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War and Peace in Veto Voting [a]

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War and Peace in Veto Voting [1]

Vladimir Gurvich

Abstract. Let $I = \{i_1, \ldots, i_n\}$ be a set of voters (players) and $A = \{a_1, \ldots, a_p\}$ be a set of candidates (outcomes). Each voter $i \in I$ has a preference P_i over the candidates. We assume that P_i is a complete order on A. The preference profile $P = \{P_i, i \in I\}$ is called a *situation.* A situation is called *war* if the set of all voters I is partitioned in two coalitions K_1 and K_2 such that all voters of K_i have the same preference, $i = 1, 2$, and these two preferences are opposite. For a simple class of veto voting schemes we prove that the results of elections in all war situations uniquely define the results for all other (*peace*) situations. In other words, the results depend only on the veto (or effectivity) function. We give several other examples from game (and from graph) theory with the same property.
Key words: veto, voting scheme, voting by veto, veto power, veto resistance, voter, candidate, player, outcome, coalition, block, effectivity function, veto function, social choice function, social choice correspondence

1 Main Theorem

We follow standard concepts and notation of veto voting theory; see, e.g., [5], [6] Chapter 6, and [8] section 8.4. Let $I = \{i_1, \dots, i_n\}$ be a set of voters (players) and $A = \{a_1, \dots, a_p\}$ be a set of candidates (outcomes). Each voter $i \in I$ has a preference (a complete order) P_i over all candidates. The set of all preferences $P = \{P_i, i \in I\}$ is called a *preference profile* or a *situation.* A situation is called *war* if the set of voters I is partitioned in two coalitions K_1 and K_2 such that all voters of K_i have the same preference, $i = 1, 2$, and these two preferences are opposite. All other situations we will call *peace.*

Further, each voter $i \in I$ has μ_i veto cards and each candidate $a \in A$ has λ_a counter-veto cards. Positive integers μ_i and λ_a are called the *veto power* of $i \in I$ and *veto resistance* of $a \in A$, respectively. The corresponding integral-valued functions. $\mu : I \to \mathbb{Z}_+$ and $\lambda : I \to \mathbb{Z}_+$ are called veto power and veto resistance distributions.

Let us define the *veto order* σ_μ as a word in the alphabet $I = \{i_1, \dots, i_n\}$ in which every letter $i \in I$ appears exactly μ_i times and hence each word σ_μ has the same length $\sum_{i \in I} \mu_i$. The triplet $(\lambda, \mu, \sigma_\mu)$ is called *veto voting scheme (VVS).* It is realized as follows. In the given order σ_μ the voters put their veto cards against the candidates until all veto cards are finished. The voters have complete information. It is forbidden to over-veto; that is as soon as a candidate a has got λ_a veto cards (s)he is eliminated and no more veto cards can be used against a. All non-eliminated candidates are elected. Obviously, this set will be empty unless total veto power is strictly less than total veto resistance; that is

$$\sum_{i \in I} \mu_i < \sum_{a \in A} \lambda_a \tag{1.1}$$

If we assume further that

$$\sum_{a \in A} \lambda_a - \sum_{i \in I} \mu_i = 1. \tag{1.2}$$

then exactly one candidate is elected in each situation. However, unlike (1.1), this assumption is not mandatory.

Remark 1. *Though majority voting procedures are much more common in the real-life committees than voting by veto, yet, the latter too has its pluses and proponents.*

"Thus, the [veto voting] procedure establishes incentives to make proposals that, although perhaps favouring oneself, stand relatively high in the other voters' preferences" [8], p.175.

"Application of veto voting will lop off the proposals lying farthest from the center of the density function defined over all optima, leaving as possible winners only a subset of proposals clustered around the center" [8], p.179.

"The essential instability implied by the majority principle results from the very sharp underlying distribution of power: any coalition has either full or zero power. In particular, a 49% minority can be exploited to death by the opposing 51% majority. To avoid this instability, we use a more flexible allocation of coalitional power where a coalition K can block up to v(K) outcomes" [6], p.119.

Let us also point out certain similarity between veto voting schemes and the well-known Colonel Blotto's games, see, e.g., [9], where the divisions play role of the veto cards and counter-veto cards.

In general, the analysis of a given voting scheme is a problem of game theory, since the voters may behave in many different, sometimes rather sophisticated, ways. However, in this paper we consider only the simplest concept of their sincere *behavior. This means that each voter $i \in I$ always puts each veto card against the worst (with respect to the preference P_i) not yet eliminated candidate. Hence, given a VVS $(\lambda, \mu, \sigma_\mu)$, a set of elected candidates $B = B(P) \subseteq A$ is uniquely defined for every situation $P = \{P_i, i \in I\}$. In fact, the voting scheme introduced above is not even implemented by a game. Indeed, given P, the behavior of each voter is prescribed uniquely; that is (s)he has only one strategy.*

In general, a mapping $S : P \to 2^A$ which assigns a set of candidates to every preference profile is called a *social choice correspondence (SCC)*, and it is called a *social choice function (SCF)* if only one candidate is elected; that is $|S(P)| = 1$ for each situation P. Thus, every veto voting scheme $(\lambda, \mu, \sigma_\mu)$ defines a SCC $S_{\lambda,\mu,\sigma_\mu}$ which is an SCF whenever (1.2) holds. The SCC or SCF generated by a veto voting scheme are called *veto SCC and SCF*, respectively.

A veto order σ_μ is called *simple* if the voters do not alternate, or more precisely, if there exists a permutation τ of I such that first the voter $\tau^{-1}(i_1)$ put all veto cards, followed by $\tau^{-1}(i_2)$, etc. Obviously, a simple veto order σ_μ is uniquely determined by μ and τ. The corresponding veto voting scheme and SCC we will call *simple* and denote by (λ, μ, τ) and $S_{\lambda,\mu,\tau}$, respectively.

In this paper we prove that each simple veto SCC is uniquely defined by the values it takes in the war situations. More precisely, the following statement holds.

Theorem 1. *Given two simple veto voting schemes $VVS' = (\lambda', \mu', \tau)$ and $VVS'' = (\lambda'', \mu'', \tau)$ that generate social choice correspondences $S' = S_{\lambda',\mu',\tau}(P)$ and $S'' = S_{\lambda'',\mu'',\tau}(P)$, respectively, if $S'(P) = S''(P)$ for each war situation P then $S'(P) = S''(P)$ for all P.*

Note, however, that we do not promote Theorem 1 to the rank of a general law of diplomacy. For example, it is not general enough just because it only holds when the two involved veto orders coincide, moreover, it must be a simple order; otherwise the claim may fail, see Example 1 below.

Further, let us remark that in a war situation the veto order (simple or not) does not matter at all. In this case all candidates are uniquely ordered and all voters are split in two coalitions that veto candidates from two opposite ends of this order. Some moderate (centrist) candidates will be elected and the set of these candidates does not depend on the order in which the voters act. More accurately these arguments are summarized as follows.

Lemma 1. *Given distributions λ, μ and two veto orders $\sigma'_\mu, \sigma''_\mu$, the equality $S_{\lambda,\mu,\sigma'_\mu}(P) = S_{\lambda,\mu,\sigma''_\mu}(P)$ holds for each war situation P.*

Yet, for the other, peace, situations the result of elections may depend on the veto order.

Example 1. *Let us consider two voters of veto power 3 and 1 and three candidates of veto resistance 1,2, and 2; that is* $I = \{i_1, i_2\}, A = \{a_1, a_2, a_3\},\ \mu_1 = 3, \mu_2 = 1,\ \lambda_1 = 1, \lambda_2 = \lambda_3 = 2$. *Note that (1.2) holds and hence this voting scheme generates an SCF. Let the preferences be* $a_1 > a_2 > a_3$ *and* $a_2 > a_1 > a_3$ *for* i_1 *and* i_2 *respectively. This profile defines a peace situation* P.

First, let us consider two simple veto orders i_1, i_1, i_1, i_2 *and* i_2, i_1, i_1, i_1. *If* i_1 *votes first then (s)he eliminates* a_3 *and puts one remaining veto card against* a_2. *Still* a_2 *is not eliminated, yet. Moreover,* a_2 *will be elected, since* i_2 *vetoes* a_1. *If* i_2 *votes first (s)he puts the veto card against* a_3. *This allows* i_1 *to eliminate both* a_3 *and* a_2. *Hence, in this case* a_1 *is elected.*

Now let us consider two veto orders i_1, i_1, i_2, i_1 *and* i_1, i_2, i_1, i_1. *These orders are not simple and they have similar pattern: first* i_1, *then* i_2, *then* i_1 *again. However, these two orders result in electing different candidates. In the first case* i_1 *eliminates* a_3, *then* i_2 *eliminates* a_1, *and* a_2 *is elected. In the second case* i_1 *puts just one veto card against* a_3, *then* i_2 *eliminates* a_3, *and now* i_1 *can eliminate* a_2 *by the two remaining veto cards, hence,* a_1 *is elected.*

Finally, let us remark that, according to Lemma 1, all four veto orders considered above would give the same result in each war situation.

2 An equivalent statement

The theorem can be equivalently reformulated as follows.

The *veto function* is defined as a mapping $V : 2^I \times 2^A \to \{0, 1\}$; that is V has two arguments: a coalition of voters $K \subseteq I$ and a block of candidates $B \subseteq A$. The equalities $V(K, B) = 1$ and $V(K, B) = 0$ mean that K can, and respectively cannot, veto B. The complementary function $E(K, B) = V(K, A \setminus B)$ is called the *effectivity function*; see [6] Section 7.2 and [10] Chapter 6.

Each pair of distributions $\mu : I \to \mathbb{Z}_+$ and $\lambda : I \to \mathbb{Z}_+$, generates a veto function $V = V_{\mu,\lambda}$

$$V(K, B) = 1 \text{ iff } \sum_{i \in K} \mu_i \geq \sum_{a \in B} \lambda_a. \tag{2.3}$$

In other words, K can veto B if the voters from K have sufficiently many veto-cards to eliminate all candidates from B. Now we can reformulate Theorem 1 in terms of veto functions as follows.

Theorem 2. *Let* $VVS' = (\lambda', \mu', \tau)$ *and* $VVS'' = (\lambda'', \mu'', \tau)$ *be two simple veto voting schemes such that they have the same simple veto order* τ *and their veto functions* $V' = V_{\mu',\lambda'}$ *and* $V'' = V_{\mu'',\lambda''}$ *are equal; that is* $V'(K, B) = V''(K, B)$ *for all* $K \subseteq I, B \subseteq A$. *Then the SCCs* $\mathcal{S}' = \mathcal{S}_{\mu',\lambda',\tau}$ *and* $S'' = \mathcal{S}_{\mu'',\lambda'',\tau}$ *are equal, too; that is* $S'(P) = S''(P)$ *for every situation* P.

To prove that Theorems 1 and 2 are equivalent we only need to show that Theorem 2 becomes trivial if we restrict ourselves to the war situations only. In other words, given a veto function, the results of elections in all war situations are uniquely defined, and vice versa. Due to Lemma 1, this is true for all (not only simple) veto orders.

Lemma 2. *Given two veto voting schemes* $VVS' = (\lambda', \mu', \sigma'_{\mu'})$ *and* $VVS'' = (\lambda'', \mu'', \sigma''_{\mu''})$ *that generate veto functions* $V' = V_{\lambda',\mu',\sigma'_{\mu'}}$, $V'' = V_{\lambda'',\mu'',\sigma''_{\mu''}}$ *and SCCs* $S' = S_{\lambda',\mu',\sigma'_{\mu'}}$, $S'' = S_{\lambda'',\mu'',\sigma''_{\mu''}}$, *the following claims are equivalent:*
(i) $V' = V''$; *that is* $V'(K, B) = V''(K, B)$ *for all* $K \subseteq I, B \subseteq A$,
(ii) $S'(P) = S''(P)$ *for every war situation* P.

Proof. . Suppose that $V' \neq V''$, say $1 = V'(K, B) \neq V''(K, B) = 0$ for some $K \subseteq I, B \subseteq A$; that is in VVS' coalition K can veto block B but in VVS'' it cannot. Consider a complete order P_0 over A such that each candidate from $A \setminus B$ is preferred to each candidate from B. Let a_0 be the best candidate from B in this order. Define a war situation P as follows. All voters from K prefer candidates according to P_0 (that is for them $A \setminus B$ is better than B) and all voters from $I \setminus K$ have the opposite preference. Then obviously, $a_0 \notin S'(P)$, since $V'(K, B) = 1$ and in VVS' coalition K can veto the whole block B including a_0. Yet, $a_0 \in S''(P)$, since $V''(K, B) = 0$; that is in VVS'' coalition K does not have enough veto power to eliminate B and hence a_0 will remain unvetoed. Thus $S'(P) \neq S''(P)$.

Vice versa, suppose that $S'(P) \neq S''(P)$ for a war situation P defined by a complete order P_0 over A and a partition $K, I \setminus K$. Without loss of generality, we can assume that $a_0 \in S''(P) \setminus S'(P)$; that is $a_0 \notin S'(P)$ and $a_0 \in S''(P)$. Let B consist of a_0 and all candidates preceding a_0 in order P_0. Then obviously, $V'(K, B) = 1$, otherwise a_0 would be elected in VVS', and $V''(K, B) = 0$, otherwise a_0 would be vetoed in VVS''. □

3 Proof of Theorem 2

In this section we will consider only simple veto orders. Then, without any loss of generality, we can assume that permutation τ is identical; that is first i_1 distributes all veto cards, then i_2, etc. In this case argument τ becomes irrelevant and we will omit it in all formulas. In particular, pair (λ, μ) already defines a voting scheme.

We will make use of the terminology from the popular game Sea Battle; see, for example, http:/www.karpolan.com/sea-battle. Given a scheme (λ, μ), a voter $i \in I$, and a candidate $a \in A$, we say that a is *killed* (or *eliminated*) by i if a is not elected and the last veto card put against a belongs to i; we say that a is *wounded* by i if i puts at least one veto card against a but i does not eliminate a, that is, either a is elected or a is eliminated later by some other voter.

Lemma 3. *A voter can kill several candidates but can wound at most one.*

Proof. . Indeed, if i votes against a then (s)he cannot switch to another candidate a' before a is eliminated. This follows from our two basic assumptions: (i) the veto order is simple and (ii) the voting is sincere. □

Let us remark that both assumptions are important. For example, if veto order is not simple then i can wound a, then another voter can eliminate a, and then i can vote again and wound some other candidate a'.

Given a VVS (λ, μ) and situation P, let us assign to each candidate a a set $W(a)$ of all voters who put at least one veto card against a. We would like to prove (by induction on $n = |I|$) that $W(a)$ depends only on P and the veto function $V(\lambda, \mu)$. However, problems appear already for $n = 1$. Given two schemes $VVS' = (\lambda', \mu')$, $VVS'' = (\lambda'', \mu'')$, and P, let us assume, for example, that a_5, a_3, and a_4 are the last 3 candidates in the preference order of i_1. Furthermore, let us assume that in VVS' i_1 kills a_5 and a_3 using up all veto cards, while in VVS'' i_1 kills a_5 and a_3 and still (s)he has more veto cards to wound (but not enough to kill) a_4, that is,

$$\mu'_{i_1} = \lambda'_{a_3} + \lambda'_{a_5} \quad \textit{and} \quad \lambda''_{a_3} + \lambda''_{a_5} < \mu''_{i_1} < \lambda''_{a_3} + \lambda''_{a_4} + \lambda''_{a_5}$$

.

Then $W'(a_4) = \emptyset$, while $W''(a_4) = \{i_1\}$ and yet we get no contradiction, since veto functions V' and V'' may be equal. It is not difficult to understand that this happens due to the equality $\mu'_{i_1} = \lambda'_{a_3} + \lambda'_{a_5}$. However, it is possible to get rid of all such equalities.

For simplicity let us denote $\sum_{i \in K} \mu_i$ by $\mu(K)$ and $\sum_{a \in B} \lambda_a$ by $\lambda(B)$ for all $K \subseteq I, B \subseteq A$.

(Let us remark that now we can rewrite equations (1.1), (1.2), and (2.3) as

$$\mu(I) < \lambda(A), \quad \lambda(A) - \mu(I) = 1, \quad \textit{and} \quad V(K, B) = 1 \quad iff \quad \mu(K) \geq \lambda(B),$$

respectively.) Let us call the scheme (λ, μ) *degenerate* if $\mu(K) = \lambda(B)$ for some pair $K \subseteq I, B \subseteq A$ and *non-degenerate* otherwise.

Lemma 4. *Given a scheme (λ, μ) and situation P, if some voter i eliminates a candidate a by the last veto card then (λ, μ) is degenerate.*

Proof. . Let $K_0 = \{i\}$ and $B_0 = \{a\}$. Furthermore, let K_1 be the set of all voters who wound a. According to Lemma 3, they cannot wound any other candidate, yet, they can kill some. Let B_1 be the set of all such candidates and let K_2 be the set of all voters who wound B_1. Again, they cannot wound any other candidate, yet, they can eliminate some. Let B_2 be the set of all such candidates, etc. Finally, let $K = \cup_{j=0}^{\infty} K_j$ and $B = \cup_{j=0}^{\infty} B_j$. (Obviously, K_j and B_j become empty when j is large enough.)

By the above construction, the voters of K vote only against candidates of B and all other voters do not vote against B. Hence $\mu(K) = \lambda(B)$ and (λ, μ) is degenerate. □

Schemes (λ', μ') and (λ'', μ'') are called *equivalent* if they define the same SCC; that is two sets of elected candidates $S_{\lambda', \mu'}(P)$ and $S_{\lambda'', \mu''}(P)$ coincide for each situation P. In particular, they coincide in all war situations and hence equivalent schemes must define the same veto function, $V_{\lambda', \mu'}(K, B) = V_{\lambda'', \mu''}(K, B)$ for all pairs K, B

Lemma 5. *For each scheme (λ, μ) there exists an equivalent non-degenerate scheme (λ', μ').*

Proof. . Let us multiply vectors λ and μ by a positive integer c and then for each pair (K, B) such that $\mu(K) = \lambda(B)$ choose a voter $i \in K$ and add 1 to the corresponding veto power. Obviously, if c is large enough, say $c > 2^{|I|+|A|}$, then the obtained scheme (λ', μ') is non-degenerate and equivalent to (λ, μ). □

Proof. of Theorem 2. Let $VVS' = (\lambda', \mu')$ and $VVS'' = (\lambda'', \mu'')$ be two schemes whose veto functions $V' = V_{\lambda',\mu'}$ and $V'' = V_{\lambda'',\mu''}$ are equal, $V' = V'' = V$. We will prove that their SCCs $S' = S_{\lambda',\mu'}$ and $S'' = S_{\lambda'',\mu''}$ are equal too. Due to Lemma 5, we may assume without loss of generality that both schemes are non-degenerate. For an arbitrary situation P and candidate a we will prove that $W'(a) = W''(a)$.

Let us truncate the list of all voters $i_1, i_2, \dots$ by the first n voters and proceed with induction on n. Let $n = 1$ and $W'(a) = \emptyset$, while $W''(a) = \{i_1\}$. Let B denote the set of all candidates worse than a in the preference order P_{i_1}. Then i_1 can veto B in VVS'' but not in VVS' and, hence, $0 = V'(\{i_1\}, B) \neq V''(\{i_1\}, B) = 1$, this is a contradiction. (Let us remark that for degenerate VVS' it would be possible that i_1 can veto B using up all veto cards. See example in the beginning of this section.)

Now let us assume that $W'_{n-1} \equiv W''_{n-1}$ but $W'_n(a) \neq W''_n(a)$, say $i_n \in W''_n(a) \setminus W'_n(a)$; that is voter i_n eliminates candidate a in VVS'' but not in VVS'. Perhaps, in VVS' i_n did not even vote against a. Yet, since VVS' is non-degenerate, there exists a candidate a' killed by i_n in VVS'' and only wounded by i_n in VVS'.

Now we can repeat, with minor modifications, the proof of Lemma 4. Let $K_0 = \{i_n\}$ and $B_0 = \{a'\}$. Furthermore, let K_1 be the set of all voters who wound a'. According to Lemma 3, they cannot wound any other candidate, yet, they can kill some. Let B_1 be the set of all candidates eliminated by K_1 and let K_2 be the set of all voters who wound B_1. Again, they cannot wound any other candidate, yet, they can kill some. Let B_2 be the set of all candidates eliminated by K_2,... etc. Finally, let $K = \cup_{j=0}^{\infty} K_j$ and $B = \cup_{j=0}^{\infty} B_j$. (Again, K_j and B_j are empty when j is large enough.) Let us also note that all sets K_j and B_j defined above are the same for both schemes VVS' and VVS'' by the induction hypothesis. Furthermore, by the above construction, all voters of K, except i_n, vote only against candidates of B and all other voters do not vote against B. Hence, K cannot veto B in VVS' but can do it in VVS''; that is $0 = V'(K, B) \neq V''(K, B) = 1$. This contradiction proves the Theorem. □

4 On properties of game structures and graphs that depend only on the corresponding veto functions.

The main result of this paper states that the SCC of a simple veto voting scheme is uniquely defined by its effectivity (or equivalently, veto) function. The following two results are similar:

(i) Nash solvability of a two-person game form g depends only on its effectivity function E_g.

(ii) The core of a normal form game (g, u) depends only on its utility function u and effectivity function E_g.

The definitions follow. Let standardly I and A be a set of voters (players) and candidates (outcomes) respectively. Let us recall that an effectivity function (EFF) is a mapping $E : 2^I \times 2^A \to \{0, 1\}$; that is E has two arguments: a coalition of voters $K \subseteq I$ and a block of candidates $B \subseteq A$. Further, $E(K, B) = 1$ (respectively, $E(K, B) = 0$) means that K can (respectively, can not) guarantee that a candidate of B will be elected. An EFF E is called *maximal (or selfdual)* if $E(K, B) = 1$ if and only if $E(I \setminus K, A \setminus B) = 0$.

Let X_i be a set of strategies of a voter $i \in I$. A *game form* is a mapping $g : X \to A$, where $X = \prod_{i \in I} X_i$. Every game form g defines an EFF E_g as follows: for a coalition $K \subseteq I$ and block $B \subseteq A$ the EFF $E_g(K, B) = 1$ if and only if K has a strategy $x_K = \{x_i \in X_i, i \in K\}$ such that $g(x_K, x_{I \setminus K}) \in B$ for every strategy $x_{I \setminus K} = \{x_i \in X_i, i \notin K\}$ of the complementary coalition $I \setminus K$.

Obviously, the implication $E_g(I \setminus K, A \setminus B) = 0$ whenever $E_g(K, B) = 1$ holds for every g.

The game form g is called *tight* if the inverse implication, $E_g(I \setminus K, A \setminus B) = 1$ whenever $E_g(K, B) = 0$, holds too. In other words, g is tight if and only if its EFF E_g is selfdual.

A utility function is a mapping $u : I \times A \to \mathbb{R}$, where $u(i, a)$ is interpreted as a profit of the voter $i \in I$ in case the candidate $a \in A$ is elected. A *normal form game* is a pair (g, u), where $g : X \to A$ and $u : I \times A \to \mathbb{R}$, are a utility function and game form, respectively.

A game form g is called Nash-solvable if for every utility function u the corresponding normal form game (g, u) has at least one Nash equilibrium in pure strategies.

An old theorem claims that a two-person game form is Nash-solvable if and only if it is tight, that is, the corresponding effectivity function is selfdual, [3, 4]. However, this theorem does not generalize the case of n-person game forms for any $n \geq 3$, see [4]. Two such game forms may have the same EFF, while one is Nash-solvable and the other one is not.

Another well-known observation, [7], claims that the core of a normal form game (g, u) depends only on its utility function u and EFF E_g.

Given a candidate $a_0 \in A$, a coalition of voters $K \subseteq I$, and a utility function $u : I \times A \to \mathbb{R}$, let $PR(K, a_0, u) = \{a \in A \mid u(i, a) > u(i, a_0) \forall i \in K\}$ denote the set of all candidates strictly and unanimously preferred to a_0 by the coalition K. Furthermore, given an EFF $E : 2^I \times 2^A \to \{0, 1\}$, obviously, K rejects a_0 whenever $E(K, PR(K, a_0, u)) = 1$, since in this case the coalition K can guarantee to all coalitionists that a candidate strictly better than a_0 will be elected.

Given a normal form game (g, u), its *core* $C(g, u)$ is defined as the set of all candidates that are not rejected by any coalition $K \subseteq I$; that is

$$C(g, u) = \{a \in A \mid E_g(K, PR(K, a, u)) = 0 \ \ \forall \ K \subseteq I\}.$$

Thus, by definition, the core $C(g, u)$ depends only on u and E_g.

It is an interesting general question: which properties of game structures depend only on their effectivity (or equivalently, veto) function.

Somewhat surprisingly, a similar situation may hold not only for game structures or voting schemes but also for quite different objects, e.g., for graphs. It was shown in [1] that some important properties of graphs are uniquely determined by the corresponding effectivity functions.

Given a graph $G = (V, E)$, we define its EFF E_G as follows. Let us assign a voter $i \in I$ to each maximal clique of G and a candidate $a \in A$ to each maximal independent set of G. By this, a coalition $K_v \subseteq I$ and a block $B_v \subseteq A$ is assigned to every vertex $v \in V$. Namely, K_v (respectively, B_v) consist of all maximal cliques (respectively, independent sets) of G that contain v. Then the EFF E_G is defined as follows: $E_G(K, B) = 1$ if $K_v \subseteq K$ and $B_v \subseteq B$ for some vertex $v \in V$ and $E_G(K, B) = 0$ otherwise. It is proved in [1] that

(i) Graph G is *perfect* if and only if its EFF E_G is *balanced* and

(ii) Graph G is *kernel-solvable* if and only if its EFF E_G is *stable*.

We refer to [1] for the definitions. Thus, whether the graph G is perfect (respectively, kernel-solvable) or not depends only on its effectivity function E_G.

In is known in cooperative game theory that balanced EFFs are stable. Hence, perfect graphs are kernel-solvable. This was conjectured by Claude Berge and Pierre Duchet in 1983. The inverse implication follows from the Strong Berge Perfect Graph Conjecture that was recently proved by Chudnovsky, Robertson, Seymour, and Thomas [2].

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