the moment cone. See [6] (Ch.VI) for the original definition and proof of its equivalence with the one given above.*

The cone of positive polynomials and the moment cone are closely related (see also [6]):

Proposition 1 $\mathcal{P}^*_{2n+1} = \bar{\mathcal{M}}_{2n+1}$.

Proof: Since $\mathcal{P}_{2n+1}$ is a closed convex cone, it is sufficient to show $\mathcal{M}^*_{n+1}$=$\mathcal{P}_{n+1}$.

Let $\mathbf{p} \in \mathcal{P}_{n+1}$ and $\mathbf{c} \in \mathcal{M}_{n+1}$. Then $\mathbf{p}^T\mathbf{c} = \mathbf{p}^T \sum_i \lambda_i(1, t_i, t_i^2, \ldots, t_i^n)^T = \sum_i \lambda_i P(t_i) \geq 0$. Thus $\mathcal{P}_{n+1} \subseteq \mathcal{M}^*_{n+1}$.

We also need to show that $\mathcal{M}^*_{n+1} \subseteq \mathcal{P}_{n+1}$. Suppose that $\mathbf{p}$ belongs to M^*_{n+1}, but not to $\mathcal{P}_{n+1}$. Since $\mathbf{p} \notin \mathcal{P}_{n+1}$, there exits $t \in \mathbb{R}$ such that $P(t) < 0$. Then we have $\mathbf{p}^T(1, t_i, t_i^2, \ldots, t_i^n)^T < 0$, which is impossible since $(1, t_i, t_i^2, \ldots, t_i^n)^T \in \mathcal{M}_{n+1}$ and $\mathbf{p} \in \mathcal{M}^*_{n+1}$. ■

1.1 Parametrization of $\mathcal{P}_{2n+1}$ and $\bar{\mathcal{M}}_{2n+1}$

Theorem 1 *For a polynomial* $\mathbf{p}$ *of degree* $2n$ *with leading coefficient* 1 *the following are equivalent:*

1. $p(t) \geq 0$ $(\forall t \in \mathbb{R})$
2. *There exist* $t_i \in \mathbb{R},\ \gamma_i \geq 0\ \ (i = 1, \ldots, n)$ *such that*
   $p(t) = (t - t_1)^2\left((t-t_2)^2\left((t-t_3)^2\left(\ldots\left((t-t_n)^2 + \gamma_n\right) + \gamma_{n-1}\right) + \gamma_{n-2}\right) + \ldots + \gamma_2\right) + \gamma_1$

Proof: The implication $(2) \Rightarrow (1)$ is trivial. Let us now assume that $p(t) \geq 0$ $(\forall t \in \mathbb{R})$ holds. The claim is trivial for $n = 0$. Assume $n \geq 1$ and let $\gamma_1 \stackrel{\text{def}}{=} \min p(t) \geq 0$. Then $p(t) - \gamma_1$ is a nonnegative polynomial with some real root t_1. By nonnegativity the root t_1 has even multiplicity, therefore $p(t) - \gamma_1 = (t - t_1)^2 q(t)$, where $q(t)$ is a nonnegative polynomial of degree $2n - 2$. The claim immediately follows by induction. ■

Notice that if $\mathbf{p}$ is on the boundary of $\mathcal{P}_{2n+1}$ we have $\gamma_1 = 0$.

Proposition 2 $\mathbf{p} \in \mathcal{P}_{2n+1} \Leftrightarrow \overleftarrow{\mathbf{p}} \in \mathcal{P}_{2n+1}$

Proof: The claim follows immediately from the fact that for $x \neq 0$ we have $\overleftarrow{p}(x) = x^{2n} p(\frac{1}{x})$. ■

The proof of the following proposition can be found in [6] (Ch.VI).

Proposition 3 *$\bar{\mathcal{M}}_{2n+1} = \mathbb{R}_+\boldsymbol{e} + \mathcal{M}_{2n+1}$, where $\boldsymbol{e} = (0, 0, \ldots, 0, 1) \in \mathbb{R}^{2n+1}$.*

Corollary 1 *Every vector $\bar{\mathbf{c}} \in \bar{\mathcal{M}}_{2n+1}$ can be written in the form*

$$\bar{\mathbf{c}} = \lambda_0 \boldsymbol{e} + \sum_{j=1}^{r} \lambda_j \mathbf{c}_{2n+1}(t_j), \tag{1}$$

where $r \geq 0,\ \lambda_j > 0,\ t_j \in \mathbb{R}\ (j = 1, \ldots, r), \lambda_0 \geq 0$.

1.2 An Equivalent Condition for Complementary Slackness

In this section we characterize pairs of primal and dual vectors satisfying the complementary slackness condition.

Theorem 2 *Let $\mathbf{p} = (p_0, \ldots, p_{2n})$ and $\bar{\mathbf{c}} = \lambda_0 \boldsymbol{e} + \sum_{j=1}^{r} \lambda_j \mathbf{c}_{2n+1}(t_j)$, $\lambda_j > 0\ \ (j = 1, \ldots, r)$, $\lambda_0 \geq 0$ be arbitrary vectors in $\mathcal{P}_{2n+1}$ and $\bar{\mathcal{M}}_{2n+1}$, respectively. Then $\mathbf{p}^{\mathsf{T}} \bar{\mathbf{c}} = 0$ (i.e. $(\mathbf{p}, \bar{\mathbf{c}}) \in C(\mathcal{P}_{2n+1})$) if and only if $p(t_j) = 0\ \ (j = 1 \ldots r)$ and $\lambda_0 p_{2n} = 0$.*

Proof: Notice that since the leading coefficient of a positive polynomial cannot be negative, $p_{2n} \geq 0$. Using the parametrization in Corollary 1 we have $\mathbf{p}^{\mathsf{T}} \bar{\mathbf{c}} = \mathbf{p}^{\mathsf{T}} \left(\lambda_0 \boldsymbol{e} + \sum_{j=1}^{r} \lambda_j \mathbf{c}_{2n+1}(t_j) \right)$ $= \lambda_0 \mathbf{p}^{\mathsf{T}} \boldsymbol{e} + \sum_{j=1}^{r} \lambda_j \mathbf{p}^{\mathsf{T}} \mathbf{c}_{2n+1}(t_j) = \lambda_0 p_{2n} + \sum_{j=1}^{r} \lambda_j p(t_j)$. Since on the right hand side we have the sum of nonnegative terms, the expression equals 0 if and only if every term is 0, i.e. $p(t_j) = 0\ \ (j = 1 \ldots r)$ and $\lambda_0 p_{2n} = 0$. As on the left hand side we have $\mathbf{p}^{\mathsf{T}} \bar{\mathbf{c}}$, the theorem immediately follows. ∎

Let us notice that since every root of a nonnegative polynomial is a local minimum, it is a multiple root with even multiplicity. Thus we can obtain the following parametrization of $C(\mathcal{P}_{2n+1})$:

Corollary 2 *Consider a pair of vectors $(\mathbf{p}, \bar{\mathbf{c}}) \in C(\mathcal{P}_{2n+1})$; by Proposition 3 $\bar{\mathbf{c}} = \lambda_0 \boldsymbol{e} + \mathbf{c}$ for some $\mathbf{c} \in \mathcal{M}_{2n+1}$ and $\lambda_0 \geq 0$ such that $\lambda_0 p_{2n} = 0$. Then there exist $\lambda_j \geq 0\ \ (j = 1 \ldots r), \gamma_j \geq 0\ \ (j = r+1 \ldots m),\ \ t_j \in \mathbb{R}\ \ (j = 1 \ldots m),\ r \leq m \leq n$ such that*

$$c_i = \sum_{j=1}^{r} \lambda_j t_j^i,\ i = 0, 1, \ldots, 2n, \tag{2}$$

$$\begin{aligned} p(t) = (t - t_1)^2 (t - t_2)^2 \ldots (t - t_r)^2 \Big(\big((t - t_{r+1})^2 \big((t - t_{r+2})^2 (\ldots \\ \ldots \big((t - t_m)^2 \big) + \gamma_m \big) + \gamma_{m-1} \big) + \ldots + \gamma_{r+2} \big) + \gamma_{r+1} \Big) \end{aligned} \tag{3}$$

$\big($in the case $m = r$ we have $p(t) = (t - t_1)^2 (t - t_2)^2 \ldots (t - t_r)^2$ $\big)$. Notice that if $\bar{\mathbf{c}} \in \mathcal{M}_{2n+1}$, we can choose $\lambda_0 = 0$.

Also, for any choice of $t_i \in \mathbb{R},\ \lambda_j \geq 0,\ \gamma_k \geq 0$ the formulas (2) and (3) yield a pair $(\mathbf{p}, \mathbf{c}) \in C(\mathcal{P}_{2n+1})$, where $\mathbf{c} \in \mathcal{M}_{2n+1}$.

For the latter sections it will be useful to rewrite (3) in the following form:

$$p(t) = p^{(m+1)}(t) + \sum_{k=r+1}^{m} \gamma_k p^{(k)}(t) \tag{4}$$

where $p^{(k)}(t) = (t-t_1)^2(t-t_2)^2\dots(t-t_{k-1})^2$. We will also use the following basic fact:

Observation 1 *$P_i^{(n)}$ denotes the coefficient of t^i in $\prod_{j=1}^n (t-t_j)^2$. The term $\prod_{j=1}^n t_j^{m_j}$ appears in $P_i^{(n)}$ with a non-zero coefficient if and only if $0 \le m_j \le 2$ $(j = 1,\dots,n)$ and $\sum_{j=1}^n m_j = 2n - i$. In this case the coefficient is equal to $\prod_{j=1}^n (-2)^{\left(m_j \pmod 2\right)}$.*

2 Main Results

We say that a matrix $(A_{ij})_{i,j=0,\dots,2n}$ is a bilinear optimality constraint if for all $(\mathbf{p}, \bar{\mathbf{c}}) \in C(\mathcal{P}_{2n+1})$ $\sum_{i,j} A_{ij} p_i c_j = 0$ holds.
A is a weak optimality constraint if $\sum_{i,j} A_{ij} p_i c_j = 0$ holds for all pairs $(\mathbf{p}, \mathbf{c}) \in C(\mathcal{P}_{2n+1})$ for which $\mathbf{c} \in \mathcal{M}_{2n+1}$.

Definition 2 *Let us define the matrix M as follows: the columns are indexed by the pairs $(0,0),(0,1),(0,2),\dots,(2n,2n)$ while the rows are indexed by vectors of the form $\mathbf{m} = (m_1, m_2, \dots, m_n) \in S_n$, where $S_n = \{0,\dots,2n+2\} \times \{0,1,2\} \times \{0,1,2\} \times \dots \times \{0,1,2\}$. The entry $M_{(i,j),\mathbf{m}}$ is the coefficient of $\prod_{k=1}^n t_k^{m_k}$ in $P_i^{(n)} t_1^j$.*

The next proposition follows directly from Observation 1.

Proposition 4 *Let $0 \le i,j \le 2n$ and $\mathbf{m} \in S_n$. Let $\Delta \stackrel{\text{def}}{=} \prod_{k=2}^n (-2)^{(m_k \pmod 2)}$ and let $(*)$ denote the condition $i - j + \sum_{k=1}^n m_k = 2n$. Then*

$$M_{(i,j),\mathbf{m}} = \begin{cases} \Delta & \text{if } j = m_1 \text{ and } (*) \text{ holds} \\ -2\Delta & \text{if } j = m_1 - 1 \text{ and } (*) \text{ holds} \\ \Delta & \text{if } j = m_1 - 2 \text{ and } (*) \text{ holds} \\ 0 & \text{otherwise} \end{cases}. \tag{5}$$

Theorem 3 *A is a weak bilinear optimality constraint if and only if* $\mathrm{vec}(A) \in \mathrm{Null}(M)$.

Proof: First let us assume that A is a weak bilinear optimality constraint. Let $t_i \in \mathbb{R}$ $(i = 1,\dots,n)$ be arbitrary numbers, $\mathbf{c} \stackrel{\text{def}}{=} \mathbf{c}_{2n+1}(t_1)$, $p(t) \stackrel{\text{def}}{=} \prod_{i=1}^n (t-t_i)^2$. Then by Corollary 2 $(\mathbf{p}, \mathbf{c}) \in C(\mathcal{P}_{2n+1})$, $\mathbf{c} \in \mathcal{M}_{2n+1}$. Therefore according to (5) we have

$$0 = \sum_{i,j} A_{ij} p_i c_j = \sum_{i,j} A_{ij} \sum_{\mathbf{m} \in S_n} M_{(i,j),\mathbf{m}} \prod_{k=1}^n t_k^{m_k} = \sum_{\mathbf{m} \in S_n} \prod_{k=1}^n t_k^{m_k} \sum_{i,j} A_{ij} M_{(i,j),\mathbf{m}}.$$

Since the above polynomial vanishes for every choice of $t_i \in \mathbb{R}$ $(i = 1, \ldots, n)$, for all $\mathbf{m} \in S_n$ the coefficient of the term $\prod_{k=1}^{n} t_k^{m_k}$ has to be zero, i.e. $\sum_{i,j} A_{ij} M_{(i,j),\mathbf{m}} = 0$. Since the left hand side is the scalar product of vec(A) with the row of M indexed by $\mathbf{m}$, vec$(A) \in$ Null(M).

Now let us assume vec$(A) \in$ Null(M). Notice that if either $\mathbf{p}$ or $\mathbf{c}$ is zero, then $\sum_{i,j} A_{ij} p_i c_j = 0$. Now consider a pair of nonzero vectors $(\mathbf{p}, \mathbf{c}) \in C(\mathcal{P}_{2n+1})$, $\mathbf{c} \in \mathcal{M}_{2n+1}$. By Corollary 2 $c_i = \sum_{j=1}^{r} \lambda_j t_j^i$, $i = 0, 1, \ldots, 2n$ for some $\lambda_j > 0$ $(j = 1, \ldots, r)$, where $1 \leq r \leq n$. Therefore

$$\sum_{i,j} A_{ij} p_i c_j = \sum_{i,j} A_{ij} p_i \sum_{k=1}^{r} \lambda_k t_k^j = \sum_{k=1}^{r} \lambda_k \sum_{i,j} A_{ij} p_i t_k^j,$$

so it suffices to show $\sum_{i,j} A_{ij} p_i t_k^j = 0$ for all $k = 1, \ldots, r$. By symmetry this is equivalent to showing $\sum_{i,j} A_{ij} p_i t_1^j = 0$. By (4) $p(t) = p^{(m+1)}(t) + \sum_{k=r+1}^{m} \gamma_k p^{(k)}(t)$, where $p^{(k)}(t) = (t - t_1)^2 (t - t_2)^2 \ldots (t - t_{k-1})^2$. Thus

$$\sum_{i,j} A_{ij} p_i t_1^j = \sum_{i,j} A_{ij} \left(p_i^{(m+1)} + \sum_{k=r+1}^{m} \gamma_k p_i^{(k)} \right) t_1^j = \sum_{i,j} A_{ij} p_i^{(m+1)} t_1^j + \sum_{k=r+1}^{m} \gamma_k \sum_{i,j} A_{ij} p_i^{(k)} t_1^j.$$

Therefore it suffices to show $\sum_{i,j} A_{ij} p_i^{(k)} t_1^j = 0$ for all $k = 2, \ldots, n+1$.

Lemma 1 *Let* $k \geq 2$, $\sum_{i,j} A_{ij} p_i^{(k+1)} t_1^j = 0$ $(\forall t_l \in \mathbb{R},\ l = 1, \ldots, k)$. *Then* $\sum_{i,j} A_{ij} p_i^{(k)} t_1^j = 0$.

Proof: Since $p^{(k+1)}(t) = (t^2 - 2t_k t + t_k^2) p^{(k)}(t)$, we have

$$\sum_{i,j} A_{ij} p_i^{(k+1)} t_1^j = t_k^2 \left(\sum_{i,j} A_{ij} p_i^{(k)} t_1^j \right) - 2 t_k t \left(\sum_{i,j} A_{ij} p_i^{(k)} t_1^j \right) + t^2 \left(\sum_{i,j} A_{ij} p_i^{(k)} t_1^j \right).$$

As t_k is not featured in $\sum_{i,j} A_{ij} p_i^{(k)} t_1^j$, in order for the above polynomial to vanish everywhere $\sum_{i,j} A_{ij} p_i^{(k)} t_1^j = 0$ must hold. ■

According to Lemma 1 it suffices to show $\sum_{i,j} A_{ij} p_i^{(n+1)} t_1^j = 0$. Since vec$(A) \in$ Null(M), for every $\mathbf{m} \in S_n$ we have $\sum_{i,j} A_{ij} M_{(i,j),\mathbf{m}} = 0$. Thus

$$\sum_{i,j} A_{ij} p_i^{(n+1)} t_1^j = \sum_{i,j} A_{ij} \sum_{\mathbf{m} \in S_n} M_{(i,j),\mathbf{m}} \prod_{k=1}^{n} t_k^{m_k} = \sum_{\mathbf{m} \in S_n} \prod_{k=1}^{n} t_k^{m_k} \sum_{i,j} A_{ij} M_{(i,j),\mathbf{m}} = 0.$$

■

Corollary 3 *The number of the linearly independent weak bilinear optimality constraints equals* $(2n+1)^2 - \text{rank}(M)$.

Theorem 4 $\text{rank}(M) = (2n+1)^2 - 4$.

Proof: To show $\text{rank}(M) \leq (2n+1)^2 - 4$ we present 4 linearly independent weak bilinear optimality constraints:

$$A_{ij}^{(1)} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$
$$A_{ij}^{(2)} = \begin{cases} 2n - i & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$
$$A_{ij}^{(3)} = \begin{cases} 2n - i & \text{if } i = j - 1 \\ 0 & \text{otherwise} \end{cases}$$
$$A_{ij}^{(4)} = \begin{cases} i & \text{if } i = j + 1 \\ 0 & \text{otherwise} \end{cases}$$

These are clearly linearly independent for $n \geq 1$, therefore by Theorem 3 it suffices to show $\text{vec}(A_{ij}^{(k)}) \in \text{Null}(M)$, i.e. that for any $\mathbf{m} \in S_n$, $1 \leq k \leq 4$: $\sum_{i,j} A_{ij}^{(k)} M_{(i,j),\mathbf{m}} = 0$. Let $\mathbf{m} \in S_n$ be fixed.

If $m_1 + \ldots + m_n \neq 2n$ then by Proposition 4 $M_{(i,i),\mathbf{m}} = 0$ $(\forall i)$, therefore $\sum_{i,j} A_{ij}^{(1)} M_{(i,j),\mathbf{m}} = \sum_i M_{(i,i),\mathbf{m}} = 0$ and $\sum_{i,j} A_{ij}^{(2)} M_{(i,j),\mathbf{m}} = \sum_i (2n - i) M_{(i,i),\mathbf{m}} = 0$.
If $m_1 + \ldots + m_n = 2n$ then we have $m_1 \geq 2$ and (again by Proposition 4) $\sum_{i,j} A_{ij}^{(1)} M_{(i,j),\mathbf{m}} = \sum_i M_{(i,i),\mathbf{m}} = M_{(m_1,m_1),\mathbf{m}} + M_{(m_1-1,m_1-1),\mathbf{m}} + M_{(m_1-2,m_1-2),\mathbf{m}} = \Delta - 2\Delta + \Delta = 0$ and $\sum_{i,j} A_{ij}^{(2)} M_{(i,j),\mathbf{m}} = \sum_i (2n-i) M_{(i,i),\mathbf{m}} = M_{(m_1,m_1),\mathbf{m}} + M_{(m_1-1,m_1-1),\mathbf{m}} + M_{(m_1-2,m_1-2),\mathbf{m}} = (2n - m_1)\Delta - 2(2n - (m_1 - 1))\Delta + (2n - (m_1 - 2))\Delta = 0$.

If $m_1 + \ldots + m_n \neq 2n + 1$ then $M_{(i,i+1),\mathbf{m}} = 0$ $(\forall i)$, therefore $\sum_{i,j} A_{ij}^{(3)} M_{(i,j),\mathbf{m}} = \sum_i (2n - i) M_{(i,i+1),\mathbf{m}} = 0$.
If $m_1 + \ldots + m_n = 2n + 1$ then we have $m_1 \geq 3$, therefore $\sum_{i,j} A_{ij}^{(3)} M_{(i,j),\mathbf{m}} = \sum_i (2n - i) M_{(i,i+1),\mathbf{m}} = (2n - (m_1 - 1)) M_{(m_1-1,m_1),\mathbf{m}} + (2n - (m_1 - 2)) M_{(m_1-2,m_1-1),\mathbf{m}} + (2n - (m_1 - 3)) M_{(m_1-3,m_1-2),\mathbf{m}} = (2n - (m_1 - 1))\Delta - 2(2n - (m_1 - 2))\Delta + (2n - (m_1 - 3))\Delta = 0$.

If $m_1 + \ldots + m_n \neq 2n - 1$ then $M_{(i,i-1),\mathbf{m}} = 0$ $(\forall i)$, therefore $\sum_{i,j} A_{ij}^{(4)} M_{(i,j),\mathbf{m}} = \sum_i i M_{(i,i-1),\mathbf{m}} = 0$.
If $m_1 + \ldots + m_n = 2n - 1$ then we have $m_1 \geq 1$. If $m_1 \geq 2$, we have $\sum_{i,j} A_{ij}^{(4)} M_{(i,j),\mathbf{m}} = \sum_i i M_{(i,i-1),\mathbf{m}} = (m_1 + 1)) M_{(m_1+1,m_1),\mathbf{m}} + m_1 M_{(m_1,m_1-1),\mathbf{m}} + (m_1 - 1) M_{(m_1-1,m_1-2),\mathbf{m}} = (m_1 + 1)\Delta - 2m_1\Delta + (m_1 - 1)\Delta = 0$ while for $m_1 = 1$ we get $\sum_{i,j} A_{ij}^{(4)} M_{(i,j),\mathbf{m}} = \sum_i i M_{(i,i-1),\mathbf{m}} = (1 + 1)\Delta - 2\Delta = 0$.

In order to show $\text{rank}(M) \geq (2n+1)^2 - 4$ we present a $(2n+1)^2 - 4$ by $(2n+1)^2 - 4$ nonsingular submatrix of M. We prove nonsingularity by rearranging the columns and rows of the submatrix to obtain an upper triangular matrix with non-zero diagonal entries. Let us first define the following sequences of indices:

$$\text{For } k = 0, \ldots, n-2 \text{ let } \begin{array}{l} \mathcal{A}_k \stackrel{\text{def}}{=} (2n-k, 0), (2n-k, 1), \ldots, (2n-k, 2n-k) \\ \mathcal{B}_k \stackrel{\text{def}}{=} (k, k), (k, k+1), \ldots, (k, 2n). \end{array} \tag{6}$$

$$\text{For } k = n-1, \ldots, 2n-2 \text{ let } \begin{array}{l} \mathcal{A}_k \stackrel{\text{def}}{=} (2n-k, 0), (2n-k, 1), \ldots, (2n-k, 2n-k-1) \\ \mathcal{B}_k \stackrel{\text{def}}{=} (k, k+1), (k, k+2), \ldots, (k, 2n). \end{array} \tag{7}$$

Let $\mathcal{C} \stackrel{\text{def}}{=} (n, n)$.

Let $\bar{S} \stackrel{\text{def}}{=} \left\{(m_2, \ldots, m_n) \in \{0, 1, 2\}^{n-1} \mid m_2 \geq m_3 \geq \ldots \geq m_n\right\}$. Notice that for all $0 \leq k \leq 2n-2$ there exists a unique $\bar{\mathbf{m}}^{(k)} \in \bar{S}$ such that $\sum_{l=2}^{n} \bar{\mathbf{m}}_l^{(k)} = k$.

$$\text{For } k = 0, \ldots, n-2 \text{ let } \mathcal{D}_k \stackrel{\text{def}}{=} \begin{pmatrix} 0 \\ \bar{\mathbf{m}}^{(k)} \end{pmatrix}, \begin{pmatrix} 1 \\ \bar{\mathbf{m}}^{(k)} \end{pmatrix}, \ldots, \begin{pmatrix} 2n-k \\ \bar{\mathbf{m}}^{(k)} \end{pmatrix}$$

$$\mathcal{E}_k \stackrel{\text{def}}{=} \begin{pmatrix} k+2 \\ \bar{\mathbf{m}}^{(2n-2-k)} \end{pmatrix}, \begin{pmatrix} k+3 \\ \bar{\mathbf{m}}^{(2n-2-k)} \end{pmatrix}, \ldots, \begin{pmatrix} 2n+2 \\ \bar{\mathbf{m}}^{(2n-2-k)} \end{pmatrix}. \tag{8}$$

$$\text{For } k = n-1, \ldots, 2n-2 \text{ let } \mathcal{D}_k \stackrel{\text{def}}{=} \begin{pmatrix} 0 \\ \bar{\mathbf{m}}^{(k)} \end{pmatrix}, \begin{pmatrix} 1 \\ \bar{\mathbf{m}}^{(k)} \end{pmatrix}, \ldots, \begin{pmatrix} 2n-k-1 \\ \bar{\mathbf{m}}^{(k)} \end{pmatrix}$$

$$\mathcal{E}_k \stackrel{\text{def}}{=} \begin{pmatrix} k+3 \\ \bar{\mathbf{m}}^{(2n-2-k)} \end{pmatrix}, \begin{pmatrix} k+4 \\ \bar{\mathbf{m}}^{(2n-2-k)} \end{pmatrix}, \ldots, \begin{pmatrix} 2n+2 \\ \bar{\mathbf{m}}^{(2n-2-k)} \end{pmatrix}. \tag{9}$$

Let $\mathcal{F} \stackrel{\text{def}}{=} \begin{pmatrix} n+1 \\ \bar{\mathbf{m}}^{(n-1)} \end{pmatrix}$.

Proposition 5 *The (rearranged) $(2n+1)^2 - 4$ by $(2n+1)^2 - 4$ submatrix of M defined by the column index sequence $\mathcal{A}_0, \mathcal{B}_0, \mathcal{A}_1, \mathcal{B}_1, \ldots, \mathcal{A}_n, \mathcal{C}, \mathcal{B}_n, \mathcal{A}_{n+1}, \mathcal{B}_{n+1}, \ldots, \mathcal{A}_{2n-2}, \mathcal{B}_{2n-2}$ and the row index sequence $\mathcal{D}_0, \mathcal{E}_0, \mathcal{D}_1, \mathcal{E}_1, \ldots, \mathcal{D}_n, \mathcal{F}, \mathcal{E}_n, \mathcal{D}_{n+1}, \mathcal{E}_{n+1}, \ldots, \mathcal{D}_{2n-2}, \mathcal{E}_{2n-2}$ is upper triangular with nonzero diagonal entries.*

Proof: Let us first observe that for every $0 \leq k \leq n-2$ the sequences $\mathcal{A}_k$, $\mathcal{B}_k$, $\mathcal{D}_k$ and $\mathcal{E}_k$ are all of length $2n-k+1$, for $n-1 \leq k \leq 2n-2$ they are all of length $2n-k$ and $\mathcal{C}$ and $\mathcal{F}$ are both of length 1. Therefore the overall length of both the column and row index sequences is indeed $1 + \sum_{k=1}^{n-2}(2n-k+1) + \sum_{k=n-1}^{2n-2}(2n-k) = 1 + 2(n-1) + 2\sum_{k=0}^{2n-2} 2n-k = 1 + 2(n-1) + (2n-1)(2n+2) = 4n^2 + 4n - 3 = (2n+1)^2 - 4$.

Now we are going to show for every column that its diagonal entry is nonzero while all entries below the diagonal are zero.

Case 1 The column is indexed by an element of some $\mathcal{A}_k,\ 0 \le k \le 2n-2$.
The column index is of the form (i, j), where $i = 2n-k$ and $0 \le j \le 2n-k$. Now consider a row index $\mathbf{m} = \begin{pmatrix} m_1 \\ \bar{\mathbf{m}}^{(k')} \end{pmatrix}$ in some $\mathcal{D}_{k'},\ k' \ge k$. By Proposition 4 the corresponding element of our column is nonzero if and only if we have $i - j + m_1 + k' = 2n$ and $j \le m_1 \le j+2$. Using the latter inequality we have $i-j+m_1+k' \ge 2n-k-j+j+k = 2n$, where equality holds if and only if $m_1 = j$ and $k' = k$, i.e. for the diagonal element.
For a row index $\mathbf{m} = \begin{pmatrix} m_1 \\ \bar{\mathbf{m}}^{(2n-2-k')} \end{pmatrix}$ in some $\mathcal{E}_{k'}(k' \ge k)$ we have $m_1 \ge k'+2$. Assume for contradiction that the corresponding element in our column is nonzero, then by Proposition 4 $m_1 \ge j$ and $i-j+m_1+2n-2-k' = 2n$. From the latter equality using $i = 2n-k$ and $m_1 \ge k'+2$ we obtain $j \ge 2n-k$ which implies $j = 2n-k$ and (by (7)) $k \le n-2$. Now $(*)$ implies $m_1 = k'+2$, thus $k' = m_1 - 2 \ge j-2 = 2n-k-2 \ge 2n-(n-2)-2 = n$, therefore (by (9)) $m_1 \ge k'+3$, contradicting $m_1 = k'+2$.
Finally we verify that the entry of our column in the row indexed by $\mathbf{m} = \begin{pmatrix} n+1 \\ \bar{\mathbf{m}}^{(n-1)} \end{pmatrix} \in \mathcal{F}$ is zero. Assume for contradiction that it is nonzero, then by Proposition 4 we have $j \le n+1$ and $i-j+n+1+(n-1) = 2n$. From the latter we get $j = i = 2n-k$, which (by (7)) implies $k \le n-2$. Therefore $n+1 \ge j = 2n-k \ge 2n-(n-2) = n+2$, which is a contradiction.

Case 2 The column is indexed by an element of some $\mathcal{B}_k,\ 0 \le k \le 2n-2$.
The column index is of the form $(k, j),\ k \le j$. Now consider a row index $\mathbf{m} = \begin{pmatrix} m_1 \\ \bar{\mathbf{m}}^{(k')} \end{pmatrix}$ in some $\mathcal{D}_{k'},\ k' > k$. Assume for contradiction that the corresponding element of our column is nonzero, then by Proposition 4 we have $m_1 \le j+2$ and $k-j+m_1+k' = 2n$. Since $k \le j$ and (by (9)) $m_1 \le 2n-k'$, we have $2n = k-j+m_1+k' \le k-k+(2n-k')+k' = 2n$. As equality must hold, we get $k = j$ (implying, by (7), $k \le n-2$) and $m_1 = 2n-k'$ (implying, by (9), $k' \le n-2$). Therefore we have $n+2 = 2n-(n-2) \le 2n-k' = m_1 \le j+2 = k+2 \le (n-2)+2 = n$, which is a contradiction.
For a row index $\mathbf{m} = \begin{pmatrix} m_1 \\ \bar{\mathbf{m}}^{(2n-2-k')} \end{pmatrix}$ in some $\mathcal{E}_{k'}(k' \ge k)$ we have $m_1 \ge k'+2$. By Proposition 4 the corresponding element of our column is nonzero if and only if $j \le m_1 \le j+2$ and $k-j+m_1+2n-2-k' = 2n$. Using the above we get $2n = k-j+m_1+2n-2-k' \le k-j+(j+2)+2n-2-k' = k-k'+2n \le 2n$. Here equality holds only for $k = k'$, $m_1 = j+2$, i.e. for the diagonal element.
Finally we verify that the entry of our column in the row indexed by $\mathbf{m} = \begin{pmatrix} n+1 \\ \bar{\mathbf{m}}^{(n-1)} \end{pmatrix} \in \mathcal{F}$ is zero. Assume for contradiction that it is nonzero, then by Proposition 4 we have $k-j+(n+1)+(n-1) = 2n$, i.e. $k = j$ (implying, by (7), $k \le n-2$) and $n+1 \le j+2 = k+2$, which implies $k \ge n-1$ contradicting $k \le n-2$.

Case 3 The column is indexed by the element $(n, n) \in \mathcal{C}$

Consider a row index $\mathbf{m} = \begin{pmatrix} m_1 \\ \bar{\mathbf{m}}^{(k)} \end{pmatrix}$ in some $\mathcal{D}_k$, $k > n$. Then $m_1 \le 2n - k - 1$, therefore $n - n + m_1 + k \le 2n - k - 1 + k = 2n - 1$, so by Proposition 4 the corresponding element of our column is 0 since $(*)$ does not hold.

The diagonal element of our column is in the row indexed by $\mathbf{m} = \begin{pmatrix} n+1 \\ \bar{\mathbf{m}}^{(n-1)} \end{pmatrix} \in \mathcal{F}$, in this case $(*)$ holds since $n - n + (n+1) + (n-1) = 2n$, also, $n \le n + 1 \le n + 2$, therefore by Proposition 4 the diagonal element is nonzero.

For a row index $\mathbf{m} = \begin{pmatrix} m_1 \\ \bar{\mathbf{m}}^{(2n-2-k)} \end{pmatrix}$ in some $\mathcal{E}_k$ $(k \ge n)$ we have (by (9)) $m_1 \ge k + 3$, so $n - n + m_1 + 2n - 2 - k \ge k + 3 + 2n - 2 - k = 2n + 1$, therefore by Proposition 4 the corresponding element of our column is 0 as $(*)$ does not hold.

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Remark 2 *It follows from Theorem 4 that all weak bilinear optimality constraints are linear combinations of* $A^{(1)}, \ldots, A^{(4)}$. *Notice that in the above proof* $A^{(1)}, \ldots, A^{(4)}$ *correspond to the following equalities:*

$p_0c_0 + p_1c_1 + \ldots + p_{2n}c_{2n} = 0$

$2np_0c_0 + (2n-1)p_1c_1 + \ldots + p_{2n-1}c_{2n-1} = 0$

$2np_0c_1 + (2n-1)p_1c_2 + \ldots + p_{2n-1}c_{2n} = 0$

$2np_{2n}c_{2n-1} + (2n-1)p_{2n-1}c_{2n-2} + \ldots + p_1c_0 = 0$

These can also be proven immediately from the parametrization of M_{2n+1}; *as we well later use a similar technique to prove cubic optimality constraints, we also present this alternative proof:*

According to (3) t_k *is a multiple root of the polynomial* p *for all* $k = 1, \ldots, r$, *therefore* $p(t_k) = p'(t_k) = 0$. *Then by (2) we have:*

$\sum_{i,j} A^{(1)}_{ij} p_i c_j = \sum_{i=0}^{2n} p_i c_i = \sum_{i=0}^{2n} p_i \sum_{k=0}^{r} \lambda_k t_k^i = \sum_{k=0}^{r} \lambda_k \sum_{i=0}^{2n} p_i t_k^i = \sum_{k=0}^{r} \lambda_k p(t_k) = 0$

$\sum_{i,j} A^{(2)}_{ij} p_i c_j = \sum_{i=0}^{2n-1} (2n-i) p_i c_i = \sum_{i=0}^{2n-1} (2n-i) p_i \sum_{k=0}^{r} \lambda_k t_k^i = \sum_{k=0}^{r} \lambda_k (2n \sum_{i=0}^{2n-1} p_i t_k^i - t_k \sum_{i=0}^{2n-1} i p_i t_k^{i-1}) = \sum_{k=0}^{r} \lambda_k (2n(p(t_k) - p_{2n} t_k^{2n}) - t_k(p'(t_k) - 2np_{2n} t_k^{2n})) = \sum_{k=0}^{r} \lambda_k (-2np_{2n} t_k^{2n} + t_k 2np_{2n} t_k^{2n-1}) = 0$

$\sum_{i,j} A^{(3)}_{ij} p_i c_j = \sum_{i=0}^{2n-1} (2n-i) p_i c_{i+1} = \sum_{i=0}^{2n-1} (2n-i) p_i \sum_{k=0}^{r} \lambda_k t_k^{i+1} = \sum_{k=0}^{r} \lambda_k t_k (2n \sum_{i=0}^{2n-1} p_i t_k^i - t_k \sum_{i=0}^{2n-1} i p_i t_k^{i-1}) = \sum_{k=0}^{r} \lambda_k t_k (2n(p(t_k) - p_{2n} t_k^{2n}) - t_k(p'(t_k) - 2np_{2n} t_k^{2n}))$
$= \sum_{k=0}^{r} \lambda_k t_k (-2np_{2n} t_k^{2n} + t_k 2np_{2n} t_k^{2n-1}) = 0$

$\sum_{i,j} A^{(4)}_{ij} p_i c_j = \sum_{i=1}^{2n} i p_i c_{i-1} = \sum_{i=1}^{2n} i p_i \sum_{k=0}^{r} \lambda_k t_k^{i-1} = \sum_{k=0}^{r} \lambda_k \sum_{i=1}^{2n} i p_i t_k^{i-1}$
$= \sum_{k=0}^{r} \lambda_k p'(t_k) = 0.$

We conclude this section by showing that $A^{(1)}, \ldots, A^{(4)}$ are bilinear optimality constraints.

Definition 3 *For a matrix* $(A_{ij})_{i,j=0,\dots,2n}$ *let* $\overleftarrow{A}_{ij} = A_{2n-i,2n-j}$.

Proposition 6 *If both* A *and* $\overleftarrow{A}$ *are weak bilinear optimality constraints then* A *is a bilinear optimality constraint.*

Proof: Consider a pair $(\mathbf{p}, \bar{\mathbf{c}}) \in C(\mathcal{P}_{2n+1})$. By Corollary 1 $\bar{\mathbf{c}} = \lambda_0 \mathbf{e} + \mathbf{c}$ for some $\lambda_0 \geq 0$ and $\mathbf{c} \in \mathcal{M}_{2n+1}$, furthermore, $\mathbf{p}^T \mathbf{c} = 0$ and $\mathbf{p}^T(\lambda_0 \mathbf{e}) = 0$. This implies $\mathbf{p}^T A \mathbf{c} = 0$, since A is a weak optimality constraint. On the other hand, $\overleftarrow{\mathbf{e}} = \mathbf{c}_{2n+1}(0) \in \mathcal{M}_{2n+1}$ and, by Proposition 2, $\overleftarrow{\mathbf{p}} \in \mathcal{P}_{2n+1}$. Therefore, since $\overleftarrow{A}$ is also a weak optimality constraint and $\overleftarrow{\mathbf{p}}^T(\lambda_0 \overleftarrow{\mathbf{e}}) = \mathbf{p}^T(\lambda_0 \mathbf{e}) = 0$, we have $\mathbf{p}^T A(\lambda_0 \mathbf{e}) = \overleftarrow{\mathbf{p}}^T \overleftarrow{A}(\lambda_0 \overleftarrow{\mathbf{e}}) = 0$, which implies $\mathbf{p}^T A \bar{\mathbf{c}} = \mathbf{p}^T A(\lambda_0 \mathbf{e}) + \mathbf{p}^T A \mathbf{c} = 0$. ■

Theorem 5 $A^{(1)}, \dots, A^{(4)}$ *are linearly independent bilinear optimality constraints and all such constraints can be obtained as linear combinations of* $A^{(1)}, \dots, A^{(4)}$.

Proof: It only remains to show that $A^{(1)}, \dots, A^{(4)}$ are optimality constraints. According to the previous proposition it suffices to prove $\overleftarrow{A}^{(i)} \in \langle A^{(1)}, \dots, A^{(4)} \rangle$ $(i = 1, \dots, 4)$. This holds since $\overleftarrow{A}^{(1)} = A^{(1)}$, $\overleftarrow{A}^{(2)} = 2nA^{(1)} - A^{(2)}$, $\overleftarrow{A}^{(3)} = A^{(4)}$ and $\overleftarrow{A}^{(4)} = A^{(3)}$. ■

3 Cubic Optimality Constraints

Since for $n \geq 2$ the bilinear optimality constraints are not sufficient to describe $C(\mathcal{P}_{2n+1})$ it is natural to look for other types of conditions. In this section we examine cubic constraints of the following form:

Definition 4 $(B_{ij})_{i,j=0,\dots,2n}$ *is a weak cubic optimality constraint of Type I if* $\sum_{i,j,k} B_{ijk} p_i c_j c_k = 0$ *holds for all* $(\mathbf{p}, \mathbf{c}) \in C(\mathcal{P}_{2n+1})$ *such that* $\mathbf{c} \in \mathcal{M}_{2n+1}$.

$(D_{ij})_{i,j=0,\dots,2n}$ *is a weak cubic optimality constraint of Type II if* $\sum_{i,j,k} D_{ijk} p_i p_j c_k = 0$ *holds for all* $(\mathbf{p}, \mathbf{c}) \in C(\mathcal{P}_{2n+1})$ *such that* $\mathbf{c} \in \mathcal{M}_{2n+1}$.

If the above implications hold for all pairs $(\mathbf{p}, \bar{\mathbf{c}}) \in \mathcal{C}(\mathcal{P}_{2n+1})$, B *and* D *are called cubic optimality constraints of Type I and II, respectively.*

Proposition 7 $B^{(\delta)}_{ijk} = \begin{cases} 1 & \text{if } i = j + k - \delta, \quad j,k < \delta \\ -1 & \text{if } i = j + k - \delta, \quad j,k \geq \delta \\ 0 & \text{otherwise} \end{cases}$, *corresponding to the equality*

$$\sum_{i,j<\delta} p_{i+j-\delta} c_i c_j - \sum_{i,j \geq \delta} p_{i+j-\delta} c_i c_j = 0,$$

is a weak cubic optimality constraint of Type I for all $\delta = -1, 0, 1, \dots, 2n+1$.

Proof:

$$\sum_{i,j<\delta} p_{i+j-\delta}c_i c_j - \sum_{i,j\geq\delta} p_{i+j-\delta}c_i c_j = \sum_{k,l}\lambda_k\lambda_l\left(\sum_{i,j<\delta} p_{i+j-\delta}t_k^i t_l^j - \sum_{i,j\geq\delta} p_{i+j-\delta}t_k^i t_l^j\right)$$

therefore it suffices to show that for any k, l:

$$\sum_{i,j<\delta} p_{i+j-\delta}t_k^i t_l^j - \sum_{i,j\geq\delta} p_{i+j-\delta}t_k^i t_l^j = 0. \tag{10}$$

Let us first assume $\delta \geq 0$. Then we have: $\sum_{i,j<\delta} p_{i+j-\delta}t_k^i t_l^j = \sum_{i=0}^{\delta-1} t_k^i \sum_{j=\delta-i}^{\delta-1} p_{i+j-\delta}t_l^j$.

Notice that $\sum_{j=\delta-i}^{\delta-1} p_{i+j-\delta}t_l^j = \sum_{j=0}^{i-1} p_j t_l^{j-i+\delta} = t_l^{\delta-i}\sum_{j=0}^{i-1} p_j t_l^j = t_l^{\delta-i}\left(p(t_l) - \sum_{j=i}^{2n} p_j t_l^j\right)$ $= -\sum_{j=i}^{2n} p_j t_l^{j-i+\delta}$. Thus, $\sum_{i,j<\delta} p_{i+j-\delta}t_k^i t_l^j = -\sum_{i=0}^{\delta-1} t_k^i \sum_{j=i}^{2n} p_j t_l^{j-i+\delta} = \sum_{i=0}^{\delta-1}\sum_{j=i}^{2n}\left(\frac{t_k}{t_l}\right)^i p_j t_l^{j+\delta}$ $= -\sum_{j=0}^{2n} p_j t_l^{j+\delta}\sum_{i=0}^{\min(j,\delta-1)}\left(\frac{t_k}{t_l}\right)^i = -\sum_{j=0}^{\delta-1} p_j t_l^{j+\delta}\sum_{i=0}^{j}\left(\frac{t_k}{t_l}\right)^i - \sum_{j=\delta}^{2n} p_j t_l^{j+\delta}\sum_{i=0}^{\delta-1}\left(\frac{t_k}{t_l}\right)^i$ $= -\sum_{j=0}^{\delta-1} p_j t_l^{j+\delta}\frac{t_k^{j+1}-t_l^{j+1}}{t_l^j(t_k-t_l)} - \sum_{j=\delta}^{2n} p_j t_l^{j+\delta}\frac{t_k^\delta - t_l^\delta}{t_l^{\delta-1}(t_k-t_l)} = -\frac{t_l^\delta}{t_k-t_l}\sum_{j=0}^{\delta-1} p_j(t_k^{j+1}-t_l^{j+1}) - \frac{t_k^\delta-t_l^\delta}{t_k-t_l}t_l\sum_{j=\delta}^{2n} p_j t_l^j$.

Similarly, $\sum_{i,j\geq\delta} p_{i+j-\delta}t_k^i t_l^j = \sum_{i=\delta}^{2n} t_k^i \sum_{j=\delta}^{2n-i-\delta} p_{i+j-\delta}t_l^j$. Notice that $\sum_{j=\delta}^{2n-i-\delta} p_{i+j-\delta}t_l^j$ $= \sum_{j=i}^{2n} p_j t_l^{j-i+\delta} = t_l^{\delta-i}\sum_{j=i}^{2n} p_j t_l^j = t_l^{\delta-i}\left(p(t_l) - \sum_{j=0}^{i-1} p_j t_l^j\right) = -\sum_{j=0}^{i-1} p_j t_l^{j-i+\delta}$. Thus $\sum_{i,j\geq\delta} p_{i+j-\delta}t_k^i t_l^j = -\sum_{i=\delta}^{2n} t_k^i\sum_{j=0}^{i-1} p_j t_l^{j-i+\delta} = -\sum_{i=\delta}^{2n}\sum_{j=0}^{i-1}\left(\frac{t_k}{t_l}\right)^i p_j t_l^{j+\delta}$ $= -\sum_{j=0}^{2n-1} p_j t_l^{j+\delta}\sum_{i=\max(\delta,j+1)}^{2n}\left(\frac{t_k}{t_l}\right)^i = -\sum_{j=0}^{\delta-1} p_j t_l^{j+\delta}\sum_{i=\delta}^{2n}\left(\frac{t_k}{t_l}\right)^i - \sum_{j=\delta}^{2n-1} p_j t_l^{j+\delta}\sum_{i=j+1}^{2n}\left(\frac{t_k}{t_l}\right)^i$ $= -\sum_{j=0}^{\delta-1} p_j t_l^{j+\delta}\left(\frac{t_k^{2n+1}-t_l^{2n+1}}{t_l^{2n}(t_k-t_l)} - \frac{t_k^\delta-t_l^\delta}{t_l^{\delta-1}(t_k-t_l)}\right) - \sum_{j=\delta}^{2n-1} p_j t_l^{j+\delta}\left(\frac{t_k^{2n+1}-t_l^{2n+1}}{t_l^{2n}(t_k-t_l)} - \frac{t_k^{j+1}-t_l^{j+1}}{t_l^j(t_k-t_l)}\right)$ $= -\frac{t_l^{\delta-2n}(t_k^{2n+1}-t_l^{2n+1})}{t_k-t_l}\sum_{j=0}^{\delta-1} p_j t_l^j + \frac{t_k^\delta-t_l^\delta}{t_k-t_l}t_l\sum_{j=0}^{\delta-1} p_j t_l^j - \frac{t_l^{\delta-2n}(t_k^{2n+1}-t_l^{2n+1})}{t_k-t_l}\sum_{j=\delta}^{2n-1} p_j t_l^j$ $+\frac{t_l^\delta}{t_k-t_l}\sum_{j=\delta}^{2n-1} p_j(t_k^{j+1} - t_l^{j+1})$.

Plugging into (10) we obtain $\sum_{i,j<\delta} p_{i+j-\delta}c_i c_j - \sum_{i,j\geq\delta} p_{i+j-\delta}c_i c_j = -\frac{t_l^\delta}{t_k-t_l}\sum_{j=0}^{2n-1} p_j(t_k^{j+1} - t_l^{j+1}) - \frac{t_k^\delta-t_l^\delta}{t_k-t_l}t_l\sum_{j=0}^{2n} p_j t_l^j +$ $\frac{t_l^{\delta-2n}(t_k^{2n+1}-t_l^{2n+1})}{t_k-t_l}\sum_{j=0}^{2n-1} p_j t_l^j = -\frac{t_l^\delta}{t_k-t_l}\left(t_k(p(t_k) - p_{2n}t_k^{2n} - tl(p(t_l) - p_{2n}t_l^{2}n)\right) - \frac{t_k^\delta-t_l^\delta}{t_k-t_l}t_l p(t_l) +$ $t_l^{\delta-2n}\frac{t_k^{2n+1}-t_l^{2n+1}}{t_k-t_l}(p(t_l) - p_{2n}t_l^{2n}) = \frac{t_l^\delta}{t_k-t_l}\left(p_{2n}(t_k^{2n+1} - t_l^{2n+1}) - t_l^{-2n}(t_k^{2n+1} - t_l^{2n+1})p_{2n}t_l^{2n}\right) = 0$, which proves our claim.

In the case $\delta = -1$ we similarly have $\sum_{i,j\geq 0} p_{i+j+1}t_k^i t_l^j = \sum_{i=0}^{2n} t_k^i \sum_{j=0}^{2n-i-1} p_{i+j+1}t_l^j = \sum_{i=0}^{2n} t_k^i \sum_{j=0}^{i} p_j t_l^{j-i-1} = \sum_{i=0}^{2n}\sum_{j=0}^{i} p_j t_l^{j-1}\left(\frac{t_k}{t_l}\right)^i$ $= \sum_{j=0}^{2n} p_j t_l^{j-1}\sum_{i=j}^{2n}\left(\frac{t_k}{t_l}\right)^i = \sum_{j=0}^{2n} p_j t_l^{j-1}\left(\frac{t_k^{2n+1}-t_l^{2n+1}}{t_l^{2n}(t_k-t_l)} - \frac{t_k^j-t_l^j}{t_l^{j-1}(t_k-t_l)}\right) = \frac{t_k^{2n+1}-t_l^{2n+1}}{t_l^{2n+1}(t_k-t_l)}p(t_l)$ $-\frac{1}{t_k-t_l}(p(t_k) - p(t_l)) = 0$.

■

We can also obtain valid cubic optimality constraints from the bilinear ones:

Proposition 8 *For any* $\alpha = 1, \ldots, 4$ *and* $h = 0, \ldots, 2n$

$$B^{(\alpha,h)}_{ijk} = \begin{cases} A^{(\alpha)}_{ik} & \text{if } h = j \\ 0 & \text{otherwise} \end{cases}$$

defines a cubic optimality constraint $B^{(\alpha,h)}$ *of both Type I and Type II.*

Proof: Let $\alpha \in \{1, \ldots, 4\}$, $h \in \{0, \ldots, 2n\}$ and $(\mathbf{p}, \bar{\mathbf{c}}) \in C(\mathcal{P}_{2n+1})$. Then, since $A^{(\alpha)}$ is a bilinear optimality constraint, $\sum_{i,j,k} B^{(\alpha,h)}_{ijk} p_i \bar{c}_j \bar{c}_k = \bar{c}_h \sum_{i,k} A^{(\alpha)}_{ik} p_i \bar{c}_k = \bar{c}_h \left(\mathbf{p}^T A^{(\alpha)} \bar{\mathbf{c}} \right) = 0$, which implies that $B^{(\alpha,h)}$ is a cubic optimality constraint of Type I. The proof for Type II is identical. ■

Remark 3 *Notice that* $\left\{B^{(-1)}, B^{(0)}, \ldots, B^{(2n+1)}\right\} \bigcup \left\{B^{(\alpha,h)} \mid \alpha = 1, \ldots, 4,\ h = 0, \ldots, 2n\right\}$ *is a linearly independent set.*

By applying an approach similar to that used to establish our main results (i.e. by computing the rank of a matrix defined analogously to M in Definition 2) we were able to obtain the following:

Proposition 9 *For* $n = 1, \ldots, 5$ *all weak cubic optimality constraints of Type I for* $C(\mathcal{P}_{2n+1})$ *can be generated as linear combinations of* $B^{(-1)}, B^{(0)}, \ldots, B^{(2n+1)}$ *and* $B^{(\alpha,h)}$ $(\alpha = 1, \ldots, 4,\ h = 0, \ldots, 2n)$.

Proposition 10 *For* $n = 1, 2, 3$ *all weak cubic optimality constraints of Type II for* $C(\mathcal{P}_{2n+1})$ *can be generated as linear combinations of* $B^{(\alpha,h)}$ $(\alpha = 1, \ldots, 4,\ h = 0, \ldots, 2n)$.

We conjecture that these observations hold for any n, and a proof of them similar to that of Theorem 4, although more involved, exists.

To conclude this section we show that $B^{(-1)}, B^{(0)}, \ldots, B^{(2n+1)}$ are cubic optimality constraints. Similarly to the bilinear case, let $\overleftarrow{B}_{ijk} = B_{2n-i,2n-j,2n-k}$. The proof of the following proposition is almost identical to that of Proposition 6:

Proposition 11 *If* B *and* $\overleftarrow{B}$ *are weak cubic optimality constraints of type I, then* B *is a cubic optimality constraint of type I.*

Theorem 6 $B^{(-1)}, B^{(0)}, \ldots, B^{(2n+1)}$ *are cubic optimality constraints of type I.*

Proof: Analogously to the bilinear case it suffices to show that for all $\delta = -1, \ldots, 2n+1$ $\overleftarrow{B}^{(\delta)} \in \left\langle \left\{B^{(-1)}, B^{(0)}, \ldots, B^{(2n+1)}\right\} \bigcup \left\{B^{(\alpha,h)} \mid \alpha = 1, \ldots, 4,\ h = 0, \ldots, 2n\right\} \right\rangle$, which holds since $B^{(\delta)} = -2B^{(1,2n-\delta)} - B^{(2n-\delta)}$.

■

4 Polynomials on a Finite Interval

Let $\mathcal{P}_{n+1}^{[a,b]} = \left\{(p_0, \ldots, p_{2n}) \in \mathbb{R}^{2n+1} \mid p(t) = p_0 + p_1 t + p_2 t^2 + \ldots + p_{2n} t^{2n} \geq 0 \ \ \forall t \in [a,b]\right\}$ denote the cone of polynomials which are nonnegative of the interval $[a,b] \subset \mathbb{R}$. It is shown in [6] (Ch.II) that the dual of this cone is $\mathcal{M}_{n+1}^{[a,b]} = \operatorname{conv}\{\mathbf{c}_{n+1}(t) \mid t \in [a,b]\}$. Similarly to the case of the real line we have:

Proposition 12 *Let* $\mathbf{c} = \sum_{j=1}^{r} \lambda_j \mathbf{c}_{n+1}(t_j),\ \lambda_j > 0 \ \ (j = 1, \ldots, r)$ *and* $\mathbf{p}$ *be nonzero boundary vectors of* $\mathcal{M}_{n+1}^{[a,b]}$ *and* $\mathcal{P}_{n+1}^{[a,b]}$*, respectively. Then* $\mathbf{p}^{\mathsf{T}}\mathbf{c} = 0$ *if and only if* $p(t_j) = 0 \ \ (j = 1 \ldots r)$.

Notice that if a polynomial $\mathbf{p} \in \mathcal{P}_{n+1}^{[a,b]}$ has a root at t then either $t \in \{a, b\}$ or t is a local minimum, therefore we have $p'(t)(t-a)(t-b) = 0$.

Theorem 7 *For a pair of vectors* $(\mathbf{p}, \mathbf{c}) \in C(\mathcal{P}_{n+1}^{[a,b]})$ *the following bilinear optimality conditions hold:*

$$\sum_{i=0}^{n-1} \left[(a+b)(n-i)p_i c_i + ab(i+1)p_{i+1}c_i - (n-i)p_i c_{i+1}\right] = 0 \tag{11}$$

$$\sum_{i=0}^{n} p_i c_i = 0.$$

Proof: If either $\mathbf{p}$ or $\mathbf{c}$ is zero, the conditions trivially hold; from now on let us assume this is not the case. The second equality is the complementarity slackness condition defining $C(\mathcal{P}_{n+1}^{[a,b]})$. For the proof of (11) let us write $\mathbf{c}$ in the form $\mathbf{c} = \sum_{k=1}^{r} \lambda_k \mathbf{c}_{n+1}(t_k),\ \lambda_k \geq 0$ $(k = 1, \ldots, r)$.

$\sum_{i=0}^{n-1} \left[(a+b)(n-i)p_i c_i + ab(i+1)p_{i+1}c_i - (n-i)p_i c_{i+1}\right] = \sum_{i=0}^{n-1} \big[(a+b)(n-i)p_i \sum_{k=1}^{r} \lambda_k t_k^i + ab(i+1)p_{i+1} \sum_{k=1}^{r} \lambda_k t_k^i - (n-i)p_i \sum_{k=1}^{r} \lambda_k t_k^{i+1}\big] = \sum_{k=1}^{r} \lambda_k \big[\sum_{i=0}^{n} n\big[(a+b)p_i t_k^i - p_i t_k^{i+1}\big] + \sum_{i=0}^{n} i p_i \big[-(a+b)t_k^i + ab t_k^{i-1} + t_k^{i+1}\big]\big] = \sum_{k=1}^{r} \lambda_k \big[n(a+b-t_k)p(t_k) + \sum_{i=0}^{n} i p_i t_k^{i-1} \left(t_k^2 - (a+b)t_k + ab\right)\big] = \sum_{k=1}^{r} \lambda_k \big[n(a+b-t_k)p(t_k) + p'(t_k)(t_k-a)(t_k-b)\big] = 0$ ■

Remark 4 *Similarly to the case of polynomials over the real line for small values of* n *we have found that all bilinear conditions are linear combinations of those in Theorem 7.*

5 Trigonometric Polynomials

A trigonometric polynomial with coefficients $\mathbf{p} = (p_0, \ldots, p_{2n})$ is defined by $p(\theta) = p_0 + \sum_{k=1}^{n} p_{2k-1}\cos(k\theta) + p_{2k}\sin(k\theta)$. In this section we look at the cone of positive trigonometric polynomials, $\mathcal{TP}_{2n+1} = \left\{\mathbf{p} \in \mathbb{R}^{2n+1} \mid p(\theta) \geq 0 \ \ \forall \theta \in [-\pi, \pi]\right\}$. It is shown in [3](Ch.IX) that the dual of $\mathcal{TP}_{2n+1}$ is the cone of trigonometric moments: $\mathcal{TM}_{2n+1} = \operatorname{conv}\{\mathbf{c}_{2n+1}(\theta) \mid \theta \in [-\pi, \pi]\}$, where $\mathbf{c}_{2n+1}(\theta) \stackrel{\text{def}}{=} (1, \cos(\theta), \sin(\theta), \cos(2\theta), \sin(2\theta), \ldots, \cos(n\theta), \sin(n\theta))^{\mathsf{T}}$.

Proposition 13 *Let* $\mathbf{c} = \sum_{j=1}^{r} \lambda_j \mathbf{c}_{2n+1}(\theta_j),\ \lambda_j > 0 \quad (j = 1, \ldots, r)$ *and* $\mathbf{p}$ *be nonzero boundary vectors of* $\mathcal{TM}_{2n+1}$ *and* $\mathcal{TP}_{2n+1}$*, respectively. Then* $\mathbf{p}^{\mathsf{T}}\mathbf{c} = 0$ *if and only if* $p(\theta_j) = 0 \quad (j = 1 \ldots r)$*. In this case we also have* $p'(\theta_j) = 0 \quad (j = 1 \ldots r)$*.*

Proof: The proof of the equivalence is almost identical to the proof of Theorem 2. The statement regarding the derivatives follows from the fact that a nonnegative function has a local minimum at its every root. ■

Remark 5 *It is known that [3](Ch.IX)* p *is a positive trigonometric polynomial of degree* n *if and only if there exists a regular polynomial* q *of degree* n *(with complex coefficients) such that* $p(\theta) = \left|q(e^{i\theta})\right|^2$*. This, together with the previous proposition, provides us with a parametrization of* $C(\mathcal{TP}_{2n+1})$ *similar to Corollary 2.*

Theorem 8 *Let* $(\mathbf{p}, \mathbf{c}) \in C(\mathcal{TP}_{2n+1})$*. Then the following bilinear optimality conditions hold:*

1. $\sum_{i=0}^{2n} p_i c_i = 0$
2. $\sum_{i=0}^{n} i(p_{2i-1}c_{2i} - p_{2i}c_{2i-1}) = 0$
3. $2np_0c_1 + (n+1)p_1c_0 + \sum_{i=1}^{n-1}(n-i)(p_{2i-1}c_{2i+1} + p_{2i}c_{2i+2}) + (n+i+1)(p_{2i+1}c_{2i-1} + p_{2i+2}c_{2i}) = 0$
4. $2np_0c_2 + (n+1)p_2c_0 + \sum_{i=1}^{n-1}(n-i)(p_{2i-1}c_{2i+2} - p_{2i}c_{2i+1}) + (n+i+1)(p_{2i+2}c_{2i-1} - p_{2i+1}c_{2i}) = 0$

Proof: Throughout the following proof we are going to use some elementary trigonometric identities without further mention.

If either $\mathbf{p}$ or $\mathbf{c}$ is zero, the conditions trivially hold; from now on let us assume this is not the case. (1.) is the complementarity slackness condition defining $C(\mathcal{TP}_{2n+1})$. For the proof of the remaining conditions let us write $\mathbf{c}$ in the form $\mathbf{c} = \sum_{k=1}^{r} \lambda_k \mathbf{c}_{2n+1}(\theta_k),\ \lambda_k \geq 0$ $(k = 1, \ldots, r)$, which implies $c_0 = \sum_{k=1}^{r} \lambda_k$, $c_{2i-1} = \sum_{k=1}^{r} \lambda_k \cos(i\theta_k)$, $c_{2i} = \sum_{k=1}^{r} \lambda_k \sin(i\theta_k)$ $(i = 1, \ldots, n)$. Recall that by Proposition 13 we have $p(\theta_k) = p'(\theta_k) = 0$.

(2.) $\sum_{i=0}^{n} i(p_{2i-1}c_{2i} - p_{2i}c_{2i-1}) = \sum_{i=0}^{n} i(p_{2i-1} \sum_{k=1}^{r} \lambda_k \sin(i\theta_k) - p_{2i} \sum_{k=1}^{r} \lambda_k \cos(i\theta_k)) = \sum_{k=1}^{r} \lambda_k \left[\sum_{i=0}^{n} i(p_{2i-1} \sin(i\theta_k) - p_{2i} \cos(i\theta_k))\right] = -\sum_{k=1}^{r} \lambda_k p'(\theta_k) = 0.$

(3.) $2np_0c_1 + (n+1)p_1c_0 + \sum_{i=1}^{n-1}(n-i)(p_{2i-1}c_{2i+1} + p_{2i}c_{2i+2}) + (n+i+1)(p_{2i+1}c_{2i-1} + p_{2i+2}c_{2i}) = 2np_0c_1 + \sum_{i=1}^{n}(n-i)(p_{2i-1}c_{2i+1} + p_{2i}c_{2i+2}) + \sum_{i=0}^{n-1}(n+i+1)(p_{2i+1}c_{2i-1} + p_{2i+2}c_{2i}) = 2np_0c_1 + \sum_{i=1}^{n}(n-i)(p_{2i-1}c_{2i+1} + p_{2i}c_{2i+2}) + (n+i)(p_{2i-1}c_{2i-3} + p_{2i}c_{2i-2}) =$

$2np_0\sum_{k=1}^{r}\lambda_k\cos(\theta_k)+\sum_{i=1}^{n}(n-i)(p_{2i-1}\sum_{k=1}^{r}\lambda_k\cos((i+1)\theta_k)+p_{2i}\sum_{k=1}^{r}\lambda_k\sin((i+1)\theta_k))+(n+i)(p_{2i-1}\sum_{k=1}^{r}\lambda_k\cos((i-1)\theta_k)+p_{2i}\sum_{k=1}^{r}\lambda_k\sin((i-1)\theta_k))=\sum_{k=1}^{r}\lambda_k\big[2np_0\cos(\theta_k)+\sum_{i=1}^{n}(n-i)(p_{2i-1}\cos((i+1)\theta_k)+p_{2i}\sin((i+1)\theta_k))+(n+i)(p_{2i-1}\cos((i-1)\theta_k)+p_{2i}\sin((i-1)\theta_k))\big]$.

We are going to show that in the above sum the term multiplied by λ_k is zero for every k: $C_k=2np_0\cos(\theta_k)+\sum_{i=1}^{n}(n-i)(p_{2i-1}\cos((i+1)\theta_k)+p_{2i}\sin((i+1)\theta_k))+(n+i)(p_{2i-1}\cos((i-1)\theta_k)+p_{2i}\sin((i-1)\theta_k))=2np_0\cos(\theta_k)+\sum_{i=1}^{n}i\big[p_{2i-1}(\cos((i-1)\theta_k)-\cos((i+1)\theta_k))-p_{2i}(\sin((i+1)\theta_k))-\sin((i-1)\theta_k))\big]+n\big[p_{2i-1}(\cos((i+1)\theta_k)+\cos((i-1)\theta_k))+p_{2i}(\sin((i+1)\theta_k))+\sin((i-1)\theta_k)))\big]=\sum_{i=1}^{n}2i(p_{2i-1}\sin(\theta_k)\sin(i\theta_k)-p_{2i}\sin(\theta_k)\cos(i\theta_k))+2n\big[p_0\cos(\theta_k)+\sum_{i=1}^{n}(p_{2i-1}\cos(\theta_k)\cos(i\theta_k)+p_{2i}\cos(\theta_k)\sin(i\theta_k))\big]=-2\sin(\theta_k)\sum_{i=1}^{n}i(p_{2i}\cos(i\theta_k)-p_{2i-1}\sin(i\theta_k))+2n\cos(\theta_k)\big[p_0+\sum_{i=1}^{n}(p_{2i-1}\cos(i\theta_k)+p_{2i}\sin(i\theta_k))\big]=-2\sin(\theta_k)p'(\theta_k)+2n\cos(\theta_k)p(\theta_k)=0$.

(4.) $2np_0c_2+(n+1)p_2c_0+\sum_{i=1}^{n-1}(n-i)(p_{2i-1}c_{2i+2}-p_{2i}c_{2i+1})+(n+i+1)(p_{2i+2}c_{2i-1}-p_{2i+1}c_{2i})=2np_0c_2+\sum_{i=1}^{n}(n-i)(p_{2i-1}c_{2i+2}-p_{2i}c_{2i+1})+\sum_{i=0}^{n-1}(n+i+1)(p_{2i+2}c_{2i-1}-p_{2i+1}c_{2i})=2np_0c_2+\sum_{i=1}^{n}(n-i)(p_{2i-1}c_{2i+2}-p_{2i}c_{2i+1})+(n+i)(p_{2i}c_{2i-3}-p_{2i-1}c_{2i-2})=2np_0\sum_{k=1}^{r}\lambda_k\sin(\theta_k)+\sum_{i=1}^{n}(n-i)(p_{2i-1}\sum_{k=1}^{r}\lambda_k\sin((i+1)\theta_k)-p_{2i}\sum_{k=1}^{r}\lambda_k\cos((i+1)\theta_k))+(n+i)(p_{2i}\sum_{k=1}^{r}\lambda_k\cos((i-1)\theta_k)-p_{2i-1}\sum_{k=1}^{r}\lambda_k\sin((i-1)\theta_k))=\sum_{k=1}^{r}\lambda_k\big[2np_0\sin(\theta_k)+\sum_{i=1}^{n}(n-i)(p_{2i-1}\sin((i+1)\theta_k)-p_{2i}\cos((i+1)\theta_k))+(n+i)(p_{2i}\cos((i-1)\theta_k)-p_{2i-1}\sin((i-1)\theta_k))\big]$.

We are going to show that in the above sum the term multiplied by λ_k is zero for every k: $C_k=2np_0\sin(\theta_k)+\sum_{i=1}^{n}(n-i)(p_{2i-1}\sin((i+1)\theta_k)-p_{2i}\cos((i+1)\theta_k))+(n+i)(p_{2i}\cos((i-1)\theta_k)-p_{2i-1}\sin((i-1)\theta_k))=2np_0\sin(\theta_k)+\sum_{i=1}^{n}i\big[p_{2i}(\cos((i+1)\theta_k)+\cos((i-1)\theta_k))-p_{2i-1}(\sin((i+1)\theta_k)+\sin((i-1)\theta_k))\big]+n\big[p_{2i-1}(\sin((i+1)\theta_k)-\sin((i-1)\theta_k))+p_{2i}(\cos((i-1)\theta_k)-\cos((i+1)\theta_k))\big]=\sum_{i=1}^{n}2i(p_{2i}\cos(\theta_k)\cos(i\theta_k)-p_{2i-1}\cos(\theta_k)\sin(i\theta_k))+2n\big[p_0\sin(\theta_k)+\sum_{i=1}^{n}(p_{2i-1}\sin(\theta_k)\cos(i\theta_k)+p_{2i}\sin(\theta_k)\sin(i\theta_k))\big]$
$=2\cos(\theta_k)\sum_{i=1}^{n}i(p_{2i}\cos(i\theta_k)-p_{2i-1}\sin(i\theta_k))+2n\sin(\theta_k)\big[p_0+\sum_{i=1}^{n}(p_{2i-1}\cos(i\theta_k)+p_{2i}\sin(i\theta_k))\big]=2\cos(\theta_k)p'(\theta_k)+2n\sin(\theta_k)p(\theta_k)=0$.

■

Remark 6 *Using the parametrization mentioned before we were able to show that for* $n=1,\dots,4$ *all bilinear optimality conditions can be generated as linear combinations of the* 4 *conditions given in Theorem 8. We conjecture that this holds for all* n *and a proof similar to that for the case of regular polynomials exists. Also, for* $n=1,\dots,3$ *we have found that, similarly to the case of regular polynomials, there are* $2n+3$ *linearly independent cubic optimality conditions not generated by the bilinear ones.*

Appendix: Non-existence of Bilinear Optimality Constraints for Some Cones

It is well known ([5]) that for symmetric cones and for cones that are linearly isomorphic to a symmetric cone a complete system (i.e. a system defining the manifold $C(\mathcal{K})$) of bilinear optimality constraints exists. For the cone $\mathcal{P}_{2n+1}$ of positive polynomials our main results characterize all bilinear optimality constraints. These (with the exception of the case of $\mathcal{P}_3$, which is isomorphic to a symmetric cone), do not form a complete system. For a wide range of other cones we were able to show that apart from the trivial $\mathbf{x}^T\mathbf{s} = 0$ no bilinear optimality constraints exist. In absence of a polynomial parametrization of $C(\mathcal{K})$ (as described in Corollary 1) for a cone $\mathcal{K} \subset \mathbb{R}^n$ we can use the following approach: For a pair $(\mathbf{x}, \mathbf{s}) \in C(\mathcal{K})$ let $\mathbf{z}_{\mathbf{x},\mathbf{s}} \stackrel{\text{def}}{=} (x_1s_1, x_1s_2, \ldots, x_ns_n)^T \in \mathbb{R}^{n^2}$. Then for any bilinear optimality constraint A the equality $\mathbf{z}_{\mathbf{x},\mathbf{s}}^T \operatorname{vec}(A) = \sum_{i,j} A_{ij}x_is_j = 0$ must hold. Therefore if we have a set of pairs $\{(\mathbf{x}_1, \mathbf{s}_1), \ldots, (\mathbf{x}_{n^2-1}, \mathbf{s}_{n^2-1})\} \subset C(\mathcal{K})$ for which the vectors $\mathbf{z}_{\mathbf{x}_1,\mathbf{s}_1}, \ldots, \mathbf{z}_{\mathbf{x}_{n^2-1},\mathbf{s}_{n^2-1}}$ are linearly independent, the set of bilinear optimality constraints has dimension 1, i.e. the only constraints are the multiples of the complementary slackness constraint $A = I_n$, corresponding to $\mathbf{x}^T\mathbf{s} = 0$. Using this method it can be shown that the following cones do not have any bilinear optimality constraints apart from the trivial one:

1. Polyhedral cones in $\mathbb{R}^3$ generated by 4 vectors in general position (these include the cone $\left\{ \begin{pmatrix} t \\ \mathbf{x} \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; \|\mathbf{x}\|_1 \le t \right\}$).

2. Polyhedral cones in $\mathbb{R}^3$ generated by 5 vectors in general position.

3. The Koecher cones $\mathcal{K}_\rho = \left\{ \begin{pmatrix} u \\ v \\ w \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; u \ge 0,\ v \ge 0,\ |w| \le u^\rho v^{1-\rho} \right\}$ for $\rho \in (0, 1),\ \rho \ne \frac{1}{2}$.

4. The cone consisting of all pairs $\left(\begin{pmatrix} a & x \\ x & c \end{pmatrix}, \begin{pmatrix} b & y \\ y & c \end{pmatrix} \right)$ of (symmetric) positive semidefinite matrices with a common corner element c.

Note that cones 2 and 3 are isomorphic to self-dual cones while cone 4 is homogeneous ([9], [10]).

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