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Tongyin Liu and András Prékopa

Abstract. Probabilistic constrained stochastic programming problems are considered with discrete random variables on the r.h.s. in the stochastic constraints. It is assumed that the random vector has multivariate Poisson, binomial or geometric distribution. We prove a general theorem that implies that in each of the above cases the c.d.f. majorizes the product of the univariate marginal c.d.f's and then use the latter one in the probabilistic constraints. The new problem is solved in two steps: (1) first we replace the c.d.f's in the probabilistic constraint by smooth logconcave functions and solve the continuous problem; (2) search for the optimal solution for the case of the discrete random variables. Numerical results are presented and comparison is made with the solution of a problem taken from literature.

Keywords: Stochastic programming, probabilistic constraints, Poisson distribution, binomial distribution, geometric distribution, incomplete gamma function, incomplete beta function.

AMS subject classification: 90C15 90C25 62E10

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1 Introduction

In this paper, we consider the stochastic programming problem:

$$\begin{array}{rl} \min & c^T x \\ \text{subject to} & \mathbf{P}(Tx \geq \xi) \geq p \\ & Ax \geq b \\ & x \geq 0, \end{array} \tag{1}$$

where $\xi = (\xi_1, \ldots, \xi_r)$ is a discrete random vector, p is a given probability ($0 < p < 1$, in practice near 1) and A, T, c, b are constant matrices and vectors with sizes $m \times n$, $r \times n$, n and m, respectively. We can write (1) in the equivalent form:

$$\begin{array}{rl} \min & c^T x \\ \text{subject to} & \mathbf{P}(\xi \leq y) \geq p \\ & Tx = y \\ & Ax \geq b \\ & x \geq 0. \end{array} \tag{2}$$

Problem (1) for general random vector ξ with stochastically dependent components was introduced in [6] and [7]. In these and other subsequent papers convexity theorems have been proved and algorithms have been proposed for the solution of problem (1) and the companion problem:

$$\begin{array}{rl} \max & \mathbf{P}(Tx \geq \xi) \\ \text{subject to} & Ax \geq b \\ & x \geq 0. \end{array} \tag{3}$$

For detailed presentation of these results, the reader is referred to [10] and [11].

For the case of a discrete ξ, the concept of p level efficient point(PLEP) has been introduced in [9]. Below, we recall this definition. Let $F(z)$ designate the probability distribution function of ξ, i.e., $F(z) = \mathbf{P}(\xi \leq z)$, $z \in \mathbf{R}^r$.

Definition 1.1. *Let $\mathcal{Z}$ be the set of possible values of ξ. A vector $z \in \mathcal{Z}$ is said to be a p level efficient point or PLEP of the probability distribution of ξ if $F(z) = \mathbf{P}(\xi \leq z) \geq p$ and there is no $y \in \mathcal{Z}$ such that $F(y) \geq p$, $y \leq z$ and $y \neq z$.*

Dentcheva, Prékopa and Ruszczyński (2000) remarked that, by a classical theorem of Dickson [1] on partially ordered sets (posets), the number of PLEP's is finite even if $\mathcal{Z}$ is not a finite set. Let $v^{(j)}$, $j \in \mathcal{J}$ be the set of PLEP's. Since

$$\{y \mid \mathbf{P}(\xi \leq y) \geq p\} = \mathcal{Z}_p = \bigcup_{j \in \mathcal{J}} \{v^{(j)} + \mathbf{R}^s_+\},$$

a further equivalent form of problem (1)is the following:

$$\begin{array}{rl} \min & c^T x \\ \text{subject to} & Tx \in \mathcal{Z}_p \\ & Ax \geq b \\ & x \geq 0. \end{array} \tag{4}$$

The first paper on problem (1) with discrete random vector ξ was published by Prékopa (1990). He presented a general method to solve problem (4), assuming that the PLEP's are enumerated. Note that problem (4) can be regarded as a disjunctive programming problem. Sen (1992) studied the set of all valid inequalities and the facets of the convex hull of the given disjunctive set implied by the probabilistic constraint in (2). Prékopa, Vizvári and Badics (1998) relaxed problem (4) in the following way:

$$\begin{array}{rl} \min & c^T x \\ \text{subject to} & Tx \geq \sum_{i=1}^{|\mathcal{J}|} v^{(i)} \mu_i \\ & \sum_{i=1}^{|\mathcal{J}|} \mu_i = 1,\ \mu_i \geq 0,\ i \in \{1, \ldots, |\mathcal{J}|\} \\ & Ax \geq b \\ & x \geq 0, \end{array} \tag{5}$$

gave an algorithm to find all the PLEP's and a cutting plane method to solve problem (5). In general, however, the number of p level efficient points for ξ is very large. To avoid the enumeration of all PLEP's, Dentcheva, Prékopa and Ruszczyński (2000) presented a cone generation method to solve problem (5). Vizvári (2002) further analyzed the above solution technique with emphasis on the choices of Lagrange multipliers and the solution of the knapsack problem that comes up as a PLEP generation technique in case of independent random variables.

Discrete random vectors in the contexts of (1) and (3) come up in many practical problems. Singh et al. [16] present and solve a chip fabrication problem, where the components of the random vector designate the number of chip sites in a wafer that produce good chips of given types. These authors solve a similar problem as (3) rather than problem (1), where the objective function in (3) is $1 - \mathbf{P}(Tx \geq \xi)$. Dentcheva et al. [4] present and solve a traffic assignment problem in telecommunication, where the problem is of type (1) and the random variables are demands for transmission. In the design of a stochastic transportation network in power systems, Prékopa and Boros [12] present a method to find the probability of the existence of a feasible flow problem, where the demands for power at the nodes of the network are integer valued random variables.

We assume that the components of the random vector ξ have all Poisson, all binomial or all geometric distributions. We use incomplete gamma, beta and exponential functions, respectively, to convexify the problem by replacing smooth distribution functions for the discrete ones such that they coincide at lattice points. Then the optimal solutions to the convex problems are used to find the optimal solution of problem (1). This is carried out by the use of a modified Hooke and Jeeves direct search method. In Section 2 we solve problem (1) under the assumption that the components of ξ are independent and the solution of (3) is discussed for the case of discrete random vector ξ. In Section 3 we look at the case where the components of ξ are partial sums of independent random variables. We prove a general theorem that implies that in our cases the multivariate c.d.f majorizes the product of the univariate marginal c.d.f's. Finally, in Section 4 numerical results, along with new applications are presented. Comparison is made between the solution of a telecommunication problem presents in [4] and our solution to the same problem.

2 The case of independent Poisson, binomial and geometric random variables

Assume that $\xi_1, \dots, \xi_r$ are independent and nonnegative integer valued. Let $F_i(z)$ be the c.d.f. of ξ_i, $i = 1, \dots, r$. Then problem (1) can be written in the following form:

$$\begin{array}{rl} \min & c^T x \\ \text{subject to} & Tx = y \\ & Ax \geq b, \quad x \geq 0 \\ & \prod_{i=1}^r F_i(y_i) \geq p. \end{array} \tag{6}$$

Note that the inequality

$$\mathbf{P}(T_i x \geq \xi_i) \geq \mathbf{P}(Tx \geq \xi) \geq p$$

implies $T_i x \geq 0$, $i = 1, \dots, r$. Thus if ξ is a discrete random vector, the above problem is equivalent to the following:

$$\begin{array}{rl} \min & c^T x \\ \text{subject to} & Tx \geq y, \quad y \in \mathbf{Z}_r^+ \\ & Ax \geq b, \quad x \geq 0 \\ & \prod_{i=1}^r F_i(y_i) \geq p. \end{array} \tag{7}$$

We solve problem (7) in such a way that to each F_i we fit a smooth c.d.f. that coincides with F_i at integers, solve the smooth problem and then search for the optimal solution of the discrete random variable problem.

2.1 Independent Poisson random variables

Let ξ be a random variable that has Poisson distribution with parameter $\lambda > 0$. The values of its c.d.f. at nonnegative integers are

$$P_n = \sum_{k=0}^{n} \frac{\lambda^k}{k!} e^{-\lambda}, \qquad n = 0, 1, \ldots.$$

Let

$$F(p;\lambda) = \int_{\lambda}^{\infty} \frac{x^p}{\Gamma(p+1)} e^{-x} dx,$$

where

$$\Gamma(p) = \int_0^{\infty} x^{p-1} e^{-x} dx, \quad \text{for} \quad p > -1.$$

It is well-known (see, e.g., Prékopa 1995) that for $n \geq 0$,

$$P_n = \sum_{k=0}^{n} \frac{\lambda^k}{k!} e^{-\lambda} = \int_{\lambda}^{\infty} \frac{x^n}{n!} e^{-\lambda} dx. \tag{8}$$

We recall the following theorem.

Theorem 2.1. *([10]) For any fixed $\lambda > 0$, the function $F(p;\lambda)$ is logconcave on the entire real line, strictly logconcave on $\{p \mid p \geq -1\}$ and $P_n = F(p, \lambda)$ for any nonnegative integer $n = p$.*

Suppose that $\xi_1, \ldots, \xi_r$ are independent Poisson random variables with parameters $\lambda_1, \cdots, \lambda_r$, respectively. To solve (7), first we consider the following problem:

$$\begin{array}{rl} \min & c^T x \\ \text{subject to} & Tx = y \\ & Ax \geq b, \quad x \geq 0 \\ & \prod_{i=1}^{r} \int_{\lambda_i}^{\infty} \frac{t^{y_i}}{\Gamma(y_i+1)} e^{-t} dt \geq p. \end{array} \tag{9}$$

Then we can rewrite problem (9) in the following form:

$$\begin{array}{rl} \min & c^T x \\ \text{subject to} & Tx = y \\ & Ax \geq b, \quad x \geq 0 \\ & \sum_{i=1}^{r} \ln \left(1 - \frac{1}{\Gamma(y_i+1)} \int_0^{\lambda_i} t^{y_i} e^{-t} dt \right) \geq \ln p. \end{array} \tag{10}$$

By Theorem 2.1, (9) is a convex nonlinear programming problem.

2.2 Independent binomial random variables

Suppose ξ has binomial distribution with parameter $0 < p < 1$. Let x be a nonnegative integer. It is known (see, e.g., Singh et. al., 1980; Prekopa 1995) that

$$\sum_{i=a}^{x} \binom{x}{i} p^i (1-p)^{(x-i)} = \frac{\int_0^p y^{a-1}(1-y)^{x-a} dy}{\int_0^1 y^{a-1}(1-y)^{x-a} dy}. \tag{11}$$

For fixed $a > 0$ define $G(a,\ x)$, as a function of the continuous variable x, by the equation (11) for $x \geq 0$ and let $G(a,\ x) = 0$ for $x < a$. We have the following Theorem.

Theorem 2.2. *([10],[16]) Let $a > 0$ be a fixed number. Then $G(a,\ x)$ is strictly increasing and strictly logconcave for $x \geq a$.*

If x is an integer then $G(a, x) = 1 - F(a-1)$ where F is the c.d.f. of the binomial distribution with parameters x and p. While Theorem 2.2 provides us with a useful tool in some applications (c.f. [16]), we need a smooth logconcave extension of F and it can not be derived from $G(a, x)$.

Let X be a binomial random variable with parameter n and p. Then, by (11), we have for every $x = 0, 1, \ldots, n-1$:

$$P(X \leq x) = \frac{\int_p^1 y^x (1-y)^{n-x-1} dy}{\int_0^1 y^x (1-y)^{n-x-1} dy}. \tag{12}$$

The function of the variable x, on the right hand side of (12), is defined for every x satisfying $-1 < x < n$. Its limit is 0, if $x \to 0$ and is 1, if $x \to n$. Let

$$F(x; n, p) = \begin{cases} 0, & \text{if } x \leq -1; \\ \frac{\int_p^1 y^x (1-y)^{n-x-1} dy}{\int_0^1 y^x (1-y)^{n-x-1} dy}, & \text{if } -1 < x < n; \\ 1, & \text{if } x \geq n. \end{cases} \tag{13}$$

We have the following:

Theorem 2.3. *The function $F(x; n, p)$ satisfies the relations*

$$\lim_{x \to -\infty} F(x; n, p) = 0, \quad \lim_{x \to \infty} F(x; n, p) = 1, \tag{14}$$

it is strictly increasing in the interval $(-1,\ n)$, has continuous derivative and is logconcave on $\mathbf{R}^1$.

Proof. We skip the proof of the relations because it requires any standard reasoning.

To prove the other assertions first transform the integral in (13) by the introduction of the new variable $\frac{y}{1-y} = t$. We obtain

$$F(x;n,p) = \frac{\int_\lambda^\infty t^x \frac{1}{(1+t)^{n+1}} dt}{\int_0^\infty t^x \frac{1}{(1+t)^{n+1}} dt}, \qquad -1 < x < n, \tag{15}$$

where $\lambda = \frac{p}{1-p}$. To prove strict monotonicity we take first derivative of this function:

$$\begin{aligned} &\frac{dF(x;n,p)}{dx} \\ &= F(x;n,p) \left(\frac{\int_\lambda^\infty t^x \ln t \frac{1}{(1+t)^{n+1}} dt}{\int_\lambda^\infty t^x \frac{1}{(1+t)^{n+1}} dt} - \frac{\int_0^\infty t^x \ln t \frac{1}{(1+t)^{n+1}} dt}{\int_0^\infty t^x \frac{1}{(1+t)^{n+1}} dt} \right) \end{aligned} \tag{16}$$

and show that it is positive if $-1 < x < n$. The derivative, with respect to λ, of the first term in the parenthesis equals

$$\begin{aligned} &\frac{d}{d\lambda} \frac{\int_\lambda^\infty t^x \ln t \frac{1}{(1+t)^{n+1}} dt}{\int_\lambda^\infty t^x \frac{1}{(1+t)^{n+1}} dt} \\ &= \frac{-\lambda^x \ln \lambda \frac{1}{(1+\lambda)^{n+1}} \int_\lambda^\infty t^x \frac{1}{(1+t)^{n+1}} dt + \lambda^x \frac{1}{(1+\lambda)^{n+1}} \int_\lambda^\infty t^x \ln t \frac{1}{(1+t)^{n+1}} dt}{\int_\lambda^\infty t^x \frac{1}{(1+t)^{n+1}} dt}. \end{aligned}$$

This is a positive value, since

$$\int_\lambda^\infty t^x \ln t \frac{1}{(1+t)^{n+1}} dt > \ln \lambda \int_\lambda^\infty t^x \frac{1}{(1+t)^{n+1}} dt.$$

Thus, the first term in the parenthesis in (16) is an increasing function of λ, which proves the positivity of the first derivative of $F(x;n,p)$ in the interval $-1 < x < n$.

The continuity of the derivative of $F(x;n,p)$ on $\mathbf{R}^1$ follows from (16). In fact the derivative is continuous if $-1 < x < n$ and by the application of a standard reasoning (similar to the one needed to prove (14)) we can show that

$$\lim_{x \to -1+0} \frac{F(x;n,p)}{dx} = \lim_{x \to n-0} \frac{F(x;n,p)}{dx} = 0.$$

It is enough to prove the logconcavity of $F(x;n,p)$ for the case of $-1 < x < n$, because the logconcavity of the function on $\mathbf{R}^1$ easily follows from it. We have the equation

$$\begin{aligned} \frac{d^2 F(x;n,p)}{dx^2} &= \frac{\int_\lambda^\infty t^x (\ln t)^2 \frac{1}{(1+t)^{n+1}} dt}{\int_\lambda^\infty t^x \frac{1}{(1+t)^{n+1}} dt} - \left(\frac{\int_\lambda^\infty t^x \ln t \frac{1}{(1+t)^{n+1}} dt}{\int_\lambda^\infty t^x \frac{1}{(1+t)^{n+1}} dt} \right)^2 \\ &- \left(\frac{\int_0^\infty t^x (\ln t)^2 \frac{1}{(1+t)^{n+1}} dt}{\int_0^\infty t^x \frac{1}{(1+t)^{n+1}} dt} - \left(\frac{\int_0^\infty t^x \ln t \frac{1}{(1+t)^{n+1}} dt}{\int_0^\infty t^x \frac{1}{(1+t)^{n+1}} dt} \right)^2 \right). \end{aligned} \tag{17}$$

Let us introduce the following p.d.f.:

$$g(t) = \frac{e^{(x+1)t}\frac{1}{(1+e^t)^{n+1}}}{\int_{-\infty}^{\infty} e^{(x+1)u}\frac{1}{(1+e^u)^{n+1}}du}, \quad -\infty < t < \infty, \tag{18}$$

where x is a fixed number satisfying $-1 < x < n$. The function $g(t)$ is exactly logconcave on the entire real line. Let X be a random variable that has p.d.f. equal to (18). Then (17) can be rewritten as

$$\begin{aligned} \frac{d^2F(x;n,p)}{dx^2} &= \frac{\int_{\ln\lambda}^{\infty} t^2 g(t)dt}{\int_{\ln\lambda}^{\infty} g(t)dt} - \left(\frac{\int_{\ln\lambda}^{\infty} tg(t)dt}{\int_{\ln\lambda}^{\infty} g(t)dt}\right)^2 \\ &\quad - \left(\frac{\int_{-\infty}^{\infty} t^2 g(t)dt}{\int_{-\infty}^{\infty} g(t)dt} - \left(\frac{\int_{-\infty}^{\infty} tg(t)dt}{\int_{-\infty}^{\infty} g(t)dt}\right)^2\right) \\ &= E(X^2 \mid X \geq \ln\lambda) - E^2(X \mid X \geq \ln\lambda) \\ &\quad - (E(X^2) - E^2(X)). \end{aligned} \tag{19}$$

Burridge (1982) has shown that if a random variable X has a logconcave p.d.f., then

$$E(X^2 \mid X \geq u) - E^2(X \mid X \geq u)$$

is a decreasing function (a proof of this fact is given in Prékopa 1995 pp.118-119). If we apply this in connection with the function (18), then we can see that the value in (19) is negative. □

Remark The proof of Theorem 2.1 is similar to the proof of Theorem 2.3. In that case the $g(t)$ function is the following:

$$g(t) = \frac{e^{(x+1)t}e^{-e^t}}{\int_{-\infty}^{\infty} e^{(x+1)u}e^{-e^u}du}, \quad -\infty < t < \infty,$$

(see Prékopa 1995, p.117).

Suppose $\xi_1, \xi_2, \ldots, \xi_r$ are independent binomial random variables with parameters $(n_1, p_1), \ldots, (n_r, p_r)$, respectively. To solve problem (7), we first solve the following problem:

$$\begin{array}{rl} \min & c^T x \\ \text{subject to} & Tx = y \\ & Ax \geq b \\ & \sum_{i=1}^{r}\left(\ln\int_{p_i}^{1} t^{y_i}(1-t)^{n_i-y_i-1}dt - \right. \\ & \qquad \left. \ln\int_0^1 t^{y_i}(1-t)^{n_i-y_i-1}dt\right) \geq \ln p \\ & x \geq 0. \end{array} \tag{20}$$

This is again a convex programming problem from Theorem 2.3.

2.3 Independent geometric random variables

Let ξ be a random variable with geometric distribution. ξ has probability function

$$P(k) \ = \ pq^{k-1} \text{ if } k = 1, 2, \dots$$

and $P(k) = 0$ otherwise, where $q = 1 - p$ and $0 < p < 1$. Its distribution function is

$$P_n = \sum_{k=1}^{n} pq^{k-1} = 1 - q^n. \tag{21}$$

A general theorem ensures but a simple direct reasoning also shows that (see, e.g. Prekopa 1995, p.110) P_n is a logconcave sequence. The continuous counterpart of the geometric distribution is the exponential distribution. If λ is the parameter of the later and $\lambda = \ln \frac{1}{q}$, then

$$1 - e^{-\lambda x} = P_n, \quad \text{for} \quad x = n. \tag{22}$$

The c.d.f. $F(x) = 1 - e^{-\lambda x}$ is strictly increasing and strictly logconcave function for $x > 0$.

Suppose the components $\xi_1, \dots, \xi_r$ of random vector ξ are independent geometric variables with parameters $p_1, \dots, p_r$, respectively. In this case, to solve problem (7), we first solve the following convex programming problem:

$$\begin{array}{rl} \min & c^T x \\ \text{subject to} & Tx = y \\ & Ax \geq b \\ & \sum_{i=1}^{r} \ln(1 - e^{-\lambda_i y_i}) \geq \ln p \\ & x \geq 0. \end{array} \tag{23}$$

For the above mentioned three convex programming problems, they can be solved by many known methods, for example, interior trust region approach [3] as it is used in MatLab 6. It may also be solved by using CPLEX if we have a numerical method to calculate incomplete gamma and beta functions. In this paper, we use MatLab 6 to solve the problems in the numerical examples.

2.4 Relations between the feasible sets of the convex programming problems and the discrete cases

First, we have the following theorem:

Theorem 2.4. *Let ξ be a discrete random variable, $F(z)$ the c.d.f of ξ, $z \in \mathbf{Z}_+$, and $\mathcal{P}$ the set of all PLEP's of ξ. Let $\tilde{F}(x)$ be a smooth function, $x \in \mathbf{R}_+$, such that $F(z) = \tilde{F}(x)$ when $x \in \mathbf{Z}_+$. Then*

$$\mathcal{P} \in \tilde{\mathcal{Z}}_p = \{x \in \mathbf{Z}_+^r \mid \tilde{F}(x) \geq p\}. \tag{24}$$

Proof. Let $\mathcal{Z}_p = \{y \in \mathbf{Z}_+ \mid \mathbf{P}(\xi \le y) \ge p\}$, where $\mathbf{P}$ is the distribution function of ξ. Then $\mathcal{P} \subseteq \mathcal{Z}_p$. Since the values of $F(z)$ and $\tilde{F}(x)$ coincide at the lattice points, $\mathcal{Z}_p = \tilde{\mathcal{Z}}_p$. □

Let ξ be an r-component random vector, and $\mathcal{P}$ the set of all PLEP's of ξ. Let

$$F(y;\lambda) = \int_\lambda^\infty \frac{x^y}{\Gamma(y+1)} e^{-x} dx, \;\; y > -1,$$

$$F(y;,n,p') = \frac{\int_{p'}^1 x^y (1-x)^{n-y-1} dx}{\int_0^1 x^y (1-x)^{n-y-1} dx}, \quad -1 < y < n,$$

and

$$G(y;\lambda) = 1 - e^{-\lambda y}, \quad y > 0$$

where $\Gamma(\cdot)$ is the gamma function and $p' \in (0,\ 1)$. Let

$$\mathcal{Z}_p^P = \{y + 1 \in \mathbf{Z}_+^r \mid \prod_{i=1}^r F(y_i;\lambda_i) \ge p, \quad \lambda_i > 0,\ i = 1,\dots,r\},$$

$$\mathcal{Z}_p^B = \{y + 1 \in \mathbf{Z}_+^r \mid \prod_{i=1}^r F(y_i;n_i,p_i') \ge p, \quad p_i' \in (0,\ 1),\ -1 < y_i < n_i,\ i = 1,\dots,r\}$$

and

$$\mathcal{Z}_p^G = \{y \in \mathbf{Z}_+^r \mid \prod_{i=1}^r G(y_i;\lambda_i) \ge p, \quad y_i > 0,\ \lambda_i = \ln \frac{1}{1-p_i},\ i = 1,\dots,r\}.$$

Then we have the following Corollary:

Corollary 2.1. *(a) If all components of ξ have independent Poisson distribution with parameters $\lambda_1, \lambda_2, \dots, \lambda_r$, respectively, then $\mathcal{P} \subseteq \mathcal{Z}_p^P$;*

(b) If all components of ξ have independent binomial distribution with parameters $(n_1,p_1'), (n_2,p_2'), \dots, (n_r,p_r')$, respectively. Then $\mathcal{P} \subseteq \mathcal{Z}_p^B$;

(c) If all components of ξ have independent geometric distribution with parameters $p_1, p_2, \dots, p_r$, respectively. Then $\mathcal{P} \subseteq \mathcal{Z}_p^G$.

Proof. The proof can be directly derived from Theorem 24. □

Let

$$\bar{\mathcal{Z}}_p^P = \{y + 1 \in \mathbf{R}_+^r \mid \prod_{i=1}^r F(y_i;\lambda_i) \ge p, \quad \lambda_i > 0,\ i = 1,\dots,r\},$$

$$\bar{\mathcal{Z}}_p^B = \{y + 1 \in \mathbf{R}_+^r \mid \prod_{i=1}^r F(y_i;n_i,p_i') \ge p, \quad p_i' \in (0,\ 1),\ -1 < y_i < n_i,\ i = 1,\dots,r\}$$

and

$$\bar{\mathcal{Z}}_p^G = \{y \in \mathbf{R}_+^r \mid \prod_{i=1}^r G(y_i;\lambda_i) \geq p, \quad y_i > 0,\ \lambda_i = \ln\frac{1}{1-p_i},\ i = 1,\dots,r\}.$$

From Theorem 2.1, Theorem 2.3 and c.d.f. of exponential distribution is logconcave, the three above sets are all convex. From Theorem 2.4, for a multivariate random vector ξ, if all the components of ξ have independent Poisson, binomial or geometric distribution, then all the PLEP's of ξ are contained in a convex set, which is obtained from incomplete gamma function, incomplete beta function or exponential distribution function, respectively.

So for problem (7), if all components of ξ have independent Poisson, or binomial or geometric distribution, we can get the corresponding relaxed convex programming problem.

2.5 Local searching method

From the optimal solutions of the relaxed problems, we use a direct search method to find the optimal solutions of the discrete optimization problems. The method is based on Hooke and Jeeves searching method (Hook and Jeeves, 1961) and in each step, we have to check the feasibility of the new trial point. To state the method, without loss of generality, we simplify problem (7) in the following way:

$$\begin{array}{rl} \min & c^T x \\ \text{subject to} & Tx \geq y,\ t_{i,j} \in \mathbf{Z} \\ & Ax \geq b \\ & \prod_{i=1}^r F_i(y_i) \geq p \\ & x \in \mathbf{Z}^+. \end{array} \tag{25}$$

Let x be the optimal solution of problem (6) and $x^* = \lfloor x \rceil$. Let $f(x) = c^T x$ and

$$\mathcal{D} = \{x \mid \prod_{i=1}^r F_i(T_i x) \geq p,\ Ax \geq b,\ x \in \mathbf{Z}^+\}$$

where T_i is the i-th row vector of T.

The modified Hooke and Jeeves direct searching method is as follows. In each searching step, it comprises two kinds of moves: *Exploratory* and *Pattern*. Let Δx_i be the step length in each of the directions $\mathbf{e}_i$, $i = 1, 2, \dots, r$.

The Method

Exploratory move

Step 0: Set $i = 1$ and compute $F = f(x^*)$ where $x^* = \lfloor x \rceil = (x_1, x_2, \ldots, x_r)$.

Step 1: Set $x := (x_1, x_2, \ldots, x_i + \Delta x_i, \ldots, x_r)$.

Step 2: If $f(x) < F$ and $x \in \mathcal{D}$ then set $F = f(x)$, $i := i + 1$; Goto **Step 1**.

Step 3: If $f(x) \geq F$ and $x \in \mathcal{D}$ then set $x := (x_1, x_2, \ldots, x_i - 2\Delta x_i, \ldots, x_r)$. If $f(x) < F$ and $x \in \mathcal{D}$, the new trial point is retained. Set $F = f(x)$, $i := i + 1$, and goto **Step 1**.
If $f(x) \geq F$ then the move is rejected, x_i remains unchanged. Set $i := i + 1$ and goto **Step 1**.

Pattern move

Step 1: Set $x = x^B + (x^B - \bar{x}^B)$, where x^B is the point arrived by the Exploratory moves, and $\bar{x}^B$ is a point which is also arrived by exploratory move in previous step where x^B is obtained from the exploratory move starting from $\bar{x}^B$.

Step 2: Starts the Exploratory move. If for the point x obtained by the Exploratory moves $f(x) < f(x^B)$ and $x \in \mathcal{D}$, then the pattern move is recommended. Otherwise x^B is the starting point and the process restarts from x^B.

Remark When we consider the discrete random variables which have Poisson, binomial or geometric distributions, we set $\Delta x_i = 1$.

2.6 Probability maximization under constraints

Now we consider problem (1) and the following problem together:

$$\begin{array}{rl} \max & \mathbf{P}(Tx \geq \xi) \\ \text{subject to} & c^T x \leq K \\ & Ax \geq b \\ & x \geq 0, \end{array} \tag{26}$$

where ξ is a random vector and K is fixed number. In [10], the relations between problem (1) and (26) are discussed.

Suppose the components of random vector ξ are independent, then the objective function of problem (26) is $h(x) = \prod_{i=1}^{r} F_i(y_i)$, where $Tx = y$. Since $F_i(y_i) > 0$, we take natural logarithm of $h(x)$ and problem (26) can be written in the following form:

$$\begin{array}{rl} \max & \ln h(x) \\ \text{subject to} & c^T x \leq K \\ & Ax \geq b \\ & x \geq 0. \end{array} \tag{27}$$

If ξ is a Poisson random vector, problem (27) can be approximated by solving the following problem:

$$\begin{array}{rl} \max & \sum_{i=1}^{r} \ln \left(1 - \frac{1}{\Gamma(y_i+1)} \int_0^{\lambda_i} t^{y_i} e^{-t} dt\right) \\ \text{subject to} & Tx = y \\ & Ax \geq b \\ & c^T x \leq K \\ & x \geq 0. \end{array} \tag{28}$$

From Theorem 2.1, the objective function of problem (28) is concave. Let x be the optimal solution of problem (27) and $x^* = \lfloor x \rceil$. Then we apply the modified Hooke and Jeeves searching method to search the optimal solution of problem (26) around x^* as described above, and $\mathcal{D}$ is replaced by

$$\mathcal{D} = \{x \mid c^x \leq K,\ Ax \geq b,\ x \in \mathbf{Z}^+\}$$

and " $<$ " and " $\leq$ " are replaced by " $>$ " and " $\geq$ ", respectively. A numerical example in Section illustrates the details of this procedure.

For the case of independent binomial and geometric random variables, it can be discussed in a similar way.

3 Inequalities for the joint probability distribution of partial sums of independent random variables

For the proof of our main theorems in this section, we need the following

Lemma 3.1. *Let $0 \leq p,\ q \leq 1,\ q = 1 - p,\ a_0 \geq a_1,\ b_0 \geq b_1,\ \ldots,\ z_0 \geq z_1$, then we have the inequality*

$$pa_0 b_0 \cdots z_0 + qa_1 b_1 \cdots z_1 \geq (pa_0 + qa_1)(pb_0 + qb_1) \ldots (pz_0 + qz_1).$$

Proof. We prove the assertion by induction. For the case of $a_0 \geq a_1$, $b_0 \geq b_1$, the assertion is

$$pa_0 b_0 + qa_1 b_1 \geq (pa_0 + qa_1)(pb_0 + qb_1).$$

This is easily seen to be the same as

$$pq(a_0 - a_1)(b_0 - b_1) \geq 0.$$

which holds true, by assumption. Looking at the general case, we can write

$$\begin{aligned} & pa_0(b_0 \cdots z_0) + qa_1(b_1 \cdots z_1) \\ \geq\ & (pa_0 + qa_1)(pb_0 \cdots z_0 + qb_1 \cdots z_1) \\ \geq\ & (pa_0 + qa_1)(pb_0 + qb_1)(pc_0 \cdots z_0 + qc_1 \cdots z_1) \dots \\ \geq\ & (pa_0 + qa_1)(pb_0 + qb_1) \dots (pz_0 + qz_1). \end{aligned}$$

Thus the lemma is proved. □

Let $A = (a_{i,k}) \neq 0$ be an $m \times r$ matrix with $0-1$ entries and $X_1, \dots, X_r$ independent, 0 - 1 valued not necessarily identically distributed random variables. Consider the transformed random variables

$$Y_i = \sum_{i=1}^{r} a_{i,k} X_k, \quad i = 1, \dots, m. \tag{29}$$

We prove the following

Theorem 3.1. *For any nonnegative integers $y_1, \dots, y_m$ we have the inequality*

$$P(Y_1 \leq y_1, \cdots, Y_m \leq y_m) \geq \prod_{i=1}^{m} P(Y_i \leq y_i). \tag{30}$$

Proof. Let $I = \{i \mid a_{i,1} = 1\}$, $\bar{I} = \{1, \dots, m\} \setminus I$, $p_1 = P(X_1 = 0)$, $q_1 = 1 - p_1$. Then we have the relation

$$\begin{aligned} & P(Y_i \leq y_i,\ i = 1, \cdots, m) \\ =\ & P\left(\sum_{j=1}^{r} a_{i,j} X_j \leq y_i, \quad i \in I, \quad \sum_{j=2}^{r} a_{i,j} X_j \leq y_i, \quad i \in \bar{I}\right) \\ =\ & P\left(\sum_{j=2}^{r} a_{i,j} X_j \leq y_i, \quad i \in I, \quad \sum_{j=2}^{r} a_{i,j} X_j \leq y_i, \quad i \in \bar{I}\right) p_1 \\ & + P\left(\sum_{j=2}^{r} a_{i,j} X_j \leq y_i - 1, \quad i \in I, \quad \sum_{j=2}^{r} a_{i,j} X_j \leq y_i, \quad i \in \bar{I}\right) q_1. \end{aligned} \tag{31}$$

We prove the assertion by induction on r. Let $y_j = \min_{i\in I} y_i$ and look at the case $r = 1$. We have that

$$\begin{aligned}
& P(Y_i \le y_i,\ i = 1, \cdots, m) \\
= \ & P(X_1 \le y_i,\ i \in I) \\
= \ & P(X_1 \le \min_{i\in I} y_i) = P(X_1 \le y_j) \\
= \ & P(Y_j \le y_j) \\
\ge \ & \prod_{i=1}^{m} P(Y_i \le y_i),
\end{aligned}$$

then the assertion holds for the case. Assume that it holds for $r - 1$. Then, using (31) and Lemma 3.1, we can write

$$\begin{aligned}
& P(Y_i \le y_i,\ i = 1, \cdots, m) \\
\ge \ & \prod_{i\in I} P\left(\sum_{j=2}^{r} a_{i,j}X_j \le y_i\right) \prod_{i\in \bar{I}} P\left(\sum_{j=2}^{r} a_{i,j}X_j \le y_i\right) p_1 \\
& + \prod_{i\in I} P\left(\sum_{j=2}^{r} a_{i,j}X_j \le y_i - 1\right) \prod_{i\in \bar{I}} P\left(\sum_{j=2}^{r} a_{i,j}X_j \le y_i\right) q_1 \\
\ge \ & \prod_{i\in I} \left[P\left(\sum_{j=2}^{r} a_{i,j}X_j \le y_i\right) p_1 + P\left(\sum_{j=2}^{r} a_{i,j}X_j \le y_i - 1\right) q_1\right] \\
& \prod_{i\in \bar{I}} P\left(\sum_{j=2}^{r} a_{i,j}X_j \le y_i\right) \\
= \ & \prod_{i\in I} P\left(\sum_{j=1}^{r} a_{i,j}X_j \le y_i\right) \prod_{i\in \bar{I}} P\left(\sum_{j=2}^{r} a_{i,j}X_j \le y_i\right) \\
= \ & \prod_{i=1}^{m} P\left(\sum_{j=1}^{r} a_{i,j}X_j \le y_i\right) \\
= \ & \prod_{i=1}^{m} P(Y_i \le y_i).
\end{aligned}$$

This prove the theorem. □

Theorem 3.2. *Let $X_1, \ldots, X_r$ be independent, binomially distributed random variables with parameter $(n_1, p_1), \ldots, (n_r, p_r)$, respectively. Then for the random variables (29), then inequality (30) holds true.*

Proof. The assertion is an immediate consequence of Theorem 3.1. □

Note that in case of Theorem 3.2, the random variables Y_i, $i = 1, \ldots, r$ are not necessarily binomially distributed. They are, however, if $p_1 = \ldots = p_r$.

Theorem 3.3. *Let $X_1, \ldots, X_r$ be independent, Poisson distributed random variables with parameter $\lambda_1, \ldots, \lambda_r$, respectively. Then for the random variables (29), then inequality (30) holds true.*

Proof. If in Theorem 3.2, we let $n_i \to \infty$, $p_i \to \infty$ such that $n_i p_i \to \lambda_i$, $i = 1, \ldots, r$, then the assertion follows from (30). □

In case of Theorem 3.3, the random variables Y_i, $i = 1, \ldots, m$ have Poisson distribution with parameter $\sum_{i=1}^{r} a_{i,h}\lambda_j$, $i = 1, \ldots m$, respectively.

Theorem 3.3 obviously remain true if $X_1, \ldots, X_r$ are independent random variables such that the c.d.f of X_i can be obtained as a suitable limit of the c.d.f of binomial distributions, $i = 1, \ldots, r$. Since m-variate normal distribution with nonnegative correlations can be obtained this way as the joint distribution of the random variables $Y_1, \ldots, Y_m$. In this case, inequality (30) is a special case of the well-known Slepian-inequality (see Slepian, 1962).

4 Numerical examples

In [4], Dentcheva, Prékopa and Ruszczyński presented an algorithm to solve a stochastic programming problem with independent random variables. In this section, we compare the optimal values and solutions by using DPR algorithm and approximation methods, respectively, to solve two numerical examples.

4.1 A vehicle routing example

Consider the vehicle routing problem in [4], which is a stochastic programming problem with independent Poisson random variables, and the constraints have prescribed probability 0.9. We use the same notations as [4] and the problem is following:

$$\begin{array}{rl} \min_{\pi\in\Pi} & \sum_{\pi\in\Pi} c(\pi)x(\pi) \\ \text{subject to} & P\left(\sum_{\pi\in\mathcal{R}(e)} x(\pi) \geq \xi(e),\ e \in \mathcal{E}\right) \geq p \\ & x(\pi) \geq 0,\ \text{integer.} \end{array} \tag{32}$$

To approximate the optimal solution of (32), we formulated it as follows:

$$\begin{array}{rl} \min & cx \\ \text{subject to} & Tx = y \\ & \sum_{i=1}^{16} \ln\left(1 - \frac{1}{\Gamma(y_i+1)} \int_0^{\lambda_i} x^{y_i} e^{-x} dx\right) \geq \ln p \\ & x \geq 0, \text{ integer,} \end{array} \tag{33}$$

where

$$c = (10\ 15\ 18\ 15\ 32\ 32\ 57\ 57\ 60\ 60\ 63\ 63\ 61\ 61\ 75\ 75\ 62\ 62\ 44),$$

$$\lambda = (2\ 3\ 2\ 2\ 1\ 1\ 2\ 1\ 4\ 2\ 4\ 3\ 2\ 3),$$

$p = 0.9$ and

$$T = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}.$$

The optimal solution of problem (32) obtained from DPR algorithm is

$$\hat{x} = (2\ 3\ 6\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 4\ 4\ 7)^T$$

and the optimal value 977 is reached at the following 0.9-level efficient point

$$\hat{v} = (6\ 7\ 6\ 7\ 5\ 4\ 7\ 4\ 8\ 6\ 8\ 7\ 7\ 7)^T.$$

By solving problem (33), the optimal value is 972.5315, which is reached at

$$\begin{aligned} x \;=\; & (1.7869,\ 3.0314,\ 5.8495,\ 0.0000,\ 0.0000,\ 0.0000, \\ & 0.0000,\ 0.0000,\ 0.0000,\ 0.0000,\ 0.0000,\ 0.0000, \\ & 0.0000,\ 0.0000,\ 0.0000,\ 0.0000,\ 3.9970,\ 4.1492,\ 6.7917) \end{aligned}$$

Let $x^* = \lfloor x \rceil$, which is exactly $\hat{x}$. By using the modified Hooke and Jeeves searching method to search around x^*, the optimal solution is remained at x^*, i.e.,

$$x^* = (2\ 3\ 6\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 4\ 4\ 7)^T.$$

Problem (33) is solved by using MatLab 6, and the time is 5 seconds in a PIII-750 CPU computer.

Now we reconsider the vehicle routing problem, suppose we have a budget $\$1,000K$, and we want to maximize the probability of vehicle routing. Then the problem is formulated in the following way:

$$\begin{array}{rl} \max & P\left(\sum_{\pi\in\mathcal{R}(e)} x(\pi) \geq \xi(e),\ e \in \mathcal{E}\right) \\ \text{subject to} & c(\pi)x(\pi) \leq 1000,\ \pi \in \Pi \\ & x(\pi) \geq 0,\ \text{integer,} \end{array} \tag{34}$$

and we use the following formulation to approximate the optimal solution of (34):

$$\begin{array}{rl} \max & \sum_{i=1}^{16} \ln\left(1 - \frac{1}{\Gamma(y_i+1)} \int_0^{\lambda_i} x^{y_i} e^{-x} dx\right) \\ \text{subject to} & Tx = y \\ & c^T x \leq 1000 \\ & x \geq 0 \text{ integer.} \end{array} \tag{35}$$

The optimal solution to problem (35) is

$$\begin{aligned} x \;=\; & (1.8475,\ 3.1173,\ 6.0080,\ 0.0000,\ 0.0000,\ 0.0000, \\ & 0.0000,\ 0.0000,\ 0.0000,\ 0.0000,\ 0.0000,\ 0.0000, \\ & 0.0000,\ 0.0000,\ 0.0000,\ 0.0000,\ 4.1117,\ 4.2633,\ 6.9857). \end{aligned}$$

The optimal probability p at this point is 0.9188 and the cost reaches the $\$1000K$. Let

$$x^* = (2\ 3\ 6\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 4\ 4\ 7)^T.$$

We apply the modified Hooke and Jeeves searching method.

Exploratory move

Step 0: $x_0 = (2\ 3\ 6\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 4\ 4\ 7)^T$, $p = 0.9017$ and $c^T x_0 = 977$.

Step 1: $x_1 = (3\ 3\ 6\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 4\ 4\ 7)^T$, $p = 0.9049$ and $c^T x_1 = 987$.

Step 2: $x_2 = (3\ 4\ 6\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 4\ 4\ 7)^T$, $p = 0.9131$ and $c^T x_2 = 1002$, which is great than 1000. so x_2 is rejected. Then check $x_3 = (3\ 2\ 6\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 4\ 4\ 7)^T$,

$p = 0.8821$ and $c^T x_2 = 972$. Since the probability at x_3 is less than the probability at x_1, we do not accept x_3.

Pattern move

Step 1: Let $x_4 = 2x_1 - x_0$. Then $x_4 = (4\ 3\ 6\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 4\ 4\ 7)^T$.

Step 2: Start the exploratory moves from x_4. First check $x_5 = (5\ 3\ 6\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 4\ 4\ 7)^T$, which is not in $\mathcal{D}$ because $c^T x_5 = 1007 > 1000$. Then try $x_6 = x_4 = (4\ 3\ 6\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 4\ 4\ 7)^T$. At x_6, $p = 0.9057$ and $c^T x_6 = 997 < 1000$. Also the probability at x_6 is the greatest one in all the tested feasible points. Repeat the procedure from x_6, finally the procedure stops at x_6. So the optimal solution to problem (34) is

$$x = (4\ 3\ 6\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 4\ 4\ 7)^T,$$

and the optimal value is $p = 0.9057$ and the cost is \$$997K$.

4.2 A stochastic network design problem

We look at the power system model presented in Prékopa (1995, Section 14.3) and formulate an optimization problem based on the special system of four nodes.

We reproduce here the graph of the system topology as shown in Figure 2, where x_1, x_2, x_3, x_4 are the power generating capacities and $\xi_1, \xi_2, \xi_3, \xi_4$ the local demands corresponding to the nodes 1, 2, 3, 4, respectively. The values y_2, y_3, y_4 are transmission capacities. The differences $\xi_i - x_i$, $i = 1, 2, 3, 4$ are called network demands.

We want to find minimum cost optimal generating capacities subject to the condition that all demands can be satisfied simultaneously on at least p probability level together with some lower and upper bounds on the x_i's.

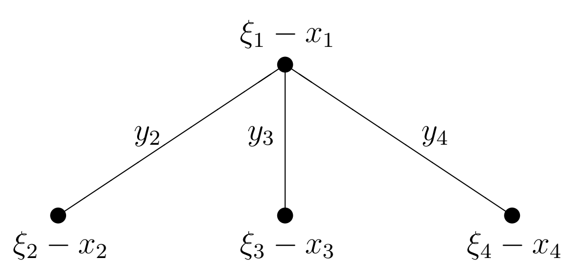

Figure 2. A power system with four nodes

A system of linear inequalities in the variables $\xi_1, \xi_2, \xi_3, \xi_4$, x_1, x_2, x_3, x_4, y_1, y_2, y_3 provides us with a necessary and sufficient condition that all demands can be satisfied, given the realized values of the random variables (see Prékopa, 1995, p.455). If we remove those which are consequences of others, then we obtain the following inequalities:

$$
\begin{array}{lcl}
\xi_1 - x_1 + \xi_2 - x_2 + \xi_3 - x_3 + \xi_4 - x_4 & \leq & 0 \\
\xi_1 - x_1 & \leq & y_2 + y_3 + y_4 \\
\xi_2 - x_2 & \leq & y_2 \\
\xi_3 - x_3 & \leq & y_3 \\
\xi_4 - x_4 & \leq & y_4 \\
\xi_1 - x_1 + \xi_2 - x_2 & \leq & y_3 + y_4 \\
\xi_1 - x_1 + \xi_3 - x_3 & \leq & y_2 + y_4 \\
\xi_1 - x_1 + \xi_4 - x_4 & \leq & y_2 + y_3 \\
\xi_1 - x_1 + \xi_2 - x_2 + \xi_3 - x_3 & \leq & y_4 \\
\xi_1 - x_1 + \xi_2 - x_2 + \xi_4 - x_4 & \leq & y_3 \\
\xi_1 - x_1 + \xi_3 - x_3 + \xi_4 - x_4 & \leq & y_2
\end{array}
\tag{36}
$$

Assume that the random variables $\xi_1, \xi_2, \xi_3, \xi_4$ are independent and ξ_i has parameters (n_i, p_i), $i = 1, 2, 3, 4$. We rewrite (36) in the following form:

$$
\begin{array}{lcl}
x_1 + x_2 + x_3 + x_4 & \geq & \xi_1 + \xi_2 + \xi_3 + \xi_4 \\
x_1 + y_2 + y_3 + y_4 & \geq & \xi_1 \\
x_2 + y_2 & \geq & \xi_2 \\
x_3 + y_3 & \geq & \xi_3 \\
x_4 + y_4 & \geq & \xi_4 \\
x_1 + x_2 + y_3 + y_4 & \geq & \xi_1 + \xi_2 \\
x_1 + x_3 + y_2 + y_4 & \geq & \xi_1 + \xi_3 \\
x_1 + x_4 + y_2 + y_3 & \geq & \xi_1 + \xi_4 \\
x_1 + x_2 + x_3 + y_4 & \geq & \xi_1 + \xi_2 + \xi_3 \\
x_1 + x_2 + x_4 + y_3 & \geq & \xi_1 + \xi_2 + \xi_4 \\
x_1 + x_3 + x_4 + y_2 & \geq & \xi_1 + \xi_3 + \xi_4
\end{array}
\tag{37}
$$

a more compact form of (37) is:

$$T_1 x + T_2 y = T z \geq \eta.$$

Our optimization problem is the following:

$$\begin{array}{rl} \min & c^T z \\ \text{subject to} & \mathbf{P}(Tz \geq \eta) \geq p \\ & z_i^{(l)} \leq z_i \leq z_i^{(u)}. \end{array} \tag{38}$$

Suppose the demands at the nodes x_1, x_2, x_3 and x_4 in Figure 2 have binomial distributions with parameters $(45, q)$, $(15, q)$, $(25, q)$ and $(30, q)$, respectively, where $p \in (0, 1)$. The unit costs for nodes are $c_1 = 15$, $c_2 = 8$, $c_3 = 12$, $c_4 = 13$, and the unit costs for links y_1, y_2, y_3 are 11, 12, 13, respectively. Let us impose the following bounds on the decision variable: $0 < x_1 \leq 45$, $0 < x_2 \leq 15$, $0 < x_3 \leq 25$ $0 < x_4 \leq 30$, $0 < y_1 \leq 10$, $0 < y_2 \leq 10$ and $0 < y_3 \leq 10$. From Theorem 3.1, instead of solving (38), we solve the following problem:

$$\begin{array}{rl} \min & 15x_1 + 8x_2 + 12x_3 + 13x_4 + 11y_1 + 12y_2 + 13y_3 \\ \text{subject to} & \ln(1 - \text{betainc}(q, x_1 + x_2 + x_3 + x_4 + 1, 115 - x_1 - x_2 - x_3 - x_4)) + \\ & \ln(1 - \text{betainc}(q, x_1 + y_1 + y_2 + y_3 + 1, 75 - x_1 - y_1 - y_2 - y_3)) + \\ & \ln(1 - \text{betainc}(q, x_2 + y_1 + 1, 25 - x_2 - y_1)) + \\ & \ln(1 - \text{betainc}(q, x_3 + y_2 + 1, 35 - x_3 - y_2)) + \\ & \ln(1 - \text{betainc}(q, x_4 + y_3 + 1, 40 - x_4 - y_3)) + \\ & \ln(1 - \text{betainc}(q, x_1 + x_2 + y_2 + y_3 + 1, 80 - x_1 - x_2 - y_2 - y_3)) + \\ & \ln(1 - \text{betainc}(q, x_1 + x_3 + y_1 + y_3 + 1, 90 - x_1 - x_3 - y_1 - y_3)) + \\ & \ln(1 - \text{betainc}(q, x_1 + x_4 + y_1 + y_2 + 1, 95 - x_1 - x_4 - y_1 - y_2)) + \\ & \ln(1 - \text{betainc}(q, x_1 + x_2 + x_3 + y_3 + 1, 95 - x_1 - x_2 - x_3 - y_3)) + \\ & \ln(1 - \text{betainc}(q, x_1 + x_2 + x_4 + y_2 + 1, 100 - x_1 - x_2 - x_4 - y_2)) + \\ & \ln(1 - \text{betainc}(q, x_1 + x_3 + x_4 + y_1 + 1, 110 - x_1 - x_3 - x_4 - y_1)) \geq \ln p \\ & 0 < x_1 \leq 45, \quad 0 < x_2 \leq 15 \\ & 0 < x_3 \leq 25, \quad 0 < x_4 \leq 30 \\ & 0 < y_1, y_2, y_3 \leq 10, \end{array}$$

where betainc is the incomplete beta function. For different p and q, the optimal solutions and costs are listed in the following table.

$(p,\ q)$	x	y	cost
(0.95, 0.80)	37.9611 14.5608 22.7719 26.9669	9.8140 10.000 10.000	1.6677e+003
(0.90, 0.80)	37.7537 14.4422 22.4671 26.6126	9.6601 9.8660 9.8145	1.6497e+003
(0.85, 0.80)	37.6570 14.3641 22.2777 26.3844	9.5544 9.7371 9.6664	1.6377e+003
(0.90, 0.75)	35.6027 13.9996 21.4929 25.3966	9.3855 9.5897 9.5288	1.5763e+003
(0.85, 0.75)	35.5101 13.9013 21.2772 25.1391	9.2583 9.4390 9.3580	1.5628e+003
(0.95, 0.70)	33.5815 13.6740 20.8184 24.5413	9.2816 9.5198 9.4804	1.5216e+003
(0.90, 0.70)	33.4251 13.4952 20.4603 24.1194	9.0659 9.2672 9.1985	1.4989e+003
(0.85, 0.70)	33.3368 13.3803 20.2229 23.8383	8.9209 9.0986 9.0090	1.4841e+003
(0.95, 0.65)	31.3732 13.1414 19.7653 23.2437	8.9485 9.1828 9.1376	1.4425e+003
(0.90, 0.65)	31.2254 12.9393 19.3791 22.7914	8.7087 8.9060 8.8309	1.4182e+003
(0.85, 0.65)	31.1419 12.8104 19.1241 22.4913	8.5491 8.7225 8.6260	1.4023e+003
(0.95, 0.60)	29.1448 12.5595 18.6638 21.8950	8.5784 8.8077 8.7573	1.3601e+003
(0.90, 0.60)	29.0063 12.3380 18.2547 21.4185	8.3187 8.5108 8.4301	1.3345e+003
(0.85, 0.60)	28.9279 12.1973 17.9856 21.1035	8.1470 8.3151 8.2128	1.3178e+003
(0.95, 0.55)	26.8978 11.9331 17.5177 20.4994	8.1745 8.3972 8.3425	1.2748e+003
(0.90, 0.55)	26.7692 11.6951 17.0908 20.0044	7.8984 8.0842 7.9992	1.2481e+003
(0.85, 0.55)	26.6962 11.5447 16.8109 19.6782	7.7171 7.8787 7.7720	1.2308e+003
(0.95, 0.50)	24.6332 11.2639 16.3290 19.0591	7.7385 7.9532 7.8951	1.1865e+003
(0.90, 0.50)	24.5150 11.0127 15.8891 18.5510	7.4495 7.6277 7.5392	1.1591e+003
(0.85, 0.50)	24.4479 10.8544 15.6015 18.2171	7.2604 7.4148 7.3047	1.1414e+003

Table 1. Computing result for different $(p,\ q)$.

5 Conclusion

We have presented efficient numerical solution techniques for probabilistic constrained stochastic programming problems with discrete random variables on the right hand sides. We have assumed that the r.h.s. random variables are partial sums of independent ones where all of them are either Poisson or binomial or geometric with arbitrary parameters. The probability that a joint constraint of one of these types is satisfied is shown to be bounded from below by the product of the probabilities of the individual constraints. The probabilistic constraint is imposed on the lower bound. Then smooth logconcave c.d.f's are fitted to the univariate discrete c.d.f's and the continuous problem is solved numerically. A search for the discrete optimal solution, around the continuous one, completes the procedure. Applications to vehicle routing and network design are presented. The former problem is taken from the literature, where a solution technique to it is also offered. Our method turns out to be significantly faster.

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