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# STRATEGIC EQUILIBRIUM VS. GLOBAL OPTIMUM FOR A PAIR OF COMPETING SERVERS

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**Abstract.** Christ and Avi-Itzhak (2002) analyze a queuing system with two competing servers who determine their service rates so as to optimize their individual utilities. The system is formulated as a two person game and the authors prove the existence of a unique Nash equilibrium which is symmetric. In this note we explore globally optimal solutions. We prove that the unique Nash equilibrium is generally strictly inferior to globally optimal solutions and that optimal solutions are symmetric and require the servers to adopt service rates that are smaller than those occurring in equilibrium. Further, given a symmetric globally optimal solution, we show how to impose linear penalties on the service rates so that the given optimal solution becomes a unique Nash equilibrium. This is accomplished by following the approach of Golany and Rothblum (2005). When service rates are not observable, we show how the same effect is achieved by imposing linear penalties on a corresponding signal.

**Key words:** Queuing systems, Nash equilibrium, linear reward schemes

# 1 Introduction

Kalai et al. [4] analyze a system in which two servers, with identical features, compete over customers that arrive to the system according to a Poisson process. Christ and Avi-Itzhak (CAI) [1] extend the analysis by incorporating the phenomenon of deserting customers. Specifically, upon arrival, each customer decides whether to join the system's common queue or to leave the system without getting service; the decision is random and the probability that a customer will join the queue is a decreasing function of the queue's length. The servers are to determine their service rates with the goal of improving their expected profits. For each server the revenue is linearly dependent on the number of customers he/she serves and the cost is a convex and increasing function of the service rate. CAI's main result is the existence of a symmetric unique Nash equilibrium in this two-person game and a characterization of this equilibrium.

While some qualitative discussion of the gap between equilibrium and optimal performance in the context of queuing systems started to emerge in the 1960s, Naor [5] was the first to propose a quantitative mechanism for the purpose of inducing a globally optimal solution in a queuing system with non-cooperative agents. For an extensive study of these issues see [3]. Recently, Golany and Rothblum (GR) [2] introduced a general framework for the use of linear rewards (and penalties) on actions of competing agents under which a given globally optimal solution satisfies first order conditions of a Nash equlibrium. The "per unit-action" rewards they use are determined by the marginal influence of agents' actions on the utilities of the other agents, evaluated at a given optimal solution.

The goal of the current note is to analyze the gap between the optimal performance of the system and its performance at the unique Nash equilibrium and to show that implementation of the GR mechanism in these settings will convert any given optimal solution into a unique Nash equilibrium. Further, it is shown that optimal performance requires reduction of service rates in comparison to the (unique) Nash equilibrium.

# 2 Analysis

We start by analyzing the gap between the equilibrium and the global optimal solutions for this game. To do so, we use the notation of CAI as follows.

**Notation**

$\lambda$ – mean arrival rate of customers.

$\mu_j$ – service rate for server $j, j = 1, 2$, satisfying $\mu_1 \geq 0$, $\mu_2 \geq 0$ and $\mu_1 + \mu_2 > 0$.

$R$ – fixed per-customer revenue.

$c(\mu)$ – cost of server $j = 1, 2$ per unit time when serving at rate $\mu$. The function $c(\mu)$ is non-negative, increasing and convex in $\mu$.

$\theta_n$ – the probability that a customer will join a queue whose length is $n$. It is assumed throughout that $\theta_0 = \theta_1 = 1, \theta_i \geq \theta_{i+1}, i = 2, 3...$ and $\lim_{i\to\infty} \theta_i = 0$.

$\alpha_j(\mu_1, \mu_2)$ – steady state proportion of the arriving customers (including those who do not join) served by server $j = 1, 2$.

Using the auxiliary variables, $\eta = \frac{\lambda}{\mu_1+\mu_2}$, $\beta_i = \prod_{n=0}^{i} \theta_n, i = 0, 1, ...$ and

$$\sigma = \sum_{i=0}^{\infty} \eta^i \beta_{i+1} \geq 1 \tag{1}$$

CAI show that the steady state proportions are given by:

$$\alpha_j(\mu_1, \mu_2) = \frac{\mu_1\mu_2 + \lambda\mu_j\sigma}{2\mu_1\mu_2 + \lambda(\mu_1 + \mu_2) + \lambda^2\sigma}, \quad j = 1, 2, \tag{2}$$

and in addition they define $\alpha_j(0, 0) = 0, j = 1, 2$.

Now, the individual expected profit function for server $j$ is:

$$\Pi_j(\mu_1, \mu_2) = R\lambda\alpha_j(\mu_1, \mu_2) - c(\mu_j), \quad j = 1, 2 \tag{3}$$

and the global profit function is:

$$\Pi(\mu_1, \mu_2) = R\lambda\alpha_1(\mu_1, \mu_2) + R\lambda\alpha_2(\mu_1, \mu_2) - c(\mu_1) - c(\mu_2). \tag{4}$$

Our analysis explores the dependence of the profit function on the selection of the service rates by the two servers with the arrival rate ($\lambda$) fixed; hence, it is important to recall that $\sigma$ depends on these rates and should be written as $\sigma(\mu_1, \mu_2)$. But, to avoid cumbersome notation, we sometimes suppress this dependence and refer simply to $\sigma$.

In order to prove the next proposition, we need the following lemma.

**Lemma 1** *If $\mu_1, \mu_2 > 0$, then $\eta > \frac{\sigma-1}{\sigma}$.*

**Proof**. Using (1), $0 < \eta, \beta_1 = 1, \beta_i \geq \beta_{i+1}\ \forall i$ and $\lim_{i\to\infty} \beta_i = 0$, we have

$$\frac{\sigma - 1}{\sigma} - \eta = \frac{(\beta_1 + \eta\beta_2 + \eta^2\beta_3 + \ldots) - 1}{\beta_1 + \eta\beta_2 + \eta^2\beta_3 + \ldots} - \eta = \eta\left[\frac{\beta_2 + \eta\beta_3 + \ldots}{\beta_1 + \eta\beta_2 + \eta^2\beta_3 + \ldots} - 1\right] \tag{5}$$

$$= \frac{\sum_{i=0}^{\infty}(\beta_{i+2} - \beta_{i+1})\eta^i}{\sigma} < 0$$

(where we emphasize that the last inequality holds strictly) . □

We say that a pair $(\mu_1, \mu_2)$ of service rates is *symmetric* if $\mu_1 = \mu_2$.

**Proposition 1** *Every globally optimal solution has symmetric service rates.*

**Proof**. Consider service rates $(\mu_1, \mu_2)$ with $\mu_1 \neq \mu_2$ and let $\mu \equiv \frac{\mu_1+\mu_2}{2}$. We will show that $\Pi(\mu, \mu) > \Pi(\mu_1, \mu_2)$, that is,

$$R\lambda[\alpha_1(\mu, \mu) + \alpha_2(\mu, \mu)] - 2c(\mu) > R\lambda[\alpha_1(\mu_1, \mu_2) + \alpha_2(\mu_1, \mu_2)] - c(\mu_1) - c(\mu_2). \tag{6}$$

First, since the cost function $c(\cdot)$ is convex, $2c(\mu) \leq c(\mu_1) + c(\mu_2)$. Also,

$$[\alpha_1(\mu, \mu) + \alpha_2(\mu, \mu)] - [\alpha_1(\mu_1, \mu_2) + \alpha_2(\mu_1, \mu_2)] \tag{7}$$

$$= \frac{2\mu^2 + 2\lambda\mu\sigma}{2\mu^2 + 2\lambda\mu + \lambda^2\sigma} - \frac{2\mu_1\mu_2 + 2\lambda\mu\sigma}{2\mu_1\mu_2 + 2\lambda\mu + \lambda^2\sigma} = \frac{2\lambda\left(\mu^2 - \mu_1\mu_2\right)(2\mu + \lambda\sigma - 2\sigma\mu)}{(2\mu^2 + 2\lambda\mu + \lambda^2\sigma)^2}\ .$$

Now, the term in the left brackets is positive since $(\mu^2 - \mu_1\mu_2) = \frac{1}{4}(\mu_1 - \mu_2)^2$ and the term in the right brackets is positive due to Lemma 1, when noting that $\sigma \geq 1$. □

Denoting $\bar{\sigma}(\mu) = \sigma(\mu, \mu)$ and its derivative $\bar{\sigma}'(\mu) = \frac{d\bar{\sigma}(\mu)}{d\mu}$, we have that

$$\frac{\partial\sigma(\mu_1, \mu_2)}{\partial\mu_1}\Big|_{(\mu_1,\mu_2)=(\mu,\mu)} = \frac{1}{2}\bar{\sigma}'(\mu) = -\sum_{i=1}^{\infty}\frac{i\lambda^i}{(2\mu)^{i+1}}\beta_{i+1} < 0. \tag{8}$$

CAI prove [1, Theorem 1] that there exists a unique Nash equilibrium solution which is symmetric. We next show that when positive, the service rates of this solution are strictly larger than the service rates of any globally optimal solution.

**Proposition 2** *Whenever the (joint) service rates at the unique Nash equilibrium are positive, they are strictly larger than the (joint) service rates of each globally optimal solution.*

**Proof**. It is proven in [1] that $A(\mu) \equiv \frac{\partial \alpha_1(\mu_1,\mu_2)}{\partial \mu_1}|_{(\mu_1,\mu_2)=(\mu,\mu)}$ is strictly decreasing in $\mu$ and that a necessary and sufficient condition for $(\mu,\mu) > (0,0)$ to be a Nash equilibrium is that it satisfies the first order conditions $R\lambda A(\mu) = c'(\mu)$. Also, a necessary (first order) condition for $(\mu,\mu) > (0,0)$ to be a globally optimal solution is that $\frac{d\Pi(\mu,\mu)}{d\mu} = 0$, that is, with $B(\mu) \equiv \frac{1}{2}\frac{d}{d\mu}[\alpha_1(\mu,\mu) + \alpha_2(\mu,\mu)]$, $R\lambda B(\mu) = c'(\mu)$. Next, observe that

$$A(\mu) = \frac{\partial \alpha_1(\mu_1,\mu_2)}{\partial \mu_1}|_{(\mu_1,\mu_2)=(\mu,\mu)} = \tag{9}$$

$$= \frac{[\mu + \lambda\bar{\sigma}(\mu) + \frac{1}{2}\lambda\mu\bar{\sigma}'(\mu)][2\mu^2 + 2\lambda\mu + \lambda^2\bar{\sigma}(\mu)] - [\mu^2 + \lambda\mu\bar{\sigma}(\mu)][2\mu + \lambda + \frac{1}{2}\lambda^2\bar{\sigma}'(\mu)]}{[2\mu^2 + 2\lambda\mu + \lambda^2\bar{\sigma}(\mu)]^2}$$

and

$$B(\mu) = \frac{1}{2}\frac{d}{d\mu}[\alpha_1(\mu,\mu) + \alpha_2(\mu,\mu)] = \frac{d}{d\mu}[\frac{\mu^2 + \lambda\mu\bar{\sigma}(\mu)}{2\mu^2 + 2\lambda\mu + \lambda^2\bar{\sigma}(\mu)}] = \tag{10}$$

$$= \frac{[2\mu + \lambda\bar{\sigma}(\mu) + \lambda\mu\bar{\sigma}'(\mu)][2\mu^2 + 2\lambda\mu + \lambda^2\bar{\sigma}(\mu)] - [\mu^2 + \lambda\mu\bar{\sigma}(\mu)][4\mu + 2\lambda + \lambda^2\bar{\sigma}'(\mu)]}{[2\mu^2 + 2\lambda\mu + \lambda^2\bar{\sigma}(\mu)]^2}.$$

Consequently,

$$B(\mu) - A(\mu) = \frac{\mu[2\mu^2 + 2\lambda\mu + \lambda^2\bar{\sigma}(\mu)] - [\mu^2 + \lambda\mu\bar{\sigma}(\mu)][2\mu + \lambda]}{[2\mu^2 + 2\lambda\mu + \lambda^2\bar{\sigma}(\mu)]^2} \tag{11}$$

$$+ \frac{\frac{1}{2}\bar{\sigma}'(\mu)\{\lambda\mu[2\mu^2 + 2\lambda\mu + \lambda^2\bar{\sigma}(\mu)] - [\mu^2 + \lambda\mu\bar{\sigma}(\mu)]\lambda^2\}}{[2\mu^2 + 2\lambda\mu + \lambda^2\bar{\sigma}(\mu)]^2}$$

$$= \frac{\lambda\mu^2[1 - 2\bar{\sigma}(\mu)] + \frac{1}{2}\bar{\sigma}'(\mu)\lambda\mu^2(2\mu + \lambda)}{[2\mu^2 + 2\lambda\mu + \lambda^2\bar{\sigma}(\mu)]^2} < 0.$$

The last inequality follows from the fact that $\beta_1 = \theta_1\theta_0 = 1$ which implies that $\sigma = \sum_{i=0}^{\infty} \eta^i \beta_{i+1} \geq \beta_1 = 1$ and from $\bar{\sigma}'(\mu) < 0$. Now, let $(\mu,\mu)$ be a globally optimal solution with $\mu > 0$ ; in particular, $R\lambda B(\mu) = c'(\mu)$. As $R\lambda A(\mu) > R\lambda B(\mu) = c'(\mu)$, as $A(\mu)$ is strictly decreasing and as $c'(\mu)$ is nondecreasing ($c(\mu)$ is convex), we have that for all $\hat{\mu} \leq \mu$, $R\lambda A(\hat{\mu}) \geq R\lambda A(\mu) > R\lambda B(\mu) = c'(\mu) \geq c'(\hat{\mu})$, assuring that the unique $\mu^*$ satisfying $R\lambda A(\mu^*) = c'(\mu^*)$ has $\mu < \mu^*$. This completes our proof (as the verification of the asserted bound when $\mu = 0$ is trite). □

A unified framework was presented in [2] for using linear rewards and penalties to convert a given globally optimal solution into a Nash equilibrium. In particular, the coefficients of the corresponding linear expressions are the marginal contributions of each agent's actions to the total welfare (expected utility) of the other agents, evaluated at the given optimal solution. Results in [6] show that first order conditions for optimality imply the Nash equilibrium first order conditions and under restrictive condition the globally optimal solution becomes a Nash equilibrium. Broader applicability of the approach was demonstrated [2] where it was further shown that, for specific models, the resulting Nash equilibrium is unique. We next demonstrate that these linear rewards (in fact, penalties) can be used to convert any given globally optimal solution of the systems considered herein into a unique Nash equilibrium.

**Proposition 3** *Let $(\hat{\mu}, \hat{\mu}) > (0, 0)$ be a globally optimal solution. Then $(\hat{\mu}, \hat{\mu})$ is a unique Nash equilibrium of the system in which the cost function $c(\mu)$ is modified by adding the linear penalty $-K\mu$ where*

$$K \equiv R\lambda[\frac{\lambda\hat{\mu}^2[1 - 2\bar{\sigma}(\hat{\mu})] + \frac{1}{2}\bar{\sigma}'(\hat{\mu})\lambda\hat{\mu}^2(2\hat{\mu} + \lambda)}{[2\hat{\mu}^2 + 2\lambda\hat{\mu} + \lambda^2\bar{\sigma}(\hat{\mu})]^2}]; \tag{12}$$

*further, $K = \frac{\partial\Pi_1(\mu_1,\mu_2)}{\mu_2}|_{(\mu_1,\mu_2)=(\hat{\mu},\hat{\mu})} < 0$.*

**Proof**. From Proposition 2 we have that the $K$ selected in (12) satisfies $K = R\lambda[B(\hat{\mu}) - A(\hat{\mu})] < 0$. Adding any linear term $L\mu$ to the cost function $c(\mu)$ preserves its convexity and at the same time does not change the evolution of the queueing process for any prescribed service rates. Consequently, [1, Theorem 1] assures the existence of a unique Nash equilibrium which is symmetric and is characterized by the (modified) first order conditions $R\lambda A(\mu) = c'(\mu) + L$.

As $(\hat{\mu}, \hat{\mu}) > (0, 0)$ is optimal, it satisfies the first order conditions $R\lambda B(\hat{\mu}) = c'(\hat{\mu})$, assuring that $R\lambda A(\hat{\mu}) = c'(\hat{\mu}) - R\lambda[B(\hat{\mu}) - A(\hat{\mu})] = c'(\hat{\mu}) - K$. So, $\hat{\mu}$ satisfies the Nash equilibrium first order conditions for the system in which linear penalty $-K\mu$ is added to the cost function $c(\mu)$, implying that $\hat{\mu}$ is the unique Nash equilibrium for this system.

To complete the proof recall that $\frac{\partial\sigma(\mu_1,\mu_2)}{\partial\mu_1}|_{(\mu_1,\mu_2)=(\mu,\mu)} = \frac{1}{2}\bar{\sigma}'(\mu)$ and observe that

$$\frac{\partial\Pi_1}{\partial\mu_2}|_{(\mu_1,\mu_2)=(\hat{\mu},\hat{\mu})} = R\lambda\frac{\partial\alpha_1}{\partial\mu_2}|_{(\mu_1,\mu_2)=(\hat{\mu},\hat{\mu})} \tag{13}$$

$$= R\lambda\{\frac{[\hat{\mu} + \frac{1}{2}\lambda\hat{\mu}\bar{\sigma}'(\hat{\mu})][2\hat{\mu}^2 + 2\lambda\hat{\mu} + \lambda^2\bar{\sigma}(\hat{\mu})] - [\hat{\mu}^2 + \lambda\hat{\mu}\bar{\sigma}(\hat{\mu})][2\hat{\mu} + \lambda + \frac{1}{2}\lambda^2\bar{\sigma}'(\hat{\mu})]}{[2\hat{\mu}^2 + 2\lambda\hat{\mu} + \lambda^2\bar{\sigma}(\hat{\mu})]^2}\}$$

$$= R\lambda\{\frac{\lambda\hat{\mu}^2(1 - 2\bar{\sigma}) + \frac{1}{2}\bar{\sigma}'(\hat{\mu})\lambda\hat{\mu}^2(2\hat{\mu} + \lambda)}{[2\hat{\mu}^2 + 2\lambda\hat{\mu} + \lambda^2\bar{\sigma}(\hat{\mu})]^2}\}. \quad \square$$

An important issue in imposing linear rewards/penalties is the observability of the agents' actions. When actions are observed only by the agents who select them, it is still possible to achieve the desired effect by imposing linear rewards/penalties on observable random signals whose expected values equal, respectively, the agents' actions. In our queueing system, the agents' actions are the service rates which may, or may not, be observed by an external observer. The next lemma provides an explicit expression for a (random) signal that has the above characteristics; it can be determined from an accounting of the service completions and from full information of the busy/idle status of the server.

**Lemma 2** *For any period in which the server operates at service rate $\mu$, the expectation of the ratio of the number of service completions to the amount of time the server was busy equals $\mu$.*

**Proof**. When observing the server only while busy, the number of service completions becomes a Poisson process with parameter $\mu$. In particular, when the server is busy during $t$ time units, the expected number of service completions is $t\mu$. □

# 3 Summary

When each server determines its service rate so as to maximize its individual profit, both end up selecting values that are larger than those that will bring the performance of the system as a whole to a global optimum. This is a consequence of the convexity of the cost functions. When both servers implement the reduced service rates prescribed by the globally optimal solution, the system loses some revenue as the queue length increases and some customers leave. But, this loss is more than compensated by the savings in the cost of maintaining the smaller service rates. Since both servers are locked in an equilibrium solution, none of them will make the first move out of it. By introducing linear penalty simultaneously on both servers, we induce them to switch from the equilibrium service rates to a new equilibrium point, which corresponds to global optimal performance. In reality, these penalties can be interpreted as a tax which the servers pay to some authority for the right to operate within the system.

# References

[1] Christ, D., B. Avi-Itzhak , 2002. Strategic equilibrium for a pair of competing servers with convex cost and balking, *Management Science*, 48, 813-820.

[2] Golany, B. and U.G. Rothblum, 2005. Inducing coordination in supply chains through linear reward schemes, *Naval Research Logistics*, forthcoming.

[3] Hassin, R. and M. Haviv, 2003. *To queue or not to queue: equilibrium behavior in queuing systems*. Kluwer Academic Publishers.

[4] Kalai, E., M.I. Kamien and M. Rubinovitz, 1992. Optimal service speeds in a competitive environment. *Management Science*, 38, 1154-1163.

[5] Naor, P. 1969. the regulation of queue size by levying tolls. *Econometrica*, 37, 15-24.

[6] Rothblum,U.G., 2005. "Optimality vs. equilibrium: inducing stability by linear rewards and penalties," submitted for publication.