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**Abstract.** Method of augmenting graphs is a general approach to solve the maximum independent set problem. As the problem is generally NP-hard, no polynomial-time algorithms are available to implement the method. However, when restricted to particular classes of graphs, the approach may lead to efficient solutions. A famous example of this nature is the matching algorithm: it finds a maximum matching in a general graph $G$, which is equivalent to finding a maximum independent set in the line graph of $G$. In the particular case of line graphs, the method reduces to finding augmenting (alternating) chains. Recent investigations of more general classes of graphs revealed many more types of augmenting graphs. In the present paper we study the problem of finding augmenting graphs different from chains. To simplify this problem, we introduce the notion of a redundant set. This allows us to reduce the problem to finding some basic augmenting graphs. As a result, we obtain a polynomial-time solution to the maximum independent set problem in a class of graphs which extends several previously studied classes including the line graphs.

**Keywords:** independent set; augmenting graph; polynomial-time algorithm

# 1 Introduction

All graphs in this paper are undirected, without loops and multiple edges. The vertex set of a graph $G$ is denoted $V(G)$ and its edge set $E(G)$. We say that a graph $G$ is $H$-free if $G$ does not contain $H$ as an induced subgraph. As usual, $C_n$ stands for a chordless cycle and $P_n$ for a chordless path (chain) on $n$ vertices. Also, $S_{i,j,k}$ is the graph represented in Figure 1. In particular, $S_{1,1,1}$ is a claw.

In a graph, an independent set is a subset of vertices no two of which are adjacent. The maximum independent set problem is that of finding in a given graph an independent set of maximum cardinality. This problem is generally NP-hard. However, when restricted to some particular classes, it can be solved in polynomial time. A remarkable example of this type is the class of line graphs, in which case the maximum independent set problem is equivalent to that of finding maximum matchings in general graphs. The solution to this problem is based on Berge's idea of augmenting (alternating) chains [4] and the celebrated Edmonds' algorithm [7] that finds augmenting chains. Lovász and Plummer observed in [13] that Edmonds' solution is "among the most involved of combinatorial algorithms".

Rephrasing Berge's idea in terms of independent sets, we can say that in a line graph an independent set is maximum if and only if there are no augmenting chains with respect to this set. As the example of Edmonds' algorithm shows, finding augmenting chains is not a trivial task. In 1980, independently Minty [15] and Sbihi [17] extended the solution of Edmonds to claw-free graphs, a class properly containing the line graphs. In conjunction with the fact that in the class of claw-free graphs augmenting chains constitute the only type of augmenting graphs this has led to a polynomial-time solution to the maximum independent set problem in that class. Recently, the problem of finding augmenting chains has been solved for some extensions of claw-free graphs [9, 12].

In general, augmenting chains are not the only type of augmenting graphs. For instance, Mosca showed in [16] that in the class of $(P_6, C_4)$-free graphs every augmenting graph is a simple augmenting tree (the graph $T_1$ represented in Figure 2). Many more types of augmenting graphs have been revealed in [2, 3, 5, 8]. With each of them, one can associate the problem of finding augmenting graphs of the type. The number of various types of augmenting graphs is generally growing with each extension of the class under review. In order to simplify the problem of finding augmenting graphs, we introduce in this paper the notion of a redundant set of vertices, which allows us to reduce this problem to augmenting graphs of some "basic" types. As a result, we develop a polynomial-time solution to the maximum independent set problem in a class of graphs which strictly extends claw-free, $(P_6, C_4)$-free and some other previously studied classes. Of our special concern are the classes of $S_{i,j,k}$-free graphs, which is motivated by the following result proved in [1].

**Theorem 1** *Let $X$ be a class of graphs defined by a finite set $M$ of forbidden induced subgraphs. If $M$ contains no graph every connected component of which is of the form $S_{i,j,k}$, then the maximum independent set problem is NP-complete in the class $X$.*

The organization of the paper is as follows. In the rest of this section we introduce some more terminology and notations. In Section 2 we present the notion of an augmenting graph

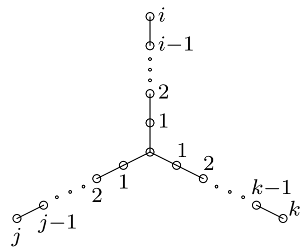


Figure 1: Graph $S_{i,j,k}$

and prove a few preliminary results. Section 3 formalizes the problem of finding augmenting graphs, introduces the notion of a redundant set of vertices and proves the main result related to this notion. In Section 4, we apply this result to graphs in a particular class that extends several previously studied classes.

In a graph $G$, the neighborhood of a vertex $v \in V(G)$ (i.e., the set of vertices of $G$ adjacent to $v$) will be denoted $N(v)$. The degree of $v$, denoted $deg(v)$, is $|N(v)|$. For a subset $U \subset V(G)$, we shall denote by $N(U)$ the neighborhood of $U$, i.e., the set of vertices of $G$ outside $U$ that have at least one neighbor in $U$. Also, $N_U(v) := N(v) \cap U$, and if $W$ is another subset of $V(G)$ then $N_W(U) := N(U) \cap W$. By $G - U$ we shall denote the graph obtained from $G$ by deleting vertices of $U$, and $G[U]$ is the subgraph of $G$ induced by $U$, i.e., $G[U] := G - (V(G) \backslash U)$.

The *duplication* of a vertex $v \in V(G)$ is the operation of addition of a new vertex $v'$ to $G$ with $N(v') = N(v)$. The graph obtained from a $P_4$ by duplicating one of its middle vertices will be called a *banner*. Notice that a banner contains a claw as an induced subgraph. Therefore, banner-free graphs constitute a generalization of claw-free graphs. However, unlike claw-free graph, the class of banner-free graphs is difficult with respect to the independent set problem, which is an immediate corollary of Theorem 1.

# 2 Augmenting Graphs

Let $G$ be a graph and $S$ an independent set in $G$. We shall call the vertices of $S$ *white* and the remaining vertices of $G$ *black*.

**Definition 1** *An* augmenting graph for $S$ in $G$ *is an induced bipartite subgraph* $H = (W, B, E)$ *of* $G$*, where* $W \cup B$ *is a bipartition of its vertex set and* $E$ *its edge set, such that:*

- $|B| > |W|$,
- $W \subseteq S$,
- $B \subseteq V(G) \backslash S$, *and*

- $N(B) \cap S \subseteq W$.

If a bipartite subgraph $H$ of $G$ is augmenting for $S$, we also say that $S$ *admits* the augmenting graph. Clearly if $H = (W, B, E)$ is an augmenting graph for $S$, then $S$ is not a maximum independent set in $G$, since the set $S' = (S - W) \cup B$ is independent and $|S'| > |S|$. We shall say that the set $S'$ is obtained from $S$ by $H$-augmentation.

Conversely, if $S$ is not a maximum independent set, and $S'$ is an independent set such that $|S'| > |S|$, then the subgraph of $G$ induced by the set $(S - S') \cup (S' - S)$ is augmenting for $S$. Therefore, the following key result holds.

**Theorem of augmenting graphs.** *An independent set $S$ in a graph $G$ is maximum if and only if there are no augmenting graphs for $S$.*

This theorem suggests the following general approach to find a maximum independent set in a graph $G$: begin with any independent set $S$ in $G$ and, as long as $S$ admits an augmenting graph $H$, apply $H$-augmentation to $S$. This approach has proven to be a useful tool to develop approximate solutions to the problem [11], to compute bounds on the independence number [6], and to solve the problem in polynomial time for graphs in special classes [15, 2]. In the present paper we focus on efficient implementations of the approach for graphs in particular classes. To this end, let us introduce some more definitions.

**Definition 2** *A bipartite graph $H = (W, B, E)$ will be called* augmenting *if there is a graph $G$ and an independent set $S$ in $G$ such that $H$ is augmenting for $S$ in $G$.*

Clearly not every bipartite graph is augmenting. For instance, a bipartite cycle cannot be augmenting, since it has equally many vertices in both parts. Moreover, without loss of generality we may exclude from our consideration those augmenting graphs which are not minimal.

**Definition 3** *An augmenting graph $H$ for a set $S$ is called* minimal *if no proper induced subgraph of $H$ is augmenting for $S$.*

Some bipartite graphs that could be augmenting are never minimal augmenting. To give an example, consider the claw $K_{1,3}$. If it is augmenting for an independent set $S$, then its subgraph obtained by deleting any vertex of degree 1 also is augmenting for $S$. The following lemma characterizes minimal augmenting graphs.

**Lemma 1** *An augmenting graph $H = (W, B, E)$ is minimal if and only if*

*(i)* $|W| = |B| - 1$;

*(ii) for every nonempty subset $A \subseteq W$, $|A| < |N(A)|$.*

*(iii) $H$ is connected.*

**Proof.** Let $H = (W, B, E)$ be a minimal augmenting graph. Condition $(i)$ is obvious. To show $(ii)$, assume $|A| \geq |N(A)|$ for some nonempty subset $A$ of $W$. Then the vertices in $(W \backslash A) \cup (B \backslash N(A))$ induce a proper subgraph of $H$ which is augmenting too. Condition $(iii)$ follows from $(i)$ and $(ii)$.

Conversely, let $H = (W, B, E)$ be an augmenting graph for an independent set $S$ of a graph $G$, satisfying $(i) - (iii)$. Assume that $H' = (W', B', E')$ is a proper induced subgraph of $H$ which also is augmenting for $S$. Then $W'$ is a proper subset of $W$ (since otherwise $|B'| \geq |W| + 1 = |B|$ and $B' = B$, implying $H' = H$). Therefore, the set $A := W \backslash W'$ is nonempty. Moreover, since $H'$ is augmenting, it follows that $N(A) \subseteq B \backslash B'$, which in its turn implies that $|N(A)| \leq |A|$, contradicting $(ii)$. ■

For a polynomial-time implementation of the augmenting graph approach in a class of graphs $X$, one has to

(a) find a complete list of (minimal) augmenting graphs in $X$,

(b) develop a polynomial-time procedure for detecting augmenting graphs from the list.

For instance, for the class of claw-free graphs, question (a) has a simple answer. Indeed, by definition, augmenting graphs are bipartite, and each vertex in a claw-free bipartite graph clearly has degree at most two. Therefore, every connected claw-free bipartite graph is either an even cycle or a chain. Cycles of even length and chains of odd length are not augmenting. Thus, every connected claw-free augmenting graph is a chain of even length. Extension of the class under consideration leads to more complicated structure of augmenting graphs. For instance, it has been shown in [3] that in the class of $(S_{1,2,3}, banner)$-free graphs (a proper extension of claw-free graphs) a minimal augmenting graph is either

- a chain of even length or
- a complete bipartite graph or
- a simple augmenting tree (graph $T_1$ in Fig. 2) or
- an augmenting plant (graph $T_3$ with $r = 0$ and $s = 1$ in Fig. 2).

Further extension to the class of $(S_{1,2,4}, banner)$-free graphs adds only finitely many new minimal augmenting graphs to this list [8]. From the theoretical point of view, any finite collection of augmenting graphs can be neglected. Moreover, we do not even need any description of such a collection. As an example exploiting this observation we prove Theorem 2 below. A corollary of this theorem will be used in Section 4, where we study the maximum independent set problem restricted to a particular class of graphs.

Let us call a *strip* any finite graph obtained from a path by repeatedly performing the duplication of vertices, and a *bracelet* any finite graph obtained in the same manner from a cycle.

**Theorem 2** *For any positive integers $n$ and $d$, there are only finitely many $S_{1,2,n}$-free connected bipartite graphs of maximum degree at most $d$ different from strips and bracelets.*

**Proof.** Let $l = (d+1)(n+2)$. There are only finitely many connected graphs of vertex degree at most $d$ which are $P_l$-free. Therefore, we assume that a connected bipartite graph $G$ of degree at most $d$ contains a longest induced path $P = (v_1, \ldots, v_r)$ with $r \geq l$.

If $G = P$, then $G$ is a strip. If $G$ is different from $P$, it must contain a vertex $v$ outside $P$, which has a neighbor on $P$.

First, suppose that $v$ has at least three neighbors on $P$. Since the degree of $v$ is at most $d$, the neighbors of $v$ divide $P$ into at most $d+1$ edge-disjoint paths, at least one of which has many edges. Then an induced $S_{1,2,n}$ can be easily found.

Second, assume that $v$ has two neighbors on $P$, say $v_i$, $v_j$. Then either $|i-j| = 2$ or $i = 1$ and $j = r$, since otherwise (similarly as above) an induced $S_{1,2,n}$ arises.

Third, suppose that $v$ has exactly one neighbor $v_i$ on $P$. Then either $i = 2$ or $i = r-1$, since otherwise either $P$ is not a longest path or $G$ contains an induced $S_{1,2,n}$.

The above discussion allows us to conclude that every vertex of $G$ outside $P$ has a neighbor on $P$, since otherwise one can find an induced $S_{1,2,n}$ in $G$. To complete the proof, we distinguish between the two following cases.

Consider first the case when a vertex $v_0 \notin V(P)$ is adjacent to $v_1$ and $v_r$, i.e., $P$ together with $v_0$ induce a cycle $C$. From the above we know that every vertex $w$ of $G$ outside $C$ has at least one and at most two neighbors on $C$. Clearly, $w$ cannot have exactly one neighbor on $C$, since otherwise $G$ contains an induced $S_{1,2,n}$. Therefore, $w$ has exactly two neighbors on $C$, and moreover, these two neighbors are of distance 2. In other words, $w$ is a duplication of a certain vertex $v_i$ on $C$. Replacing $v_i$ with $w$ we obtain another cycle $C'$, and all the above arguments can be applied with respect to $C'$. It is now not difficult to see that $G$ is a bracelet.

Finally, we analyze the case when every vertex $v$ of $G$ outside $P$ is adjacent either to $v_2$ or to $v_{r-1}$ or to two vertices of distance 2 in $P$. In other words, $v$ is a duplicate of a vertex of $P$, and therefore $G$ is a strip. ■

# 3 The Problem of Finding Augmenting Graphs

In its most general form, the problem of finding augmenting graphs can be formulated as follows:

AUGMENTATION
**Instance:** A graph $G$, and a maximal independent set $S$ in $G$.
**Problem:** Find an augmenting graph for $S$ whenever $S$ admits one.

From NP-hardness of the independent set problem and the Theorem of augmenting graphs we conclude that

**Claim 1** *The problem* AUGMENTATION *is NP-hard.*

Since in its whole generality the problem is intractable, we introduce a hierarchy of sub-problems and study the computational complexity of the problems in this hierarchy. For a class $\mathcal{A}$ of augmenting graphs, let us consider the following problem:

AUGMENTATION($\mathcal{A}$)
**Instance:** A graph $G$, and a maximal independent set $S$ in $G$.
**Problem:** Find an augmenting graph for $S$ whenever $S$ admits an augmenting graph from $\mathcal{A}$.

Note that we don't require the output graph to belong to $\mathcal{A}$. If $\mathcal{A}$ is the class of all augmenting graphs, then the problem AUGMENTATION($\mathcal{A}$) coincides with the problem AUGMENTATION and hence is intractable. However, it becomes polynomial-time solvable, for instance, if $\mathcal{A}$ contains only finitely many graphs. Between these two extremes there are infinitely many intermediate classes of augmenting graphs and respective problems.

The following notion is a helpful tool for establishing reducibility among some of these problems.

**Definition 4** *In an augmenting graph $H = (W, B, E)$ a subset of vertices $U$ will be called* redundant *if*

- $|U \cap W| = |U \cap B|$,
- *$H$ contains no edges between black vertices of $U$ and vertices of $H - U$.*

**Theorem 3** *Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be two classes of augmenting graphs. If there is a constant $k$ such that for every graph $H = (W, B, E) \in \mathcal{A}_2$ there is a redundant subset $U$ of size at most $k$ such that $H - U \in \mathcal{A}_1$, then the problem* AUGMENTATION*($\mathcal{A}_2$) is polynomially reducible to the problem* AUGMENTATION*($\mathcal{A}_1$).*

**Proof.** Let *Augment*$_1(G, S)$ be a procedure that solves the problem AUGMENTATION($\mathcal{A}_1$) for a graph $G$ and an independent set $S$. We assume that the procedure outputs a subset $V'$ of $V(G)$ such that $G[V']$ is augmenting for $S$ whenever $S$ admits an augmenting graph from $\mathcal{A}_1$ (and perhaps even if this is not the case). If no augmenting graph is found, then $V' = \emptyset$.

To prove the theorem we present procedure *Augment*$_2(G, S)$ that solves the problem AUGMENTATION($\mathcal{A}_2$):

**Procedure** *Augment*$_2(G, S)$

**Input:** A graph $G$ and an independent set $S$ in $G$.
**Output:** A subset $V'$ of $V(G)$ such that $G[V']$ is augmenting for $S$ whenever $S$ admits an augmenting graph from $\mathcal{A}_2$. If no augmenting graph is found, then $V' = \emptyset$.

**begin**
 **for all** ($U \subseteq V(G)$ of size at most $k$ such that
  $B_0 := U \cap (V(G) \backslash S)$ is an independent set in $G$,
  $|B_0| = |U \cap S|$ and $N_G(B_0) \cap (S \backslash U) = \emptyset$)
 **do**

[*remove the neighbors of* $B_0$ *in* $V(G)\backslash S$]
let $G' = G - N_G(B_0) \cap (V(G)\backslash S)$;
[*try to solve the problem* AUGMENTATION$(\mathcal{A}_1)$]
let $T = Augment_1(G' - U,\ S\backslash U)$;
**if** $(T \neq \emptyset)$ [*we have an augmenting graph for* $S$]
**then** return $U \cup T$;
return $\emptyset$;
**end;**

Suppose $S$ admits an augmenting graph $H = (W, B, E) \in \mathcal{A}_2$. Then, according to the theorem's assumption, $H$ contains a redundant set $U$ of size at most $k$ such that $H - U \in \mathcal{A}_1$. It is not difficult to see that the graph $H - U$ is augmenting for $S\backslash U$ in $G' - U$. Therefore, procedure $Augment_1$ must output a nonempty set $T$. Consequently, procedure $Augment_2$ also outputs a nonempty set $U \cup T$. Obviously, $G[U \cup T]$ is a bipartite graph. Moreover, since $U$ is a redundant set, the graph $G[U \cup T]$ is augmenting for $S$ even if $G[T]$ does not coincide with $H - U$. Therefore, whenever $S$ admits an augmenting graph from $\mathcal{A}_2$, procedure $Augment_2$ finds an augmenting graph. To this end, it inspects polynomially many subsets of vertices of the input graph, which results in polynomially many calls of the procedure $Augment_1$. Therefore, the problem AUGMENTATION$(\mathcal{A}_2)$ is polynomially reducible to the problem AUGMENTATION$(\mathcal{A}_1)$. ■

*Remark.* Note that if in the definition of a redundant set we drop the second condition, then the procedure in the above theorem may fail to work: it may happen that even though $T$ from the procedure induces an augmenting graph for $S\backslash U$ in $G' - U$, the graph induced by $T \cup U$ may not be augmenting for $S$. In particular, if $S'$ denotes the set of white neighbors of black vertices of $U$ in the graph $G' - U$ and if $T$ is augmenting for $S\backslash U$ in $G' - U$, then $T \cup U$ is augmenting for $S$ if and only if $S' \subseteq V(T)$.

In the following section, we use the above results in order to develop a polynomial-time solution to the maximum independent set problem in a certain class of graphs which extends several previously studied cases.

# 4 Application to $(S_{1,2,5}, banner)$-free Graphs

We study the maximum independent set problem in the class of $(S_{1,2,5}, banner)$-free graphs. This class generalizes both $(S_{1,2,4}, banner)$-free and $(P_8, banner)$-free graphs studied in [8], as well as $(S_{1,2,3}, banner)$-free, $(P_7, banner)$-free graphs, $(P_6, C_4)$-free, $(P_5, banner)$-free and claw-free graphs studied earlier in [3, 16, 14, 15]. With the help of Theorems 2,3 and some previously obtained results we prove that the maximum independent set problem in the class of $(S_{1,2,5}, banner)$-free graphs can be solved in polynomial time. The following simple lemma can be found in [3].

**Lemma 2** *A connected bipartite banner-free graph containing a* $C_4$ *is complete bipartite.*

According to this lemma, the problem of finding augmenting graphs in the class under consideration splits into two subproblems:

(A) finding $(S_{1,2,5}, C_4)$-free augmenting graphs;

(B) finding complete bipartite augmenting graphs.

A solution to problem (B) in the class of *banner*-free graphs has been proposed in [3]. In the rest of this section we analyze problem (A). To this end, we further decompose it into two subproblems:

(A.1) finding $(S_{1,2,5}, C_4)$-free augmenting graphs of bounded vertex degree;

(A.2) finding $(S_{1,2,5}, C_4)$-free augmenting graphs containing a vertex of high degree.

From Theorem 2 we derive the following conclusion.

**Corollary 1** *In the class of $(S_{1,2,5}, C_4)$-free graphs there are finitely many minimal augmenting graphs of bounded vertex degree different from chains.*

**Proof.** Let $H$ be an $(S_{1,2,5}, C_4)$-free minimal augmenting graph. According to Theorem 2, we can assume without loss of generality that $H$ is either a strip or a bracelet. Notice that a cycle cannot be an augmenting graph and the duplication of any vertex of a cycle leads to an induced $C_4$. Therefore, $H$ is a strip. We assume that $H$ is obtained from a path $P$ by duplicating some (possibly no) vertices of $P$. As before, no vertex of degree 2 on $P$ can be duplicated, since otherwise a $C_4$ would arise. And if an end point of $P$ was duplicated, then $H$ is not a minimal augmenting graph. Therefore, $H = P$ is an augmenting chain. ■

Finding augmenting chains in $(S_{1,2,j}, banner)$-free graphs is a polynomially solvable task for any fixed $j$ [12]. Therefore, we proceed to subproblem (A.2). First of all, let us show that without loss of generality we may restrict ourselves to augmenting graphs containing a *black* vertex of high degree.

**Lemma 3** *If a minimal augmenting $(S_{1,2,5}, C_4)$-free graph $H$ contains no black vertex of degree more than $k$, then the degree of each white vertex is at most $2k + 1$.*

**Proof.** Assume that $H$ contains a white vertex $a$ of degree more than $2k + 1$. Denote by $A_j$ the set of vertices of $H$ of distance $j$ from $a$. Since $H$ is minimal, at most one vertex of $A_1$ has no neighbors in $A_2$, and because of $C_4$-freeness, every vertex of $A_2$ has exactly one neighbor in $A_1$. Therefore $|A_2| \geq 2k + 1$. Again by the minimality of $H$, every vertex of $A_2$ has a neighbor in $A_3$. If $H$ contains no black vertex of degree more than $k$, then $|A_3| \geq 3$.

Suppose $A_4$ contains a (white) vertex $x$ and let $y$ be its neighbor in $A_3$. Due to the minimality of $H$, $x$ has at least one more black neighbor, say $z$. If $deg(y), deg(z) \leq k$, then $A_2$ contains a vertex non-adjacent both to $y$ and $z$, and hence there is an induced $S_{1,2,5}$ in $H$. Therefore, $A_4$ is empty.

Let $x \in A_3$. By Lemma 1 and Hall's theorem [10] we know that the subgraph $H - x$ has a perfect matching $M$. For a subset $U \subset V(H - x)$ of vertices of the same color, we denote by $m(U)$ the set of vertices of the opposite color matched with vertices of $U$ with respect to $M$. Denote $A := A_1$, $B := m(A)$, $C := A_2 - (B \cup \{a\})$, $D := m(C)$. If $deg(x) \leq k$, $B$ contains at least $k + 1$ vertices each of which has a neighbor in $D$.

Consider a vertex $d_1 \in D$ such that $m(a_1) \neq a$, where $a_1$ is the only neighbor of $m(d_1)$ in $A$. If $deg(d_1) \leq k$, there is a vertex $b \in B$ such that $b$ is not adjacent to $d_1$, $b$ has a neighbor $d_2$ in $D$, and $m(b)$ is not adjacent to $m(d_1)$.

Assume now that there is an edge $m(d_2)d_1$. Then obviously $m(d_1)$ is not adjacent to $d_2$. Since $|A| = |B| > 2k + 1$, vertices $d_1, d_2$ have at most $2k - 2$ neighbors in $B$, while vertices $m(d_1), m(d_2)$ have at most 2 neighbors in $A$, there is a couple of vertices $a_2, a_3 \in A$ such that $m(a_2) \neq a$, $m(a_2)$ is non-adjacent to $d_1, d_2$ and there are no edges between $a_2, a_3$ and $m(d_1), m(d_2)$. But now the vertices $d_2, m(d_2), d_1, m(d_1), a_1, a, a_2, m(a_2), a_3$ induce an $S_{1,2,5}$ in $H$. This contradiction shows that $m(d_2)$ cannot be adjacent to $d_1$.

An analogous argument shows that $m(d_1)$ is not adjacent to $d_2$. But now, the vertices $m(d_2), d_2, b, m(b), a, a_1, m(d_1), d_1, m(a_1)$ induce an $S_{1,2,5}$. This contradiction completes the proof of the lemma. ■

The above lemma permits us to restrict ourselves to augmenting graphs containing a black vertex $x$ of "sufficiently large" degree $k$. Figure 2 represents all "basic" families of augmenting graphs of this type. The meaning of the word "basic" in the above sentence is specified in the following theorem.

**Theorem 4** *Let $G$ be an $(S_{1,2,5}, banner)$-free graph. The problem of finding in $G$ a minimal augmenting $(S_{1,2,5}, C_4)$-free graph with a black vertex of degree at least 6 can be reduced in polynomial time to the problem of finding in $G$ one of the augmenting graphs $T_1, \ldots, T_6$ represented in Figure 2.*

## 4.1 Proof of Theorem 4

Throughout the section we shall denote by $H$ a minimal augmenting $(S_{1,2,5}, C_4)$-free graph with a black vertex $x$ of degree $k \geq 6$, by $A = \{a_1, \ldots, a_k\}$ the neighborhood of $x$ and by $C$ the remaining white vertices of $H$, i.e., those that are not in $A$. According to Lemma 1 and Hall's theorem [10], the subgraph $H - x$ has a perfect matching. For a subset of vertices $U \subseteq V(H - x)$ of the same color, we shall denote by $m(U)$ the set of vertices of the opposite color matched with the vertices in $U$. In particular, $B := m(A)$, and $D := m(C)$. Also, let $C_1$ denote the subset of vertices of $C$ that are adjacent to $p$ vertices of $B$, for some $0 < p < k$, and $C_0$ the subset of vertices of $C$ that have no neighbors in $B$. Finally, $D_0 := m(C_0)$ and $D_1 := m(C_1)$. Since $H$ is $C_4$-free, we know that

(0) $C - (C_0 \cup C_1)$ contains at most one vertex, and any vertex of $D$ is adjacent to at most one vertex of $A$.

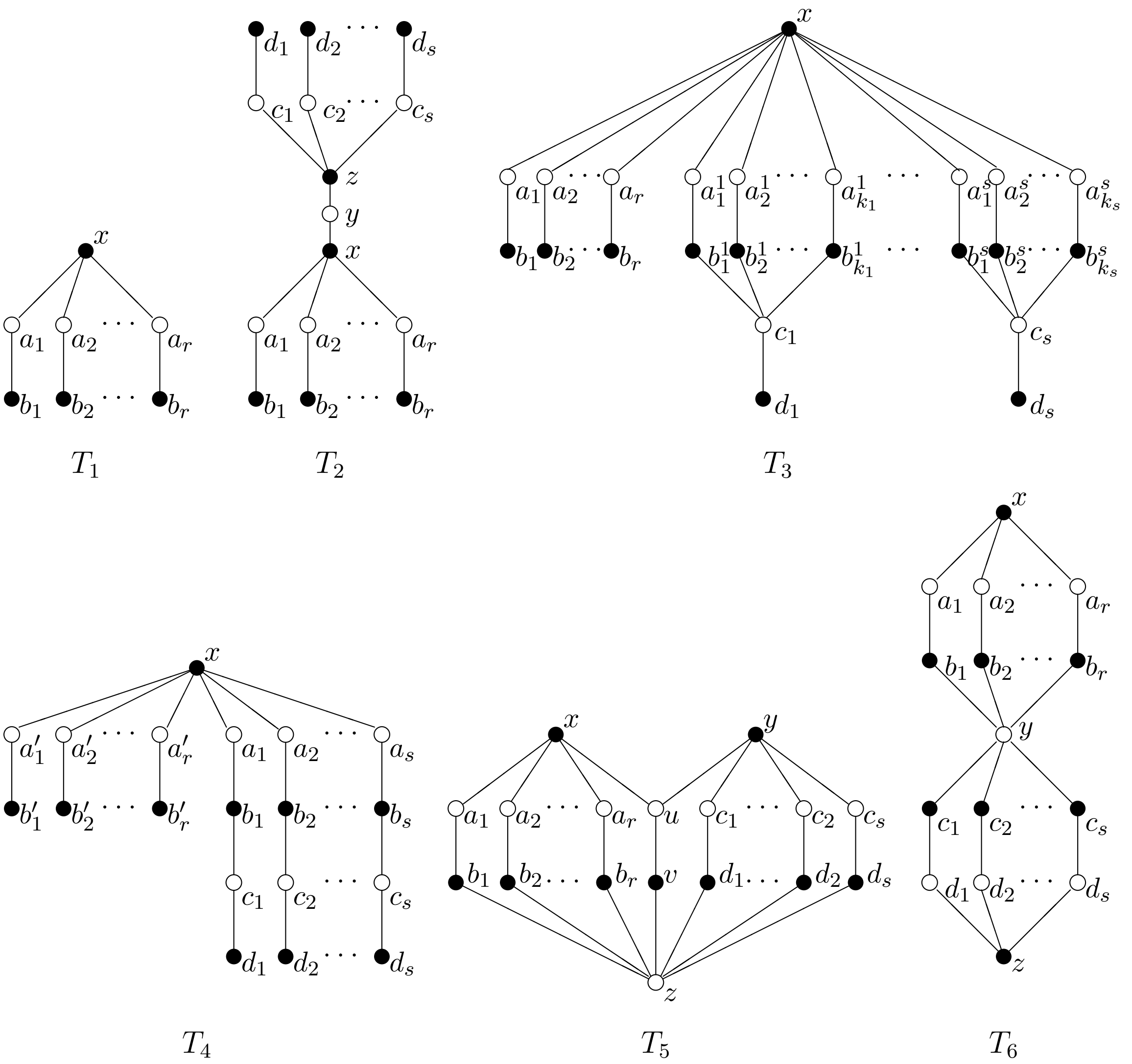


Figure 2: "Basic" families of augmenting $(S_{1,2,5}, banner)$-free graphs

**Lemma 4** *If $H$ contains a vertex $y \in C$ which is adjacent to every vertex of $B$, then either $H = T_5$ or $H = T_6$ or $H$ contains a redundant set $U$ of size at most 10 such that either $H - U = T_1$ or $H - U = T_4$.*

**Proof.** Assume first that $C_1 \neq \emptyset$. Since $H$ is $C_4$-free we know that

(1) $y$ has no neighbors in $D_1$, and $m(y)$ has no neighbors in $A \cup C_1$;

(2) every vertex of $C_1$ has exactly one neighbor in $B$.

Also, from $S_{1,2,5}$-freeness of $H$ we can derive that

(3) Any two vertices of $C_1$ have different neighbors in $B$. Indeed, $c_1, c_2 \in C_1$ both are adjacent to $b \in B$, then for any $a_1, a_2 \in A$ different from $m(b)$ and non-adjacent to $m(c_1)$, the subgraph induced by vertices $a_1, x, a_2, m(a_2), y, b, c_1, m(c_1), c_2$ is isomorphic to $S_{1,2,5}$.

(4) $H[C_1 \cup D_1]$ is an induced matching. Indeed, if for a couple of vertices $c_1, c_2 \in C_1$ there is an edge $c_1m(c_2)$, then for the neighbor $b_1 \in B$ of $c_1$ and a vertex $b_2 \in B$ such that $b_2$ is non-adjacent to $c_1, c_2$ and $m(b_2)$ is non-adjacent to $m(c_1), m(c_2)$, the subgraph induced by vertices $b_2, m(b_2), x, m(b_1), b_1, c_1, m(c_2), m(c_1)$ is isomorphic to $S_{1,2,5}$.

(5) No vertex of $C_0$ has a neighbor in $D_1$. Indeed, if $c \in C_0$ is adjacent to $d \in D_1$, then for the vertex $b \in B$ adjacent to $m(d)$ and any two vertices $a_1, a_2 \in A$ different from $m(b)$ and non-adjacent to $d$, the subgraph induced by vertices $c, d, m(d), b, m(b), x, a_1, m(a_1), a_2$ is isomorphic to $S_{1,2,5}$.

(6) No vertex of $C_1$ has a neighbor in $D_0$. Indeed, if $c \in C_1$ is adjacent to $d \in D_0$, then for the vertex $b \in B$ adjacent to $c$ and any two vertices $a_1, a_2 \in A$ different from $m(b)$ and non-adjacent to $d$, the subgraph induced by vertices $m(d), d, c, , b, m(b), x, a_1, m(a_1), a_2$ is isomorphic to $S_{1,2,5}$.

(7) $|C_0| \leq 1$. Indeed, assume $|C_0| \geq 2$. Due to the minimality of $H$ and (5), $C_0$ must have a vertex $z$ adjacent to $m(y)$. Then no other vertex of $C_0$ is adjacent to $m(y)$ by analogy with (3). Assume there is a vertex $c_0 \in C_0$ different from $z$ which is adjacent to $m(z)$. Since $C_1 \neq \emptyset$, we may consider a vertex $c_1 \in C_1$ and its neighbor $b \in B$. If $m(z)$ is not adjacent to $m(b)$, then $H$ contains an $S_{1,2,5}$ induced by vertices $c_0, m(z), z, m(y), y, b, c_1, m(c_1), m(b)$. If $m(z)$ is adjacent to $m(b)$, then for any vertex $a \in A$ different from $m(b)$, $H$ contains either an $S_{1,2,5}$ induced by vertices $y, m(a), a, x, m(b), m(z), c_0, m(c_0), z$ (if $y$ is not adjacent to $m(c_0)$) or an $S_{1,2,5}$ induced by vertices $m(c_0), y, m(a), a, x, m(b), b, c_1, m(z)$ (if $y$ is adjacent to $m(c_0)$). This contradiction shows that no vertex of $C_0$ other than $z$ is adjacent to $m(z)$. But now $|C_0 - z| = |N(C_0 - z)|$, which contradicts the minimality of $H$. Therefore, $|C_0| \leq 1$.

(8) If $|C_1| \geq 2$, then no vertex of $D_1$ has a neighbor in $A$. To the contrary, assume a vertex $d \in D_1$ has a neighbor $a \in A$, and let $c \in C_1$ be a vertex different from $m(d)$. Also, denote by $b_1 \in B$ the neighbor of $m(d)$ and by $b_2 \in B$ the neighbor of $c$. Then vertices $x, a, d, m(d), b_1, y, b_2, c, m(y)$ induce an $S_{1,2,5}$ in $H$.

From the above list of claims we conclude that if $|C_1| \geq 2$, then $U_1 := \{y, m(y)\} \cup C_0 \cup D_0 \cup N_A(D_0) \cup m(N_A(D_0))$ is a redundant subset of size at most 6 such that $H - U_1 = T_4$. If $|C_1| \leq 1$, then $U_2 := U_1 \cup C_1 \cup D_1 \cup N_A(D_1) \cup m(N_A(D_1))$ is a redundant subset of size at most 10 such that $H - U_2 = T_1$.

Now assume that $C_1 = \emptyset$. If in addition $C_0 = \emptyset$, then $H - \{y, m(y)\} = T_1$.

If $C_0 \neq \emptyset$, then due to the minimality of $H$ there must exist a vertex $z \in C_0$ adjacent to $m(y)$. Then

- no other vertex of $C_0$ is adjacent to $m(y)$ by analogy with (3).
- $y$ is not adjacent to $m(z)$.

Denote by $C_0'$ the set of vertices of $C_0 - \{z\}$ adjacent to $m(z)$ and let $C_0'' := C_0 - (C_0' \cup \{z\})$. Then $y$ is adjacent to every vertex in $m(C_0')$, since otherwise for any vertex $d \in m(C_0')$ non-adjacent to $y$, any vertex $a \in A$ non-adjacent both to $m(z)$ and $d$, and any vertex $b \in B$ different from $m(a)$, we have $H[d, m(d), m(z), z, m(y), y, m(a), a, b] = S_{1,2,5}$. Consequently, no vertex of $m(C_0')$ has a neighbor in $A$, since otherwise a $C_4$ arises.

If $C_0'' \neq \emptyset$, then it must contain a vertex $c$ that has a neighbor $d \in m(C_0')$, since otherwise $|N(C_0'')| = |C_0''|$. But then for any two vertices $a_1, a_2$ non-adjacent to $d$ and $m(c)$, we have $H[a_1, x, a_2, m(a_2), y, d, c, m(c), m(d)] = S_{1,2,5}$. Therefore, $C_0'' = \emptyset$. Now if $m(z)$ has no neighbors in $A$ then $H = T_6$, and if $m(z)$ has a neighbor in $A$ then $H = T_5$. ■

From now on, we assume that $C = C_0 \cup C_1$, i.e., every vertex of $C$ has a non-neighbor in $B$. This implies

**Claim 1** *Every vertex of $D_1$ has at most one neighbor in $C_0$.*

**Proof.** Assume a vertex $d_1 \in D_1$ has at least two neighbors $c_0', c_0'' \in C_0$. Denote $c_1 := m(d_1)$, $B' := N(c_1) \cap B$ and $B'' := B - B'$. By definition, $B' \neq \emptyset$ and $B'' \neq \emptyset$. Let $a_1 \in m(B')$ and $a_2, a_3$ be any two other vertices of $A$ non-adjacent to $d_1$. Without loss of generality assume that $a_2$ is not adjacent to $m(c_0')$. If in addition $a_1$ is not adjacent to $m(c_0')$, then $H[a_2, x, a_1, m(a_1), c_1, d_1, c_0', m(c_0'), c_0''] = S_{1,2,5}$. If $a_1$ is adjacent to $m(c_0')$, then $H[c_0'', d_1, c_0', m(c_0'), a_1, x, a_2, m(a_2), a_3] = S_{1,2,5}$. This contradiction shows that every vertex of $D_1$ has at most one neighbor in $C_0$. ■

**Claim 2** *Every vertex of $C_0$ has at least one neighbor in $D_1$.*

**Proof.** Denote by $C_0'$ the vertices of $C_0$ that have neighbors in $D_1$ and $C_0'' := C_0 - C_0'$. If $C_0''$ is not empty, it must contain a vertex $c_0''$ that has a neighbor $d_0' \in m(C_0')$, since otherwise $|C_0''| = |N(C_0'')|$ contradicting the minimality of $H$. Denote $c_0' := m(d_0')$, $d_1 \in D_1$

a neighbor of $c_0'$, $c_1 := m(d_1)$, $B' := N(c_1) \cap B$ and $B'' := B - B'$. If $d_1$ has a neighbor $a_1 \in A$, then for any two vertices $a_2, a_3 \in A$ different from $a_1$ and non-adjacent to $d_0'$, $H[c_0'', d_0', c_0', d_1, a_1, x, a_2, m(a_2), a_3] = S_{1,2,5}$. If $d_1$ has no neighbors in $A$, then for any $a_1 \in m(B')$, $a_2 \in m(B'')$ and $a_3 \in A$ different from $a_1$, $H[c_0', d_1, c_1, m(a_1), a_1, x, a_2, m(a_2), a_3] = S_{1,2,5}$. Therefore, $C'' = \emptyset$ and the claim is proved. ■

A natural consequence of the two preceding claims is the following corollary.

**Corollary 2** *If $k \geq 6$, then $|C_0| \leq |C_1|$.*

**Lemma 5** *If $|C_1| \leq 3$, then $H$ contains a redundant set $U$ of size at most 24 such that $H - U = T_1$.*

**Proof.** Let $|C_1| \leq 3$. The above corollary then implies $|C| \leq 6$, and therefore also $|D| \leq 6$. Due to the $C_4$-freeness of $H$, every vertex of $D$ has at most one neighbor in $A$, so that $|N_A(D)| \leq 6$. Now it is easy to see that the set $U := C \cup D \cup N_A(D) \cup m(N_A(D))$ is a redundant set of size at most 24 such that $H - U = T_1$. ■

From now on we assume that $|C_1| \geq 4$.

**Lemma 6** *Let $|C_1| \geq 4$ and $C_0 = \emptyset$.*

*(a) If there are vertices $y, z \in C_1$ such that $N_B(y) \cap N_B(z) \neq \emptyset$, then $H = T_2$ or $H = T_5$ or $H$ contains a redundant set $U$ of size at most 4 such that $H - U = T_1$ or $H - U = T_2$.*

*(b) If for any two vertices $y, z \in C_1$, $N_B(y) \cap N_B(z) = \emptyset$, then $H = T_3$ or $H$ contains a redundant set $U$ of size at most 4 such that $H - U = T_3$.*

**Proof.** To prove (a), denote by $b_1$ a common neighbor of $y$ and $z$ in $B$.

*Case* 1. Assume first that $y$ has one more neighbor in $B$, say $b_2$. Suppose $B$ contains a vertex $b_3$ non-adjacent both to $y$ and $z$. Then $m(b_3)$ is adjacent to $m(y)$, since otherwise $H[b_3, m(b_3), x, m(b_2), y, b_1, z, m(y)] = S_{1,2,5}$. This implies that $m(b_3)$ is adjacent to $m(z)$, since otherwise $H[m(z), z, b_1, y, m(y), m(b_3), x, a, b_3] = S_{1,2,5}$, where $a \in A$ is a vertex non-adjacent to $b_1, m(y), m(z)$. Therefore, $B$ contains at most one vertex adjacent neither to $y$ nor to $z$, since otherwise a $C_4$ arises. From this and the fact that $|B| \geq 6$ we conclude without loss of generality that $y$ has at least three neighbors in $B$. But now the vertices $b_3, m(b_3), m(z), z, b_1, y, b_2, m(b_2), b_4$ induce an $S_{1,2,5}$ in $H$, where $b_4$ is one more neighbor of $y$. This contradiction shows that $N_B(y) \cup N_B(z) = B$. Moreover, since every vertex of $C_1$ has a non-neighbor in $B$, the vertex $z$ also has at least two neighbors in $B$. As a result, we conclude that if $C_1$ contains a vertex $c$ different from $y$ and $z$, then $N_B(c) \cup N_B(y) = B = N_B(c) \cup N_B(z)$ and $|N_B(c)| \geq 2$. But then $H$ contains an induced $C_4$. Therefore, $C_1 = \{y, z\}$. Moreover, because of $S_{1,2,5}$-freeness the vertices of $D_1$ cannot have neighbors in $A$, which means that $C_1 \cup D_1$ is a redundant set of size 4, and $H - (C_1 \cup D_1) = T_1$.

*Case* 2. Now we assume that $N_B(c) = \{b_1\}$ for any vertex $c \in C_1$ adjacent to $b_1$. Then for any such a vertex, $m(c)$ has no neighbors in $A$. Indeed, if $a_1 \in A$ is a neighbor of $m(c)$, then for any vertices $a_2, a_3 \in A$ different from $m(b_1)$ and non-adjacent to $m(c)$, we have $H[c', b_1, c, m(c), a_1, x, a_3, m(a_3), a_2] = S_{1,2,5}$, where $c'$ is another vertex of $C_1$ adjacent to $b_1$. Therefore, if every vertex of $C_1$ is adjacent to $b_1$, then $H = T_2$.

Suppose now that $C_1$ has a vertex $c'$ non-adjacent to $b_1$, and let $b_2$ denote a neighbor of $c'$ in $B$. Then $c'$ is adjacent to $m(y)$ (and similarly to $m(c)$ for every vertex $c \in C_1$ adjacent to $b_1$), since otherwise $H[c', b_2, m(b_2), x, m(b_1), b_1, y, m(y), z] = S_{1,2,5}$. Consequently, $c'$ is adjacent to every vertex $b \in B$ different from $b_1$, since otherwise $H[b, m(b), x, m(b_2), b_2, c', m(y), y, m(z)] = S_{1,2,5}$. Together with $C_4$-freeness of $H$ this implies that $c'$ is the only vertex of $C_1$ non-adjacent to $b_1$. If in addition $m(c')$ has no neighbors in $A$, then $\{c', m(c')\}$ is a redundant set in $H$ and $H - \{c', m(c')\} = T_2$. If $m(c')$ has a neighbor in $A$, then this neighbor must be $m(b_1)$, in which case $H = T_5$.

Now we proceed to the proof of (b). If for each vertex $d \in D_1$, the vertex $m(d)$ is the only neighbor of $d$, then $H = T_3$. Therefore, we assume that a vertex $d \in D_1$ has a neighbor $y \neq m(d)$.

Let first $y$ belong to $C_1$. Since $|C_1| \geq 4$, there is a vertex $a \in A$ such that $a$ is not adjacent to $d$, and $m(a)$ is adjacent neither to $m(d)$ nor to $y$. But then for any neighbor $b \in B$ of $m(d)$ and any vertex $a' \in A - \{a, m(b)\}$ non-adjacent to $d$, we have $H[y, d, m(d), b, m(b), x, a, m(a), a'] = S_{1,2,5}$.

Now assume that $y \in A$. Let first $y$ be the only neighbor of $m(y)$, i.e., let $\{y, m(y)\}$ be a redundant set. If there is a vertex $d' \in D_1$ non-adjacent to $y$, then

$$H[d', m(d'), b, m(b), x, y, d, m(d), m(y)] = S_{1,2,5}\,,$$

where $b \in B$ is a neighbor of $m(d')$. If every vertex of $D_1$ is adjacent to $y$, then they have no other neighbors in $A$ and hence $H - \{y, m(y)\} = T_3$.

Now suppose that $m(y)$ is adjacent to a vertex $c \in C_1$. Then every vertex $d'$ of $D_1$ (different from $m(c)$) also is adjacent to $y$, since otherwise for any neighbor $b \in B$ of $m(d')$, we have $H[d', m(d'), b, m(b), x, y, d, m(d), m(y)] = S_{1,2,5}$. Therefore, if $\{y, m(y), c, m(c)\}$ is a redundant set in $H$, then $H - \{y, m(y), c, m(c)\} = T_3$. In order to show that the set $\{y, m(y), c, m(c)\}$ is redundant, assume by contradiction that $m(c)$ has a neighbor $a \in A$. Without loss of generality let $m(a)$ be non-adjacent to $m(d)$. But then $H[m(a), a, m(c), c, m(y), y, d, m(d$ $S_{1,2,5}$, where $d'$ is any vertex of $D_1$ different from $d$ and $m(c)$. ■

**Lemma 7** *Let $|C_1| \geq 4$. If $C_0 \neq \emptyset$, then $U := C_0 \cup D_0$ is a redundant set of size 2 such that $H - U = T_5$.*

**Proof.** Assume that $C_0 \neq \emptyset$. Then it must contain a vertex $c$ which has a neighbor $d$ in $D_1$, since otherwise $|N(C_0)| = |C_0|$. Let $b \in B$ be a non-neighbor and $b' \in B$ a neighbor of $m(d)$. If $d$ is not adjacent to $m(b)$, then for any vertex $a \in A$ different from $m(b), m(b')$ and non-adjacent to $d$, we have $H[c, d, m(d), b', m(b'), x, m(b), b, a] = S_{1,2,5}$.

Now assume that for any non-neighbor $b \in B$ of $m(d)$, the vertex $d$ is adjacent to $m(b)$. Since $d$ may have at most one neighbor in $A$, we conclude that $m(d)$ is adjacent to all but one vertex $b$ in $B$.

This implies that $|C_0| \leq 1$. Indeed, suppose $C_0$ contains one more vertex, say $c'$. Then due to the minimality of $H$, either $c'$ has a neighbor in $D_1$ or we can assume without loss of generality that $c'$ is adjacent to $m(c)$. If $c'$ has a neighbor $d' \in D_1$ different from $d$, then analogously $m(d')$ is adjacent to all but one vertex of $B$, which leads to an induced $C_4$, since $|B| \geq 6$. If $c'$ is adjacent to $d$, then for any two vertices $a_1, a_2 \in A$ non-adjacent to $d$ and $m(c)$ such that $m(a_2)$ is adjacent to $m(d)$, we have $H[a_1, x, a_2, m(a_2), m(d), d, c, m(c), c'] = S_{1,2,5}$. Finally, if $c'$ is adjacent to $m(c)$, then for a vertex $a_1 \in A$ non-adjacent to $m(c), m(c')$, we have either $H[m(c'), c', m(c), c, d, m(b), x, a_1, b] = S_{1,2,5}$ (if $m(b)$ is not adjacent to $m(c')$) or $H[m(c'), c', m(c), c, d, m(d), m(a_1), a_1, b'] = S_{1,2,5}$, where $b' \in B$ (if $m(b)$ is adjacent to $m(c')$). This contradiction shows that $C_0 = \{c\}$.

Assume $m(c)$ has a neighbor $c_1 \in C_1$. Obviously $c_1 \neq m(d)$ and $c_1$ is not adjacent to $d$. In addition, $c_1$ has a neighbor $y \neq m(c)$ non-adjacent to $m(d)$. Indeed, if $c_1$ is adjacent to $b$ then $y = b$, and if $c_1$ is not adjacent to $b$ then it has a neighbor in $B - \{b\}$, in which case $y = m(c_1)$. But then $H[y, c_1, m(c), c, d, m(d), b', m(b'), b''] = S_{1,2,5}$, where $b'$ and $b''$ are two vertices in $B - \{b\}$ non-adjacent to $c_1$ (observe that $c_1$ may have at most one neighbor in $B - \{b\}$) such that $m(b')$ is not adjacent to $y$ and $m(c)$. This contradiction shows that $m(c)$ has no neighbors in $C_1$.

Assume now that $m(c)$ is adjacent to a vertex $a \in A$ different from $m(b)$. Since $|C_1| \geq 4$, we may consider three vertices $c_1, c_2, c_3 \in C_1$ different from $m(d)$. Remember that $c_j$ ($j = 1, 2, 3$) must have a neighbor in $B$, but it cannot have more than one neighbor in $B - \{b\}$. To avoid an induced $C_4$ we conclude that $m(c_j)$ cannot be adjacent both to $c$ and $a$. If $m(c_j)$ is adjacent to $a$, then for any vertex $b'' \in B - \{b\}$ non-adjacent to $c_j$ we have $H[b'', m(d), d, c, m(c), a, m(c_j), c_j, x)] = S_{1,2,5}$. If $m(c_j)$ is adjacent to $c$, then for any vertex $a' \in A - \{a, m(b)\}$ such that $a'$ is non-adjacent to $m(c_j)$ and $m(a')$ is not adjacent to $c_j$ we have $H[m(a'), a', x, a, m(c), c, m(c_j), c_j, d)] = S_{1,2,5}$. Therefore, we conclude that $m(c_j)$ is adjacent neither to $a$ nor to $c$. If in addition $c_j$ has a neighbor $b' \in B - \{b, m(a)\}$, then $H[c, m(c), a, m(a), m(d), b', c_j, m(c_j), m(b')] = S_{1,2,5}$. If two vertices $c_i$ and $c_j$ are both adjacent to $b$, then for any vertex $a' \in A - \{m(b)\}$ such that $m(a')$ is not adjacent to $c_i, c_j$ and $a'$ is not adjacent to $m(c_i)$ we have $H[m(d), m(a'), a', x, m(b), b, c_i, m(c_i), c_j] = S_{1,2,5}$. The only case is left is when two vertices $c_i$ and $c_j$ are both adjacent to $m(a)$. Then for any two vertices $a_1, a_2 \in A - \{a, m(b)\}$ non-adjacent to $m(c_i)$ we have $H[a_1, x, a_2, m(a_2), m(d), m(a), c_i, m(c_i), c_j] = S_{1,2,5}$. This contradiction shows that the only possible neighbor of $m(c)$ in $A$ is $m(b)$.

Due to the $C_4$-freeness of $H$, $m(c)$ cannot be adjacent to $m(b)$. Therefore, we conclude that $c$ is the only neighbor of $m(c)$.

Assume now that there is a vertex $c_1 \in C_1$ such that $c_1$ and $m(d)$ have a common neighbor in $B$, say $b_1$. Then $c_1$ is not adjacent to $b$, for otherwise the vertex set $\{m(c), c, d, m(b), b, c_1, b_1, m(b_1), m(c$ would induce an $S_{1,2,5}$ in $H$. Also, $m(c_1)$ is not adjacent to $m(b)$, since otherwise $H[b, m(b), m(c_1), c_1, b_1, m$ $S_{1,2,5}$ for any two distinct vertices $b_2$, $b_3 \in B - \{b, b_1\}$ such that $m(b_2)$ is not adjacent to $m(c_1)$. But now, an $S_{1,2,5}$ arises on the vertex set $\{m(c_1), c_1, b_1, m(d), d, m(b), x, a_1, b\}$, where $a_1 \in A$ is any vertex different from $m(b_1)$ and non-adjacent to $m(c_1)$. This contradiction shows that $N_B(c_1) = \{b\}$ for every vertex $c_1 \in C_1 - \{m(d)\}$.

Now, observe that for every vertex $c_1 \in C_1 - \{m(d)\}$, $m(c_1)$ is adjacent to $m(d)$. Indeed, if this is not the case, we have $H[m(d), m(a), a, x, m(b), b, c_1, m(c_1), c_2] = S_{1,2,5}$ for any vertex $a \in A$ different than $m(b)$ and non-adjacent to $m(c_1)$ and any $c_2 \in C_1 - \{m(d), c_1\}$. Therefore, we conclude that $H - \{c, m(c)\} = T_5$. ■

## 4.2 Finding Augmenting Graphs $T_1, \dots, T_6$

Now we present polynomial-time algorithms for finding augmenting graphs from the six basic families represented in Figure 2. To this end, we first check whether $G$ contains a certain small induced subgraph (a so-called *initial structure*) which is contained in every large enough graph from a family $T_i$ under consideration, and then try to extend it to the whole augmenting graph. For clarity of the proofs, we shall use the labeling of vertices of augmenting graphs $T_1, \dots, T_6$ as represented in Figure 2.

Given a black vertex $b$, we will denote by $W(b) = N(b) \cap S$ the set of white neighbors of $b$. For a nonnegative integer $i$, we denote by $B^i$ the set of all black vertices having exactly $i$ white neighbors. The independence number of $G$ (i.e., the size of a maximum independent set in $G$) is denoted $\alpha(G)$.

**Lemma 8** *If $G$ contains no augmenting $P_3$, then a simple augmenting tree $T_1$ (if any) can be found in polynomial time.*

**Proof.** If $G$ contains no augmenting $P_3$ but contains an augmenting $T_1$, then $r \geq 2$, where $r$ is the number of leaves in $T_1$ (see Figure 2). Therefore, we first have to check if $G$ contains an induced $P_5 = (b_1, a_1, x, a_2, b_2)$ with $\{b_1, b_2\} \subseteq B^1$. If $G$ contains no such an initial structure, then it contains no augmenting $T_1$. If such a structure exists, then we proceed as follows.

Let us denote $A = W(x) \backslash \{a_1, a_2\}$, and for $a \in A$, let $K(a)$ denote the set of black neighbors of $a$ which are in $B^1$, and which are not adjacent to any of $\{x, b_1, b_2\}$. Notice that a desired simple augmenting tree exists only if $K(a) \neq \emptyset$ for all $a$ in $A$. Finally, let $V' = \cup_{a \in A} K(a)$.

Consider any vertex $a$ in $A$. If $K(a)$ contains two non-adjacent vertices $b$ and $b'$, then $b$, $a$ and $b'$ induce an augmenting $P_3$ in $G$, a contradiction. Hence, each $K(a)$ induces a clique in $G$. It follows that a desired simple augmenting tree exists if and only if $\alpha(G[V']) = |A|$.

It is easy to see that $G[V']$ must be $P_5$-free. Indeed, consider an induced $P_4 = (p_1, p_2, p_3, p_4)$ in $G[V']$, and let $a \in A$ be such that $p_1 \in K(a)$. None of the vertices $p_3$ and $p_4$ is adjacent to $a$, since $K(a)$ is a clique. But now, $p_2 \in K(a)$, since otherwise the vertex set $\{p_4, p_3, p_2, p_1, a, x, a_2, b_2, a_1\}$ induces an $S_{1,2,5}$ in $G$. Hence, if $G[V']$ contains a $P_4 = (p_1, p_2, p_3, p_4)$, then $p_1$ and $p_2$ have a common white neighbor, while $p_2$ and $p_3$ do not have a common white neighbor. This implies that $G[V']$ is $P_5$-free.

Since the independence number of a $(P_5, banner)$-free graph can be computed in polynomial time (see e.g. [14, 8]), we can efficiently compute $\alpha = \alpha(G[V'])$. If $\alpha < |A|$, we conclude that $G$ contains no simple augmenting tree containing the above initial structure. Otherwise, we may choose one vertex from each clique $K(a)$ to obtain a simple augmenting tree. ■

**Lemma 9** *If $G$ contains no augmenting $P_3$ or $P_7$, then an augmenting $T_2$ (if any) can be found in polynomial time.*

**Proof.** We may restrict ourselves to finding a $T_2$ with $r, s \geq 2$, since any $T_2$ with, say, $r = 1$ either equals to $P_7$ or contains a redundant subset $U = \{a_1, b_1\}$ such that $T_2 - U = T_1$.

As an initial structure, consider the subgraph of $T_2$ (see Figure 2) induced by vertices $a_1, a_2, b_1, b_2, c_1, c_2, d_1, d_2, x, y, z$ such that $\{b_1, b_2, d_1, d_2\} \subseteq B^1$.

Let us denote $A = (W(x) \cup W(z)) \backslash \{a_1, a_2, c_1, c_2, y\}$, and for $a \in A$, let $K(a)$ denote the set of black neighbors of $a$ which are in $B^1$, and which are not adjacent to any of $\{x, z, b_1, b_2, d_1, d_2\}$. Note that due to the $C_4$-freeness of the augmenting graph, $(W(x) \cap A) \cap (W(z) \cap A) = \emptyset$, and that a desired augmenting $T_2$ exists only if $K(a) \neq \emptyset$ for all $a$ in $A$. Finally, let $V' = \cup_{a \in A} K(a)$.

Consider any vertex $a$ in $A$. If $K(a)$ contains two non-adjacent vertices $b$ and $b'$, then $b$, $a$ and $b'$ induce an augmenting $P_3$ in $G$, a contradiction. Hence, each $K(a)$ induces a clique in $G$. It follows that a desired augmenting $T_2$ exists if and only if $\alpha(G[V']) = |A|$.

We now show that $G[V']$ is $P_3$-free. Suppose, to the contrary, that $(p_1, p_2, p_3)$ is an induced $P_3$ in $G[V']$. Let $a \in A$ be such that $p_1 \in K(a)$. Since $K(a)$ is a clique, $p_3$ is not adjacent to $a$. This implies that $a_2$ is not adjacent to $a$, since otherwise $p_2$ and $p_3$ should have a common white neighbor different from $a$, which is impossible since $p_2 \in B^1$. Without loss of generality we may assume that $a$ is adjacent to $x$, but not to $z$. But now the vertex set $\{p_2, p_1, a, x, y, z, a_2, b_2, a_1\}$ induces an $S_{1,2,5}$ in $G$, a contradiction.

Hence, $G[V']$ is a disjoint union of cliques, and the independence number $\alpha = \alpha(G[V'])$ can be trivially computed. If $\alpha < |A|$, we conclude that $G$ contains no augmenting $T_2$ containing the above initial structure. Otherwise, we may choose one vertex from each clique $K(a)$ to obtain an augmenting $T_2$. ■

**Lemma 10** *If $G$ contains no augmenting $P_3$, then an augmenting $T_3$ (if any) can be found in polynomial time.*

**Proof.** We may restrict ourselves to finding a $T_3$ with $s \geq 2$, since any $T_3$ with $s \in \{0, 1\}$ is either a simple augmenting tree $T_1$ or contains a redundant subset $U$ of size 2 such that $T_3 - U = T_1$.

As an initial structure, consider the subgraph of $T_3$ (see Figure 2) induced by vertices $d_1, c_1, b_1^1, a_1^1, x, a_1^2, b_1^2, c_2, d_2$ such that $\{b_1, b_2\} \subseteq B^2$ and $\{d_1, d_2\} \subseteq B^1$.

Let us denote $A = W(x) \backslash \{a_1, a_2\}$, and for $a \in A$, let $K(a)$ denote the set of black neighbors $b$ of $a$ which are in $B^1 \cup B^2$, which are not adjacent to any of $\{x, b_1^1, b_1^2, d_1, d_2\}$, and such that if $b \in B^2$ then $G$ contains a pair of adjacent vertices $c(b)$ and $d(b)$ such that $W(b) = \{a, c(b)\}$, $d(b) \in B^1$, and $d(b)$ is not adjacent to any of $\{x, b_1^1, b_1^2, d_1, d_2, b\}$.

Note that due to the $C_4$-freeness of the augmenting graph, the sets $K(a_1)$ and $K(a_2)$ are disjoint for any two distinct $a_1, a_2 \in A$, and that a desired augmenting $T_3$ exists only if $K(a) \neq \emptyset$ for all $a$ in $A$. Finally, let $V' = \cup_{a \in A} K(a)$.

Consider any vertex $a$ in $A$. Suppose $K(a)$ contains two non-adjacent vertices $b$ and $b'$. If $b, b' \in B^1$, then $b$, $a$ and $b'$ induce an augmenting $P_3$ in $G$, a contradiction. Now assume $b \in B^2$, and let $\{c(b), d(b)\}$ be a pair of adjacent vertices such that $W(b) = \{a, c(b)\}$, $d(b) \in B^1$, and $d(b)$ is not adjacent to any of $\{x, b_1^1, b_1^2, d_1, d_2, b\}$. Since $b \in B^2$, it cannot be adjacent to both $c_1$ and $c_2$; without loss of generality we may assume that $b$ is not adjacent to $c_1$, i.e., $c(b) \neq c_1$. But now the vertex set $\{d_1, c_1, b_1^1, a_1^1, x, a, b, c(b), b'\}$ induces an $S_{1,2,5}$ in $G$, a contradiction.

Therefore, each $K(a)$ induces a clique in $G$. It follows that $\alpha(G[V']) = |A|$ is a necessary condition for the existence of an augmenting $T_3$ extending the initial structure.

Let us now show that $G[V']$ must be $P_5$-free. Indeed, consider an induced $P_4 = (p_1, p_2, p_3, p_4)$ in $G[V']$, and let $a \in A$ be such that $p_1 \in K(a)$. None of the vertices $p_3$ and $p_4$ is adjacent to $a$, since $K(a)$ is a clique. But now, $p_2 \in K(a)$, since otherwise the vertex set $\{p_4, p_3, p_2, p_1, a, x, a_1^2, b_1^2, a_1^1\}$ induces an $S_{1,2,5}$ in $G$. Hence, if $G[V']$ contains a $P_4 = (p_1, p_2, p_3, p_4)$, then $p_1$ and $p_2$ have a common white neighbor, while $p_2$ and $p_3$ do not have a common white neighbor. This implies that $G[V']$ is $P_5$-free.

Since the stability number of a $(P_5, banner)$-free graph can be computed in polynomial time, we can efficiently compute $\alpha = \alpha(G[V'])$. If $\alpha < |A|$, we conclude that $G$ contains no augmenting $T_3$ containing the above initial structure.

If $\alpha = |A|$, let $B$ denote the set of vertices obtained by choosing one vertex $b$ from each clique $K(a)$, and let $C = \{c(b) : b \in B \cap B^2\}$ and $D = \{d(b) : b \in B \cap B^2\}$. Consider the induced subgraph $H$ of $G$, obtained by adding to the initial structure all the vertices from $A \cup B \cup C \cup D$. Now, to see that $H$ is an augmenting $T_3$ we only need to show that $B \cup D$ is an independent set. Since $\alpha = |A|$, $B$ forms an independent set in $G$. By definition of $d(b)$, no $b \in B \cap B^2$ is adjacent to $d(b)$. Suppose there is an edge connecting a vertex $b$ from $B$ to a $d(b')$ for some $b' \in B \cap B^2$. Let $a'$ denote the unique common white neighbor of $x$ and $b'$. Now, the vertex set $\{b, d(b'), c(b'), b', a', x, a_1, b_1, a_2\}$ induces an $S_{1,2,5}$ in $G$, a contradiction. Similarly, if there is an edge connecting vertices $d, d' \in D$, then the vertex set $\{d, d', c(b'), b', a', x, a_1, b_1, a_2\}$ (where $d' = d(b')$ and $a'$ is the unique common white neighbor of $x$ and $b'$) induces an $S_{1,2,5}$ in $G$. This contradiction completes the proof of the lemma. ∎

**Lemma 11** *If $G$ contains no augmenting $P_3$, then an augmenting $T_4$ (if any) can be found in polynomial time.*

**Proof.** This follows immediately from the fact that every $T_4$ is a special case of a $T_3$, and from the proof of Lemma 10. ∎

**Lemma 12** *An augmenting $T_5$ (if any) can be found in polynomial time.*

**Proof.** We may restrict ourselves to finding a $T_5$ with $r, s > 0$ and $r \geq 2$, since a $T_5$ with $r = 0$ contains a redundant set $U$ of size 4 such that $T_5 - U = T_1$, and a $T_5$ with $r = s = 1$ can be found in polynomial time.

As an initial structure, consider the subgraph of $T_5$ (see Figure 2) induced by vertices $a_1, a_2, b_1, b_2, c_1, d_1, u, v, x, y, z$ such that $\{b_1, b_2, v, d_1\} \subseteq B^2$.

Let us denote $A_x = W(x) \backslash \{a_1, a_2, u\}$, $A_y = W(y) \backslash \{u, c_1\}$, and for $a \in A := A_x \cup A_y$, let $K(a)$ denote the set of common neighbors of $a$ and $z$ which are in $B^2$, and which are not adjacent to any of $\{x, y, b_1, b_2, v, d_1\}$.

Note that it follows from the $C_4$-freeness of the augmenting graph that the sets $A_x$ and $A_y$ are disjoint, and that for every $a \in A$, $K(a)$ is a clique. Finally, let $V'_x = \cup_{a \in A_x} K(a)$, $V'_y = \cup_{a \in A_y} K(a)$, and $V' = V'_x \cup V'_y$. From the definition of the sets $K(a)$ it follows that $V'_x \cap V'_y = \emptyset$. Moreover, a desired augmenting $T_5$ exists if and only if $\alpha(G[V']) = |A|$.

Let us show that $G[V']$ is a disjoint union of cliques. First, we observe that each of $G[V'_x]$ and $G[V'_y]$ is a disjoint union of cliques. Indeed, suppose that $(p_1, p_2, p_3)$ is an induced $P_3$ in $G[V'_x]$. Let $a \in A_x$ be such that $p_1 \in K(a)$. Since $K(a)$ is a clique, $p_3$ is not adjacent to $a$. This implies that $p_2$ is not adjacent to $a$, since otherwise $p_2$ and $p_3$ should have a common white neighbor different from $a$, which is impossible since $p_2 \in B^2$ and $W(p_2) = \{a, z\}$. But now the vertex set $\{p_3, p_2, p_1, a, x, u, v, y, c_1\}$ induces an $S_{1,2,5}$ in $G$, a contradiction. Also, there are no edges between $V'_x$ and $V'_y$, for if there are vertices $a \in A_x$ and $a' \in A_y$ with $bb' \in E$ for some $b \in K(a)$ and $b' \in K(a')$, then the vertex set $\{y, a', b', b, a, x, a_1, b_1, a_2\}$ induces an $S_{1,2,5}$ in $G$.

Hence, the independence number $\alpha = \alpha(G[V'])$ can be trivially computed. If $\alpha < |A|$, we conclude that $G$ contains no augmenting $T_5$ containing the above initial structure. Otherwise, we may choose one vertex from each clique $K(a)$ to obtain an augmenting $T_5$. ■

**Lemma 13** *If $G$ contains no augmenting $P_3$ or $P_7$, then an augmenting $T_6$ (if any) can be found in polynomial time.*

**Proof.** We may restrict ourselves to finding a $T_6$ with $r \geq 2$, since a $T_6$ with $r = s = 1$ is a $P_7$.

As an initial structure, consider the subgraph of $T_5$ (see Figure 2) induced by vertices $a_1, a_2, b_1, b_2, c_1, d_1, x, y, z$ such that $\{b_1, b_2, c_1\} \subseteq B^2$ and such that $x$ and $z$ have no common white neighbors.

Let us denote $A_x = W(x) \backslash \{a_1, a_2\}$, $A_z = W(z) \backslash \{d_1\}$ and for $a \in A := A_x \cup A_z$, let $K(a)$ denote the set of common neighbors of $a$ and $b$ which are in $B^2$, and which are not adjacent to any of $\{x, b_1, b_2, c_1, z\}$.

Note that $A_x \cap A_z = \emptyset$ by assumption. Also, due to the $C_4$-freeness of the augmenting graph, each $K(a)$ for $a \in A$ is a clique. Finally, let $V'_x = \cup_{a \in A_x} K(a)$, $V'_z = \cup_{a \in A_z} K(a)$, and $V' = V'_x \cup V'_z$. It follows that a desired augmenting $T_6$ exists only if $\alpha(G[V']) = |A|$.

Let us show that $G[V']$ is a $S_{1,1,2}$-free graph. Indeed, suppose that $\{p_1, p_2, p_3, p_4, p_5\}$ induces an $S_{1,1,2}$ in $G[V']$ (with a $P_4$ on $\{p_1, p_2, p_3, p_4\}$ and an additional edge $p_2p_5$), and let $a \in A$ be such that $p_1 \in K(a)$. Since $K(a)$ is a clique, none of $p_3$, $p_4$, $p_5$ is adjacent to $a$. Now, $p_2$ must be adjacent to $a$, or an $S_{1,2,5}$ arises on the vertex set $\{b_1, a_1, x, a, p_1, p_2, p_3, p_4, p_5\}$ (if $a \in A_x$) or on the vertex set $\{d_1, c_1, z, a, p_1, p_2, p_3, p_4, p_5\}$ (if $a \in A_z$). By symmetry, $p_2$ and $p_5$ also share a common white neighbor $a' \in A$ different from $a$. But this contradicts the $C_4$-freeness of the augmenting graph.

Since the independence number of an $S_{1,1,2}$-free graph can be computed in polynomial time (see e.g. [2]), we can efficiently compute $\alpha = \alpha(G[V'])$. If $\alpha < |A|$, we conclude that $G$ contains no augmenting $T_6$ containing the above initial structure. Otherwise, we may choose one vertex from each clique $K(a)$ to obtain an augmenting $T_6$. ■

As a consequence of the above lemmas and the results from Section 3, we obtain the following conclusion.

**Theorem 5** *The maximum independent set problem can be solved in polynomial time in the class of* $(S_{1,2,5}, banner)$*-free graphs.*

# References


[1] V.E. Alekseev, The effect of local constraints on the complexity of determination of the graph independence number. *Combinatorial-algebraic methods in applied mathematics*, Gorkiy Univ. Press, Gorky, 3-13, 1982 (in Russian).

[2] V.E. Alekseev, A polynomial algorithm for the independent set problem in fork-free graphs, *Dicsrete Appl. Math.*, 135 (2004), 3–16.

[3] V.E. Alekseev and Vadim V. Lozin, Augmenting graphs for independent sets, *Dicsrete Appl. Math.*, 145 (2004) 3–10.

[4] C. Berge, Two theorems in graph theory. *Proc. Nat. Acad. Sci. USA*, 43 (1957) 842–844.

[5] R. Boliac and V.V. Lozin, An augmenting graph approach to the stable set problem in $P_5$-free graphs, *Discrete Applied Mathematics*, 131 (2003) 567-575.

[6] T. Denley, The independence number of graphs with large odd girth. *Electron. J. Combinatorics*, 1:Research Paper 9, 12 pp, 1994.

[7] J. Edmonds, Path, trees, and flowers, *Canadian J. Mathematics*, 17 (1965) 449–467.

[8] M.U. Gerber, A. Hertz and V.V. Lozin, Stable sets in two subclasses of banner-free graphs. *Discrete Applied Mathematics*, 132 (2004) 121–136.

[9] M.U. Gerber, A. Hertz and V.V. Lozin, Finding augmenting chains in graphs without a skew star, *J. Combinatorial Theory B*, accepted.

[10] P.Hall, On representatives of subsets, J. London Math. Soc., 10 (1935) 26–30.

[11] M.M. Halldórsson, Approximating discrete collections via local improvements. *Proceedings of the Sixth SAIM-ACM Symposium on Discrete Algorithms (San Francisco, CA, 1995)*, 160–169, 1995.

[12] A. Hertz, V. Lozin and D. Schindl, Finding augmenting chains in extensions of claw-free graphs, *Information Processing Letters*, 86 (2003) 311–316.

[13] L. LOVÁSZ and M.D. PLUMMER, Matching Theory, *Ann. Discrete Math*, 29. North-Holland Publishing Co., Amsterdam; Akadmiai Kiad (Publishing House of the Hungarian Academy of Sciences), Budapest, 1986. xxvii+544 pp.

[14] V.V. Lozin, Stability in $P_5$- and banner-free graphs, *European J. Oper. Research*, 125 (2000) 292–297.

[15] G.J. Minty, On maximal independent sets of vertices in claw-free graphs. *J. Combinatorial Theory*, Ser.B, 28 (1980) 284–304.

[16] R. Mosca, Independent sets in certain $P_6$-free graphs, *Discrete Appl. Math.*, 92 (1999) 177–191.

[17] N. Sbihi, Algorithme de recherche d'un independent de cardinalité maximum dans un graphe sans étoile. *Discrete Mathematics*, 29 (1980) 63–76.