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Discrete Moment Problem with the Given Shape of the Distribution

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Discrete Moment Problem with the Given Shape of the Distribution

Ersoy Subasi Mine Subasi András Prékopa

Abstract. Discrete moment problems with finite, preassigned supports and given shapes of the distributions are formulated and used to obtain sharp upper and lower bounds for expectations of convex functions of discrete random variables as well as probabilities that at least one out of n events occurs. The bounds are based on the knowledge of some of the power moments of the random variables involved, or the binomial moments of the number of events which occur. The bounding problems are formulated as LP's and dual feasible basis structure theorems as well as the application of the dual method provide us with the results. Applications in PERT and reliability are presented.
Key words: Power Moment Problem, Binomial Moment Problem, Linear Programming, Logconcavity.

1 Introduction

Let ξ be a random variable, the possible values of which are known to be the nonnegative numbers $z_0 < z_1 < ... < z_n$. Let $p_i = P(\xi = z_i), \quad i = 0, 1, ..., n$. Suppose that these probabilities are unknown but either the power moments $\mu_k = E(\xi^k), \ k = 1, ..., m$ or the binomial moments $S_k = E\left[\binom{\xi}{k}\right], \ k = 1, ..., m$, where $m < n$ are known.

The starting points of our investigation are the following linear programming problems

$$min(max)\{f(z_0)p_0 + f(z_1)p_1 + ... + f(z_n)p_n\}$$

subject to

$$\begin{aligned} p_0 + p_1 + ... + p_n &= 1 \\ z_0 p_0 + z_1 p_1 + ... + z_n p_n &= \mu_1 \\ z_0^2 p_0 + z_1^2 p_1 + ... + z_n^2 p_n &= \mu_2 \\ &\vdots \\ z_0^m p_0 + z_1^m p_1 + ... + z_n^m p_n &= \mu_m \\ p_0 \geq 0, p_1 \geq 0, ..., p_n &\geq 0, \end{aligned} \tag{1.1}$$

$$min(max)\{f(z_0)p_0 + f(z_1)p_1 + ... + f(z_n)p_n\}$$

subject to

$$\begin{aligned} p_0 + p_1 + ... + p_n &= 1 \\ z_0 p_0 + z_1 p_1 + ... + z_n p_n &= S_1 \\ \binom{z_0}{2} p_0 + \binom{z_1}{2} p_1 + ... + \binom{z_n}{2} p_n &= S_2 \\ &\vdots \\ \binom{z_0}{m} p_0 + \binom{z_1}{m} p_1 + ... + \binom{z_n}{m} p_n &= S_m \\ p_0 \geq 0, p_1 \geq 0, ..., p_n &\geq 0. \end{aligned} \tag{1.2}$$

Problems (1.1) and (1.2) are called the power and binomial moment problems, respectively and have been studied extensively in Prékopa (1988; 1989a, b; 1990), Boros and Prékopa (1989). The two problems can be transformed into each other by the use of a simple linear transformation (see Prékopa, 1995, Section 5.6). Let $A; a_0, a_1, ..., a_n$ and b denote the matrix of the equality constraints in either problem, its columns and the vector of the right-hand side values. We will alternatively use the notation f_k instead of $f(z_k)$.

In this paper we specialize problems (1.1) and (1.2) in the following manner.

(1) In case of problem (1.1) we assume that the function f has positive divided differences of order $m+1$, where m is some fixed nonnegative integer satisfying $0 \leq m \leq n$. The optimum values of problem (1.1) provides us with sharp lower and upper bounds for $E[f(\xi)]$.

(2) In case of problem (1.2) we assume that $z_i = i,\ i = 0, ..., n$ and $f_0 = 0,\ f_i = 1,\ i = 1, ..., n$. The problem can be used in connection with arbitrary events $A_1, ..., A_n$, to obtain sharp lower and upper bounds for the probability of the union. In fact if we define $S_0 = 1$ and

$$S_k = \sum_{1 \leq i_1 < ... < i_k \leq n} P(A_{i_1} ... A_{i_k}), \quad k = 1, ..., n,$$

then by a well-known theorem (see, e.g., Prékopa, 1995) we have the equation

$$S_k = E\left[\begin{pmatrix} \xi \\ k \end{pmatrix}\right], \quad k = 1, ..., n, \tag{1.3}$$

where ξ is the number of those events which occur. The equality constraints in (1.2) are just the same as $S_0 = 1$ and the equations in (1.3) for $k = 1, ..., m$ and the objective function is the probability of $\xi \geq 1$ under the distribution $p_0, ..., p_n$. The distribution, however, is allowed to vary subject to the constraints, hence the optimum value of problem (1.2) provide us with the best possible bounds for the probability $P(\xi \geq 1)$, given $S_1, ..., S_m$.

For small m values ($m \leq 4$) closed form bounds are presented in the literature. For power moment bounds see Prékopa (1990, 1995). Bounds for the probability of the union have been obtained by Fréchet (1940, 1943) when $m = 1$, Dawson and Sankoff (1967) when $m = 2$, Kwerel (1975) when $m \leq 3$, Boros and Prékopa (1989) when $m \leq 4$. In the last two paper bounds for the probability that at least r events occur, are also presented. For other closed form probability bounds see Prékopa (1995), Galambos and Simonelli (1996). Prékopa (1988, 1989, 1990) discovered that the probability bounds based on the binomial and power moments of the number of events that occur, out of a given collection $A_1, ..., A_n$, can be obtained as optimum values of discrete moment problems (DMP) and showed that for arbitrary m values simple dual algorithms solve problems (1.1) and (1.2) if f is of type (1) or (2) (and if $f_r = 1,\ f_k = 0,\ \ k \neq r$).

In this paper we formulate and use moment problems with finite, preassigned support and with given shape of the probability distribution to obtain sharp lower and upper bounds for unknown probabilities and expectations of higher order convex functions of discrete random variables. We assume that the probability distribution $\{p_i\}$ is either decreasing (Type 1) or increasing (Type 2) or unimodal with a known modus (Type 3). The reasoning goes along the lines presented in above cited papers by Prékopa.

In Section 2 some basic notions and theorems are given. In Section 3 and 4 we use the dual feasible basis structure theorems (Prékopa, 1988, 1989, 1990) to obtain sharp bounds for

$E[f(\xi)]$ and $P(\xi \geq r)$ in case of problems, where the first two moments are known. In Section 5 we give numerical examples to compare the sharp bounds obtained by the original binomial moment problem and the sharp bounds obtained by the transformed problems: Type 1, Type 2 and Type 3. In Section 6 we present three examples for the application of our bounding technique, where shape information about the unknown probability distribution can be used.

2 Basic Notions and Theorems

Let f be a function on the discrete set $Z = \{z_0, ..., z_n\}$, $z_0 < z_1 < ... < z_n$. The first order divided differences of f are defined by

$$[z_i, z_{i+1}]f = \frac{f(z_{i+1}) - f(z_i)}{z_{i+1} - z_i}, \qquad i = 0, 1, ..., n-1.$$

The kth order divided differences are defined recursively by

$$[z_i, ..., z_{i+k}]f = \frac{[z_{i+1}, ..., z_{i+k}]f - [z_i, ..., z_{i+k-1}]f}{z_{i+1} - z_i}, \qquad k \geq 2.$$

The function f is said to be kth order convex if all of its kth divided differences are positive.

Theorem 1(Prékopa, 1988, 1989a,1989b, 1990) *Suppose that all (m+1)st divided differences of the function $f(z)$, $z \in \{z_0, z_1, ..., z_n\}$ are positive. Then, in problems* (1.1) *and* (1.2)*, all bases are dual nondegenerate and the dual feasible bases have the following structures, presented in terms of the subscripts of the basic vectors:*

	$m+1$ *even*	$m+1$ *odd*
min problem	$\{j, j+1, ..., k, k+1\}$	$\{0, j, j+1, ..., k, k+1\}$
max problem	$\{0, j, j+1, ..., k, k+1, n\}$	$\{j, j+1, ..., k, k+1, n\}$,

where in all parentheses the numbers are arranged in increasing order.

The following theorem is a conclusion of the dual feasible basis structure theorem by Prékopa (1988, 1989a, 1989b, 1990) where $r = 1$.

Theorem 2 Suppose that $f_0 = 0$, $f_i = 1$, $i = 1, ..., n$. Then every dual feasible basis subscript set has one of the following structures:

minimization problem, $m+1$ *even*

•$\{0, i, i+1, ..., j, j+1, k, k+1, ..., t, t+1, n\}$;

minimization problem, $m+1$ *odd*

•$\{0, i, i+1, ..., j, j+1, k, k+1, ..., t, t+1\}$;

maximization problem, $m+1$ *even*

•$I \subset \{1, ..., n\}$, $if \;\; n-1 \geq m$,

•$\{1, i, i+1, ..., j, j+1, k, k+1, ..., t, t+1, n\}$, $if \;\; 1 \leq n-1$,

•$\{0, 1, i, i+1, ..., j, j+1, k, k+1, ..., t, t+1\}$, $if \;\; 1 \leq n$;

maximization problem, $m+1$ *even*
$\bullet I \subset \{1,...,n\}, \quad if \quad n-1 \geq m,$
$\bullet \{1,i,i+1,...,j,j+1,k,k+1,...,t,t+1\}, \quad if \quad 1 \leq n,$
$\bullet \{0,1,i,i+1,...,j,j+1,k,k+1,...,t,t+1,n\}, \quad if \quad 1 \leq n-1,$ *where in all parentheses the numbers are arranged in increasing order. Those bases for which* $I \subset \{1,...,n\}$ *are dual nondegenerate in the maximization problem if* $n > m+1$. *The bases in all other cases are dual nondegenerate.*

3 The Case of the Power Moment Problem

In this section we consider the power moment problem (1.1). We assume that the distribution is either decreasing or increasing or unimodal with a known and fixed modus. We give sharp lower and upper bounds for $E[f(\xi)]$ in case of three problem types: the probabilities $p_0,...,p_n$ are (1) decreasing, (2) increasing; (3) form a unimodal sequence.

3.1 TYPE 1: $p_0 \geq p_1 \geq ... \geq p_n$

We assume that the probabilities $p_0,...,p_n$ are unknown but satisfy the above inequalities and the divided differences of order $m+1$ of the function f are positive. Let us introduce the variables $v_i, \ i=0,1,...,n$ as follows:

$$p_0 - p_1 = v_0, \ ..., \ p_{n-1} - p_n = v_{n-1}, \ p_n = v_n.$$

Our assumptions imply $v_0, \ ..., \ v_n \geq 0$. If we write up problem (1.1) by the use of $v_0,...,v_n$, then we obtain

$$min(max)\{f_0v_0 + (f_0+f_1)v_1 + ... + (f_0+...+f_n)v_n\}$$

subject to

$$a_0v_0 + (a_0+a_1)v_1 + ... + (a_0+...+a_n)v_n = b \tag{3.1}$$

$$v_0 \geq 0, v_1 \geq 0, ..., v_n \geq 0 \ ,$$

where $a_i = (1, z_i, ..., z_i^m)^T, \quad i=0,...,n$ and $b = (1, \mu_1, ..., \mu_m)^T$.

Let A be the $(m+1) \times n$ coefficient matrix. In order to use Theorem 1, we need to show that all minors of order $m+1$ from A and all minors of order $m+2$ from $\begin{pmatrix} f^T \\ A \end{pmatrix}$ are positive, where $f = (f_0, \ f_0+f_1, ..., f_0+...+f_n)^T$. To show that the first assertion is true we consider the $(m+1) \times (m+1)$ determinant taken from A:

$$\left| \begin{array}{cccc} a_0+...+a_{i_1} & a_0+...+a_{i_2} & \dots & a_0+...+a_{i_{m+1}} \end{array} \right|, \tag{3.2}$$

where $0 \leq i_1 < ... < i_{m+1} \leq n$. One can easily show that the determinant (3.2) is equal to

$$\left| \begin{array}{cccc} a_0+...+a_{i_1} & a_{i_1+1}+...+a_{i_2} & \dots & a_{i_m+1}+...+a_{i_{m+1}} \end{array} \right|$$

$$= \begin{vmatrix} i_1+1 & i_2-i_1-1 & \dots & i_{m+1}-i_m-1 \\ z_0+...+z_{i_1} & z_{i_1+1}+...+z_{i_2} & \dots & z_{i_m+1}+...+z_{i_{m+1}} \\ \vdots & \vdots & \ddots & \vdots \\ z_0^m+...+z_{i_1}^m & z_{i_1+1}^m+...+z_{i_2}^m & \dots & z_{i_m+1}^m+...+z_{i_{m+1}}^m \end{vmatrix}. \tag{3.3}$$

The determinant (3.43) can be written as a sum of nonnegative determinants. Notice that some of these terms are Vandermonde determinants. Thus, any $(m+1)\times(m+1)$ determinant taken from the matrix of the equality constraints is positive.

Similarly, the following $(m+2)\times(m+2)$ determinants taken from $\begin{pmatrix} f^T \\ A \end{pmatrix}$ are positive, where f is the same as before:

$$\begin{vmatrix} f_0+...+f_{i_1} & f_0+...+f_{i_2} & \dots & f_0+...+f_{i_{m+1}} \\ a_0+...+a_{i_1} & a_0+...+a_{i_2} & \dots & a_0+...+a_{i_{m+1}} \end{vmatrix} \tag{3.4}$$

and this is same as

$$\begin{vmatrix} f_0+...+f_{i_1} & f_{i_1+1}+...+f_{i_2} & \dots & f_{i_m+1}+...+f_{i_{m+1}} \\ a_0+...+a_{i_1} & a_{i_1+1}+...+a_{i_2} & \dots & a_{i_m+1}+...+a_{i_{m+1}} \end{vmatrix}$$

$$= \begin{vmatrix} f_0+...+f_{i_1} & f_{i_1+1}+...+f_{i_2} & \dots & f_{i_m+1}+...+f_{i_{m+1}} \\ i_1+1 & i_2-i_1-1 & \dots & i_{m+1}-i_m-1 \\ z_0+...+z_{i_1} & z_{i_1+1}+...+z_{i_2} & \dots & z_{i_m+1}+...+z_{i_{m+1}} \\ \vdots & \vdots & \ddots & \vdots \\ z_0^m+...+z_{i_1}^m & z_{i_1+1}^m+...+z_{i_2}^m & \dots & z_{i_m+1}^m+...+z_{i_{m+1}}^m \end{vmatrix}. \tag{3.5}$$

Since f has positive divided differences of order $m+1$, the determinant (3.5) is positive. Thus, we can use Theorem 1 to find the dual feasible bases in problem (3.1).

Below we present, by the use of Theorem 1, the sharp lower and upper bounds for $E[f(\xi)]$ for the case of $m=1$ and $m=2$, respectively.

***Case* 1.** $m=1$

If $m=1$, then the LP in (1.1) is equivalent to

$$min(max)\{f_0v_0+(f_0+f_1)v_1+...+(f_0+...+f_n)v_n\}$$

subject to

$$\begin{aligned} v_0+2v_1+...+(n+1)v_n &= 1 \\ z_0v_0+(z_0+z_1)v_1+...+(z_0+...+z_n)v_n &= \mu_1 \\ v_0\geq 0, v_1\geq 0, ..., v_n &\geq 0. \end{aligned} \tag{3.6}$$

Let A be the coefficient matrix of the equality constraints in (3.6). Since $m+1$ is even, by Theorem1, the dual feasible basis of the minimization problem in (3.8), that we designate by B_{min}, has the following form:

$$B_{min} = (a_j,\ a_{j+1}),$$

where $0 \le j \le n-1$.

Similarly, by Theorem 1, the only dual feasible basis of the maximization problem in (3.8), that we designate by B_{max}, is given by

$$B_{max} = (a_0,\ a_n).$$

First we notice that B_{min} is also primal feasible if the following conditions are satisfied:

$$(j+1)v_j + (j+2)v_{j+1} = 1$$

$$(z_0 + ... + z_j)v_j + (z_0 + ... + z_{j+1})v_{j+1} = \mu_1$$
$$v_j \ge 0, \quad v_{j+1} \ge 0 \ .$$

Therefore, by solving the equations given above, B_{min} is an optimal basis if the index j is determined by the inequality

$$\frac{z_0 + ... + z_j}{j+1} \le \mu_1 \le \frac{z_0 + ... + z_{j+1}}{j+2} \ . \tag{3.7}$$

In the maximization problem (3.6) there is just one dual feasible basis. It follows that it must also be primal feasible.

In case of $m = 1$ the bounding of $E[f(\xi)]$ is based on the knowledge of μ_1. Using the optimal bases B_{min} and B_{max}, we obtain the following sharp lower and upper bounds for $E[f(\xi)]$:

$$\frac{\sum_{i=0}^{j+1} z_i - (j+2)\mu_1}{(j+1)z_{j+1} - \sum_{i=0}^{j} z_i}[f_0 + ... + f_j] + \frac{(j+1)\mu_1 - \sum_{i=0}^{j} z_i}{(j+1)z_{j+1} - \sum_{i=0}^{j} z_i}[f_0 + ... + f_{j+1}]$$
$$\le\ E[f(\xi)]\ \le$$
$$\frac{(n+1)\mu_1 - \sum_{i=0}^{n} z_i}{(n+1)z_0 - \sum_{i=0}^{n} z_i} f_0 + \frac{\mu_1 - z_0}{(n+1)z_0 - \sum_{i=0}^{n} z_i}[f_0 + ... + f_n]\ , \tag{3.8}$$

where j satisfies (3.7).

***Case* 2.** $m = 2$

If $m = 2$, then the third order divided differences of f are positive. The bounding of $E[f(\xi)]$ is based on the knowledge of μ_1 and μ_2.

In case of $m = 2$, problem (1.1) is equivalent to the following problem:

$$min(max)\{f_0 v_0 + (f_0 + f_1)v_1 + ... + (f_0 + ... + f_n)v_n\}$$

subject to

$$v_0 + 2v_1 + ... + (n+1)v_n = 1$$
$$z_0 v_0 + (z_0 + z_1)v_1 + ... + (z_0 + ... + z_n)v_n = \mu_1 \tag{3.9}$$
$$z_0^2 v_0 + (z_0^2 + z_1^2)v_1 + ... + (z_0^2 + ... + z_n^2)v_n = \mu_2$$

$$v_0 \geq 0, v_1 \geq 0, ..., v_n \geq 0 \ .$$

Let A be the coefficient matrix of the equality constraints in (3.9). Since $m+1$ is odd, any dual feasible basis in the minimization (maximization) problem is of the form:

$$B_{min} = (a_0, a_i, a_{i+1}) \quad (B_{max} = (a_j, a_{j+1}, a_n))\,.$$

Let us introduce the notations:

$$\Sigma_{i,j}^2 = i\sum_{t=i}^{j} z_t^2 - (j-i+1)\sum_{t=0}^{i-1} z_t^2$$

$$\Sigma_{i,j} = i\sum_{t=i}^{j} z_t - (j-i+1)\sum_{t=0}^{i-1} z_t$$

$$\sigma_{i,j}^2 = \sum_{t=i}^{j} z_t^2 - (j-i+1)z_{i-1}^2$$

$$\sigma_{i,j} = \sum_{t=i}^{j} z_t - (j-i+1)z_{i-1}$$

$$\gamma_{i,j}^2 = \sum_{t=i}^{j} z_t - (j-i+1)z_{j+1}^2$$

$$\gamma_{i,j} = \sum_{t=i}^{j} z_t - (j-i+1)z_{j+1}$$

$$\alpha_{i,j}^2 = (n-j)\sum_{t=i}^{j} z_t^2 - (j-i+1)\sum_{t=j+1}^{n} z_t^2$$

$$\alpha_{i,j} = (n-j)\sum_{t=i}^{j} z_t - (j-i+1)\sum_{t=j+1}^{n} z_t. \tag{3.10}$$

The basis B_{min} is also primal feasible if i is determined by the inequality:

$$\frac{iz_0^2 - \sum_{t=1}^{i} z_t^2}{iz_0 - \sum_{t=1}^{i} z_t} \leq \frac{\mu_2 - z_0^2}{\mu_1 - z_0} \leq \frac{(i+1)z_0^2 - \sum_{t=1}^{i+1} z_t^2}{(i+1)z_0 - \sum_{t=1}^{i+1} z_t}\ . \tag{3.11}$$

Similarly, the basis B_{max} is also primal feasible if j is determined by the inequality

$$\frac{\Sigma_{j+1,n}^2}{\Sigma_{j+1,n}} \leq \frac{(n+1)\mu_2 - \sum_{s=0}^{n} z_s^2}{(n+1)\mu_1 - \sum_{s=0} z_s} \leq \frac{\Sigma_{j+2,n}^2}{\Sigma_{j+2,n}}\ . \tag{3.12}$$

Using the bases B_{min} and B_{max} we have the following lower and upper bounds for $E[f(\xi)]$:

$$\frac{(\mu_1 - z_0)\sigma_{1,i+1}^2 - (\mu_2 - z_0^2)\sigma_{1,i+1}}{\sigma_{1,i+1}^2\sigma_{1,i} - \sigma_{1,i+1}\sigma_{1,i}^2}(\sum_{t=1}^{i} f_t - if_0) + \frac{(\mu_2 - z_0^2)\sigma_{1,i} - (\mu_1 - z_0)\sigma_{1,i}^2}{\sigma_{1,i+1}^2\sigma_{1,i} - \sigma_{1,i+1}\sigma_{1,i}^2}(\sum_{t=1}^{i+1} f_t - (i+1)f_0)$$

$$\leq E[f(\xi)] \leq \tag{3.13}$$

$$\frac{\Sigma_{j+2,n}^2\left[(n+1)\mu_1 - \sum_{s=0}^n z_s\right] - \Sigma_{j+2,n}\left[(n+1)\mu_2 - \sum_{s=0}^n z_s^2\right]}{\Sigma_{j+1,n}\Sigma_{j+2,n}^2 - \Sigma_{j+2,n}\Sigma_{j+1,n}^2}\left(\frac{n\sum_{s=0}^j f_s - \sum_{s=j+1}^n f_s}{n+1}\right)$$

$$+\frac{\Sigma_{j+1,n}\left[(n+1)\mu_2 - \sum_{s=0}^n z_s^2\right] - \Sigma_{j+1,n}^2\left[(n+1)\mu_1 - \sum_{s=0}^n z_s\right]}{\Sigma_{j+1,n}\Sigma_{j+2,n}^2 - \Sigma_{j+2,n}\Sigma_{j+1,n}^2}\left(\frac{n\sum_{s=0}^{j+1} f_s - \sum_{s=j+2}^n f_s}{n+1}\right) .$$

The above lower and upper bounds are sharp, because they are the optimum values of minimization and maximization problems (3.9), respectively.

3.2 TYPE 2: $p_0 \leq p_1 \leq ... \leq p_n$

We assume that the above inequalities are satisfied and f has positive divided differences of order $m+1$. If we introduce the new variables $v_i,\ i = 0, 1..., n$ as follows:

$$p_0 = v_0,\ p_1 - p_0 = v_1,\ ...,\ p_n - p_{n-1} = v_n,$$

then problem (1.1) can be written as

$$min(max)\{(f_0 + ... + f_n)v_0 + (f_1 + ... + f_n)v_1 + ... + f_n v_n\}$$

subject to

$$(a_0 + ... + a_n)v_0 + (a_1 + ... + a_n)v_1 + ... + a_n v_n = b \tag{3.14}$$

$$v_0 \geq 0, v_1 \geq 0, ..., v_n \geq 0 .$$

All minors of order $m+1$ from A and all minors of order $m+2$ from $\begin{pmatrix} f^T \\ A \end{pmatrix}$ are positive, where A is the coefficient matrix of the equality constraints in problem (3.14) and $f = (f_0 + ... + f_n, f_1 + ... + f_n, ..., f_n)^T$. In order to show that the first assertion is true we consider the following $(m+1) \times (m+1)$ determinant taken from A ($0 \leq i_1, ..., i_{n+1} \leq n$):

$$\left|\ a_{i_1} + ... + a_n \quad a_{i_2} + ... + a_n \quad \ldots \quad a_{i_{m+1}} + ... + a_n\ \right| \tag{3.15}$$

One can easily show that the determinant (3.15) is equal to

$$\left|\ a_{i_1} + ... + a_{i_2 - 1} \quad a_{i_2} + ... + a_{i_3 - 1} \quad \ldots \quad a_{i_{m+1}} + ... + a_n\ \right|$$

$$= \begin{vmatrix} i_2 - i_1 & i_3 - i_2 & \dots & i_n - i_{m+1} - 1 \\ z_{i_1} + ... + z_{i_2 - 1} & z_{i_2} + ... + z_{i_3 - 1} & \dots & z_{i_{m+1}} + ... + z_n \\ \vdots & \vdots & \ddots & \vdots \\ z_{i_1}^m + ... + z_{i_2 - 1}^m & z_{i_2}^m + ... + z_{i_3 - 1}^m & \dots & z_{i_{m+1}}^m + ... + z_n^m \end{vmatrix} . \quad (3.16)$$

The determinant (3.16) can be written as a sum of nonnegative determinants. Notice that some of the terms are Vandermonde determinants. Thus any $(m+1) \times (m+1)$ determinant taken from the matrix of the equality constraints is positive.

We can handle, in a similar way, the following $(m+2) \times (m+2)$ determinant taken from $\begin{pmatrix} f^T \\ A \end{pmatrix}$:

$$\begin{vmatrix} f_{i_1} + ... + f_n & f_{i_2} + ... + f_n & \dots & f_{i_{m+1}} + ... + f_n \\ a_{i_1} + ... + a_n & a_{i_2} + ... + a_n & \dots & a_{i_{m+1}} + ... + a_n \end{vmatrix} . \quad (3.17)$$

It is equal to

$$\begin{vmatrix} f_{i_1} + ... + f_{i_2 - 1} & f_{i_2} + ... + f_{i_3 - 1} & \dots & f_{i_{m+1}} + ... + f_n \\ a_{i_1} + ... + a_{i_2 - 1} & a_{i_2} + ... + a_{i_3 - 1} & \dots & a_{i_{m+1}} + ... + a_n \end{vmatrix}$$

$$= \begin{vmatrix} f_{i_1} + ... + f_{i_2 - 1} & f_{i_2} + ... + f_{i_3 - 1} & \dots & f_{i_{m+1}} + ... + f_n \\ i_2 - i_1 & i_3 - i_2 & \dots & i_n - i_{m+1} - 1 \\ z_{i_1} + ... + z_{i_2 - 1} & z_{i_2} + ... + z_{i_3 - 1} & \dots & z_{i_{m+1}} + ... + z_n \\ \vdots & \vdots & \ddots & \vdots \\ z_{i_1}^m + ... + z_{i_2 - 1}^m & z_{i_2}^m + ... + z_{i_3 - 1}^m & \dots & z_{i_{m+1}}^m + ... + z_n^m \end{vmatrix} . \quad (3.18)$$

Since f has positive divided differences of order $m+1$, the determinant (3.18) is positive. Thus, we can use Theorem 1 to find the dual feasible bases.

We give the sharp lower and upper bounds for $E[f(\xi)]$ in case of $m = 1$ and $m = 2$.

***Case* 1.** $m = 1$

If $m = 1$, then problem (3.14) is equivalent to

$$min(max)\{(f_0 + ... + f_n)v_0 + (f_1 + ... + f_n)v_1 + ... + f_n v_n\}$$

subject to

$$\begin{gathered} (n+1)v_0 + nv_1 + ... + v_n = 1 \\ (z_0 + ... + z_n)v_0 + (z_1 + ... + z_n)v_1 + ... + z_n v_n = \mu_1 \\ v_0 \geq 0, v_1 \geq 0, ..., v_n \geq 0 \ . \end{gathered} \quad (3.19)$$

Let A be the coefficient matrix of the equality constraints in (3.19). Since $m+1$ is even, by Theorem 1, the dual feasible bases in the minimization and maximization problems in (3.19) are

$$B_{min} = (a_j, \ a_{j+1}), \quad B_{max} = (a_0, \ v_n),$$

respectively, where $0 \leq j \leq n-1$.

The basis B_{min} is also primal feasible if j is determined by the inequality

$$\frac{z_j + ... + z_n}{n-j+1} \le \mu_1 \le \frac{z_{j+1} + ... + z_n}{n-j} \ . \tag{3.20}$$

The basis B_{max} is the only dual feasible basis, hence it must also be primal feasible.

In case of $m = 1$ the bounds of $E[f(\xi)]$ are based on the knowledge of μ_1 and we get the following sharp lower and upper bounds for $E[f(\xi)]$:

$$\frac{\sum_{t=j+1}^{n} z_t - (n-j)\mu_1}{\sigma_{j+1,n}}[f_j + ... + f_n] - \frac{\sum_{t=j}^{n} z_t - (n-j+1)\mu_1}{\sigma_{j+1,n}}[f_{j+1} + ... + f_n]$$
$$\le \ E[f(\xi)] \ \le$$
$$\frac{\mu_1 - z_n}{\gamma_{0,n-1}}[f_0 + ... + f_n] + \frac{(n+1)\mu_1 - \sum_{t=0}^{n} z_t}{\gamma_{0,n-1}} f_n \ . \tag{3.21}$$

The bounds are sharp if j satisfies (3.20).

***Case* 2.** $m = 2$

If $m = 2$, then the third order divided differences of f are positive and problem (3.14) is equivalent to

$$min(max)\{(f_0 + ... + f_n)v_0 + (f_1 + ... + f_n)v_1 + ... + f_n v_n\}$$

subject to

$$\begin{aligned} (n+1)v_0 + nv_1 + ... + v_n &= 1 \\ (z_0 + ... + z_n)v_0 + (z_1 + ... + z_n)v_1 + ... + z_n v_n &= \mu_1 \\ (z_0^2 + ... + z_n^2)v_0 + (z_1^2 + ... + z_n^2)v_1 + ... + z_n^2 v_n &= \mu_2 \\ v_0 \ge 0, v_1 \ge 0, ..., v_n \ge 0 \ . \end{aligned} \tag{3.22}$$

The bounding of $E[f(\xi)]$ is based on the knowledge of μ_1 and μ_2.

Let A be the coefficient matrix of the equality constraints in (3.22). Since $m+1$ is odd, any dual feasible bases in the minimization and maximization problems are of the form

$$B_{min} = (a_0, a_i, a_{i+1}\}, \quad B_{max} = (a_j, a_{j+1}, a_n),$$

where $1 \le i \le n-1, \quad 0 \le j \le n-2$.

B_{min} is also primal feasible if

$$\frac{\Sigma_{i,n}^2}{\Sigma_{i,n}} \le \frac{(n+1)\mu_2 - \sum_{t=0}^{n} z_t^2}{(n+1)\mu_1 - \sum_{t=0}^{n} z_t} \le \frac{\Sigma_{i+1,n}^2}{\Sigma_{i+1,n}} \tag{3.23}$$

and B_{max} is also primal feasible if

$$\frac{\gamma_{j,n-1}^2}{\gamma_{j,n-1}} \le \frac{\mu_2 - z_n^2}{\mu_1 - z_n} \le \frac{\gamma_{j+1,n-1}^2}{\gamma_{j+1,n-1}} \ . \tag{3.24}$$

It follows that we have the following lower and upper bounds for $E[f(\xi)]$:

$$\frac{[(n+1)\mu_1 - \sum_{t=0}^n z_t]\Sigma_{i+1,n}^2 - [(n+1)\mu_2 - \sum_{t=0}^n z_t^2]\Sigma_{i+1,n}}{\Sigma_{i,n}\Sigma_{i+1,n}^2 - \Sigma_{i,n}^2\Sigma_{i+1,n}}\left[f_i + ... + f_n - (n-i+1)\sum_{t=0}^n f_t\right]$$

$$+\frac{[(n+1)\mu_2 - \sum_{t=0}^n z_t^2]\Sigma_{i,n} - [(n+1)\mu_1 - \sum_{t=0}^n z_t]\Sigma_{i,n}^2}{\Sigma_{i,n}\Sigma_{i+1,n}^2 - \Sigma_{i,n}^2\Sigma_{i+1,n}}\left[f_{i+1} + ... + f_n - (n-i)\sum_{t=0}^n f_t\right]$$

$$\leq E[f(\xi)] \leq \tag{3.25}$$

$$\frac{(\mu_1 - z_n)\gamma_{j+1,n-1}^2 - (\mu_2 - z_n^2)\gamma_{j+1,n-1}}{\gamma_{j,n-1}\gamma_{j+1,n-1}^2 - \gamma_{j+1,n-1}\gamma_{j,n-1}^2}\left[\sum_{s=j}^n f_s - (n-j+1)f_n\right]$$

$$+\frac{(\mu_2 - z_n^2)\gamma_{j,n-1} - (\mu_1 - z_n)\gamma_{j,n-1}^2}{\gamma_{j,n-1}\gamma_{j+1,n-1}^2 - \gamma_{j+1,n-1}\gamma_{j,n-1}^2}\left[\sum_{s=j+1}^n f_s - (n-j)f_n\right] ,$$

where $\Sigma_{i,j},\ \Sigma_{i,j}^2,\ \gamma_{i,j},\ \gamma_{i,j}^2$ are defined in (3.10) and i satisfies (3.23) and j satisfies (3.24).

3.3 TYPE 3: $p_0 \leq p_1 \leq ... \leq p_k \geq p_{k+1} \geq ... \geq p_n$

Now we assume that the distribution is unimodal with a known modus z_k, i.e., $p_0 \leq p_1 \leq ... \leq p_k \geq p_{k+1} \geq ... \geq p_n$. We also assume that f has positive divided differences of order $m+1$. Let us introduce the variables $v_i,\ i = 0, 1, ..., k, ...n$ as follows:

$$v_0 = p_0,\ v_1 = p_1 - p_0,\ ...,\ v_k = p_k - p_{k-1},$$

$$v_{k+1} = p_{k+1} - p_{k+2},\ ...,\ v_{n-1} = p_{n-1} - p_n,\ v_n = p_n\ .$$

Then the problem (1.1) can be written as

$$min(max)\{(f_0 + ... + f_k)v_0 + (f_1 + ... + f_k)v_1 + ... + f_k v_k + f_{k+1}v_{k+1} + ... + (f_{k+1} + ... + f_n)v_n\}$$

subject to

$$(a_0 + ... + a_k)v_0 + (a_1 + ... + a_k)v_1 + ... + a_k v_k + a_{k+1}v_{k+1} + ... + (a_{k+1} + ... + a_n)v_n = b$$

$$v_0 \geq 0, v_1 \geq 0, ..., v_n \geq 0. \tag{3.26}$$

Let A be the coefficient matrix of the equality constraints of problem (3.26). One can easily show, by the method that we applied in Section 3.1 and 3.2, that all minors of order

$m+1$ from A and all minors of order $m+2$ from $\begin{pmatrix} f^T \\ A \end{pmatrix}$ are positive, where $f = (f_0 + ... + f_k, f_1 + ... + f_k, ..., f_k, f_{k+1}, ..., f_{k+1} + ... + f_n)^T$.

Thus, if the distribution is unimodal and we know where it takes the largest value, then we can find the dual feasible bases for problem (1.1). This allows for obtaining the bounding formulas for small m values.

***Case* 1.** $m = 1$

If $m = 1$, then the problem in (3.26) is equivalent to the problem:

$$min(max)\{(f_0 + ... + f_k)v_0 + (f_1 + ... + f_k)v_1 + ... + f_k v_k + f_{k+1}v_{k+1} + ... + (f_{k+1} + ... + f_n)v_n\}$$

subject to

$$(k+1)v_0 + kv_1 + ... + v_k + v_{k+1} + 2v_{k+2} + ... + (n-k)v_n = 1$$

$$(z_0 + ... + z_k)v_0 + (z_1 + ... + z_k)v_1 + ... + z_k v_k + z_{k+1}v_{k+1} + ... + (z_{k+1} + z_n)v_n = \mu_1$$

$$v_0 \geq 0, v_1 \geq 0, ..., v_n \geq 0 \ . \tag{3.27}$$

Let A be the coefficient matrix of the equality constraints in (3.27). Since $m+1$ is even, by the use of Theorem 1, a dual feasible basis of the minimization problem in (3.27) is in the form

$$B_{min} = (a_j, a_{j+1}),$$

where $0 \leq j \leq n-1$.

The only dual feasible basis of the maximization problem (3.27) is

$$B_{max} = (a_0, a_n) \ .$$

B_{min} is also primal feasible if

$$\frac{\sum_{t=j}^{k} z_t}{k-j+1} \leq \mu_1 \leq \frac{\sum_{t=j+1}^{k} z_t}{k-j} \qquad \text{if} \quad j+1 \leq k \tag{3.28}$$

$$\frac{\sum_{t=k+1}^{j} z_t}{j-k} \leq \mu_1 \leq \frac{\sum_{t=k+1}^{j+1} z_t}{j-k+1} \qquad \text{if} \quad j \geq k+1 \tag{3.29}$$

$$z_k \leq \mu_1 \leq z_{k+1} \qquad \text{if} \quad j = k. \tag{3.30}$$

If $j+1 \leq k$ and (3.28) is satisfied, then the sharp lower and upper bounds for $E[f(\xi)]$ are as follows:

$$\frac{\sum_{t=j+1}^{k} z_t - (k-j)\mu_1}{(k-j+1)\sum_{t=j+1}^{k} z_t - (k-j)\sum_{t=j}^{k} z_t} \sum_{t=j}^{k} f_t + \frac{(k-j+1)\mu_1 - \sum_{t=j}^{k} z_t}{(k-j+1)\sum_{t=j+1}^{k} z_t - (k-j)\sum_{t=j}^{k} z_t} \sum_{t=j+1}^{k} f_t$$

$$\leq \ E[f(\xi)] \ \leq \tag{3.31}$$

$$\frac{\sum_{t=k+1}^{n} z_t-(n-k)\mu_1}{\Sigma_{k+1,n}}\sum_{t=0}^{k} f_t+\frac{(k+1)\mu_1-\sum_{t=0}^{k} z_t}{\Sigma_{k+1,n}}\sum_{t=k+1}^{n} f_t \ .$$

If $j \geq k+1$ and (3.29) is satisfied, then the sharp lower and upper bounds for $E[f(\xi)]$ are as follows:

$$\frac{\sum_{t=k+1}^{j+1} z_t-(j-k+1)\mu_1}{(j-k)\sum_{t=k+1}^{j+1} z_t-(j-k+1)\sum_{t=k+1}^{j} z_t}\sum_{t=k+1}^{j} f_t+\frac{(j-k)\mu_1-\sum_{t=k+1}^{j} z_t}{(j-k)\sum_{t=k+1}^{j+1} z_t-(j-k+1)\sum_{t=k+1}^{j} z_t}\sum_{t=k+1}^{j+1} f_t$$

$$\leq \ E[f(\xi)] \ \leq \tag{3.32}$$

$$\frac{\sum_{t=k+1}^{n} z_t-(n-k)\mu_1}{\Sigma_{k+1,n}}\sum_{t=0}^{k} f_t+\frac{(k+1)\mu_1-\sum_{t=0}^{k} z_t}{\Sigma_{k+1,n}}\sum_{t=k+1}^{n} f_t \ .$$

If $j = k$ and (3.30) is satisfied, then the sharp lower and upper bounds for $E[f(\xi)]$ are the following:

$$\frac{z_{k+1}-\mu_1}{z_{k+1}-z_k}f_k+\frac{\mu_1-z_k}{z_{k+1}-z_k}f_{k+1}$$

$$\leq \ E[f(\xi)] \ \leq \tag{3.33}$$

$$\frac{\sum_{t=k+1}^{n} z_t-(n-k)\mu_1}{\Sigma_{k+1,n}}\sum_{t=0}^{k} f_t+\frac{(k+1)\mu_1-\sum_{t=0}^{k} z_t}{\Sigma_{k+1,n}}\sum_{t=k+1}^{n} f_t \ ,$$

where $\Sigma_{i,j}$ is given in (3.10).

Case 2. $m = 2$

In case of $m = 2$, problem in (3.26) is equivalent to the problem:

$$min(max)\{(f_0+...+f_k)v_0+(f_1+...+f_k)v_1+...+f_kv_k+f_{k+1}v_{k+1}+...+(f_{k+1}+...+f_n)v_n\}$$

subject to

$$\begin{aligned}
(k+1)v_0+kv_1+...+v_k+v_{k+1}+2v_{k+2}+...+(n-k)v_n&=1\\
(z_0+...+z_k)v_0+(z_1+...+z_k)v_1+...+z_kv_k+z_{k+1}v_{k+1}+...+(z_{k+1}+z_n)v_n&=\mu_1\\
(z_0^2+...+z_k^2)v_0+(z_1^2+...+z_k^2)v_1+...+z_k^2v_k+z_{k+1}^2v_{k+1}+...+(z_{k+1}^2+z_n^2)v_n&=\mu_2\\
v_0\geq 0, v_1\geq 0, ..., v_n\geq 0 \ .
\end{aligned} \tag{3.34}$$

The third order divided differences of f are positive. The bounds of $E[f(\xi)]$ are based on the knowledge of μ_1 and μ_2.

Let A be the coefficient matrix of the equality constraints in (3.32). Since $m+1$ is odd, any dual feasible bases in the minimization and in the maximization problems are of the form

$$B_{min} = (a_0, a_i, a_{i+1}) \quad \text{and} \quad B_{max} = (a_j, a_{j+1}, a_n),$$

where $1 \le i \le n-1, \quad 0 \le j \le n-2$.

The basis B_{min} is also primal feasible if one of the following conditions is satisfied:

$$\frac{\Sigma^2_{i,k}}{\Sigma_{i,k}} \le \frac{(k+1)\mu_2 - \sum_{t=0}^k z_t^2}{(k+1)\mu_1 - \sum_{t=0}^k z_t} \le \frac{\Sigma^2_{i+1,k}}{\Sigma_{i+1,k}}, \qquad i+1 \le k \tag{3.35}$$

$$\frac{\Sigma^2_{k+1,i}}{\Sigma_{k+1,i}} \le \frac{(k+1)\mu_2 - \sum_{t=0}^k z_t^2}{(k+1)\mu_1 - \sum_{t=0}^k z_t} \le \frac{\Sigma^2_{k+1,i+1}}{\Sigma_{k+1,i+1}}, \qquad i \ge k+1 \tag{3.36}$$

$$\frac{\gamma^2_{0,k-1}}{\gamma_{0,k-1}} \le \frac{(k+1)\mu_2 - \sum_{t=0}^k z_t^2}{(k+1)\mu_1 - \sum_{t=0}^k z_t} \le \frac{\gamma^2_{0,k}}{\gamma_{0,k}}, \qquad i = k, \tag{3.37}$$

where $\Sigma_{i,j}, \ \Sigma^2_{i,j}, \ \gamma_{i,j}, \ \gamma^2_{i,j}$ are defined in (3.10).

The basis B_{max} is also primal feasible if one of the following conditions is satisfied:

$$\frac{\alpha^2_{j,k}}{\alpha_{j,k}} \le \frac{(n-k)\mu_2 - \sum_{t=k+1}^n z_t^2}{(n-k)\mu_1 - \sum_{t=k+1}^n z_t} \le \frac{\alpha^2_{j+1,k}}{\alpha_{j+1,k}}, \qquad j+1 \le k \tag{3.38}$$

$$\frac{\alpha^2_{k+1,j}}{\alpha_{k+1,j}} \le \frac{(n-k)\mu_2 - \sum_{t=k+1}^n z_t^2}{(n-k)\mu_1 - \sum_{t=k+1}^n z_t} \le \frac{\alpha^2_{k+1,j+1}}{\alpha_{k+1,j+1}}, \qquad j \ge k \tag{3.39}$$

$$\frac{\sigma^2_{k+1,n}}{\sigma_{k+1,n}} \le \frac{(n-k)\mu_2 - \sum_{t=k+1}^n z_t^2}{(n-k)\mu_1 - \sum_{t=k+1}^n z_t} \le \frac{\sigma^2_{k+2,n}}{\sigma_{k+2,n}}, \qquad j = k, \tag{3.40}$$

where $\sigma_{i,j}, \ \sigma^2_{i,j}, \ \alpha_{i,j}, \ \alpha^2_{i,j}$ are defined in (3.10).

We have the following sharp lower bound for $E[f(\xi)]$:

- If $i+1 \le k$,

$$\frac{1}{k+1}\sum_{t=0}^k f_t$$

$$+\frac{\Sigma^2_{i+1,k}[(k+1)\mu_1 - \sum_{t=0}^k z_t] - \Sigma_{i+1,k}[(k+1)\mu_2 - \sum_{t=0}^k z_t^2]}{\Sigma_{i,k}\Sigma^2_{i+1,k} - \Sigma_{i+1,k}\Sigma^2_{i,k}} \left[\sum_{t=i}^k f_t - \frac{\sum_{t=0}^k f_t}{k+1}\right)]$$

$$+\frac{\Sigma_{i,k}[(k+1)\mu_2 - \sum_{t=0}^k z_t^2] - \Sigma^2_{i,k}[(k+1)\mu_1 - \sum_{t=0}^k z_t]}{\Sigma_{i,k}\Sigma^2_{i+1,k} - \Sigma_{i+1,k}\Sigma^2_{i,k}} \left[\sum_{t=i+1}^k f_t - \frac{\sum_{t=0}^k f_t}{k+1}\right].$$

(3.41)

- If $i \geq k+1$,

$$\frac{1}{k+1}\sum_{t=0}^{k} f_t$$

$$+\frac{\Sigma_{k+1,i+1}^2[(k+1)\mu_1 - \sum_{t=0}^{k} z_t] - \Sigma_{k+1,i+1}[(k+1)\mu_2 - \sum_{t=0}^{k} z_t^2]}{\Sigma_{k+1,i}\Sigma_{k+1,i+1}^2 - \Sigma_{k+1,i}^2\Sigma_{k+1,i+1}}\left[\sum_{t=k+1}^{i} f_t - \frac{(i-k)\sum_{t=0}^{k} f_t}{k+1}\right]$$

$$+\frac{\Sigma_{k+1,i}[(k+1)\mu_2 - \sum_{t=0}^{k} z_t^2] - \Sigma_{k+1,i+1}^2[(k+1)\mu_1 - \sum_{t=0}^{k} z_t]}{\Sigma_{k+1,i}\Sigma_{k+1,i+1}^2 - \Sigma_{k+1,i}^2\Sigma_{k+1,i+1}}\left[\sum_{t=k+1}^{i+1} f_t - \frac{(i-k+1)\sum_{t=0}^{k} f_t}{k+1}\right].$$

(3.42)

- If $i = k$,

$$\frac{1}{k+1}\sum_{t=0}^{k} f_t$$

$$+\frac{\gamma_{0,k}^2[(k+1)\mu_1 - \sum_{t=0}^{k} z_t] - \gamma_{0,k}[(k+1)\mu_2 - \sum_{t=0}^{k} z_t^2]}{\gamma_{0,k-1}\gamma_{0,k}^2 - \gamma_{0,k}\gamma_{0,k-1}^2}\left[f_k - \frac{\sum_{t=0}^{k} f_t}{k+1}\right]$$

$$+\frac{\gamma_{0,k-1}[(k+1)\mu_2 - \sum_{t=0}^{k} z_t^2] - \gamma_{0,k-1}^2[(k+1)\mu_1 - \sum_{t=0}^{k} z_t]}{\gamma_{0,k-1}\gamma_{0,k}^2 - \gamma_{0,k}\gamma_{0,k-1}^2}\left[f_{k+1} - \frac{\sum_{t=0}^{k} f_t}{k+1}\right].$$

(3.43)

The sharp upper bound for $E[f(\xi)]$ can be given as follows:

- If $j+1 \leq k$,

$$\frac{1}{n-k}\sum_{t=k+1}^{n} f_t$$

$$+\frac{\alpha_{j+1,k}^2[(n-k)\mu_1 - \sum_{t=k+1}^{n} z_t] - \alpha_{j+1,k}[(n-k)\mu_2 - \sum_{t=k+1}^{n} z_t^2]}{\alpha_{j,k}\alpha_{j+1,k}^2 - \alpha_{j+1,k}\alpha_{j,k}^2}\left[\sum_{t=j}^{k} f_t - \frac{(k-j+1)\sum_{t=k+1}^{n}}{n-k}\right]$$

$$\frac{\alpha_{j,k}[(n-k)\mu_2 - \sum_{t=k+1}^{n} z_t^2] - \alpha_{j,k}^2[(n-k)\mu_1 - \sum_{t=k+1}^{n} z_t]}{\alpha_{j,k}\alpha_{j+1,k}^2 - \alpha_{j+1,k}\alpha_{j,k}^2}\left[\sum_{t=j+1}^{k} f_t - \frac{(k-j)\sum_{t=k+1}^{n}}{n-k}\right].$$

(3.44)

- If $j \geq k+1$,

$$\frac{1}{n-k}\sum_{t=k+1}^{n} f_t$$
$$+\frac{\alpha^2_{k+1,j+1}[(n-k)\mu_1 - \sum_{t=k+1}^{n} z_t] - \alpha_{k+1,j+1}[(n-k)\mu_2 - \sum_{t=k+1}^{n} z_t^2]}{\alpha_{k+1,j}\alpha^2_{k+1,j+1} - \alpha_{k+1,j+1}\alpha^2_{k+1,j}} \left[\sum_{t=k+1}^{j} f_t - \frac{(j-k)\sum_{t=k+1}^{n}}{n-k}\right]$$
$$+\frac{\alpha_{k+1,j}[(n-k)\mu_2 - \sum_{t=k+1}^{n} z_t^2] - \alpha^2_{k+1,j}[(n-k)\mu_1 - \sum_{t=k+1}^{n} z_t]}{\alpha_{k+1,j}\alpha^2_{k+1,j+1} - \alpha_{k+1,j+1}\alpha^2_{k+1,j}} \left[\sum_{t=k+1}^{j+1} f_t - \frac{(j-k+1)\sum_{t=k+1}^{n}}{n-k}\right] . \tag{3.45}$$

- If $j = k$,

$$\frac{1}{n-k}\sum_{t=k+1}^{n} f_t$$
$$+\frac{\sigma^2_{k+2,n}[(n-k)\mu_1 - \sum_{t=k+1}^{n} z_t] - \sigma_{k+2,n}[(n-k)\mu_2 - \sum_{t=k+1}^{n} z_t^2]}{\sigma_{k+1,n}\sigma^2_{k+2,n} - \sigma^2_{k+1,n}\sigma_{k+2,n}} \left[f_k - \frac{\sum_{t=k+1}^{n} f_t}{n-k}\right]$$
$$+\frac{\sigma_{k+1,n}[(n-k)\mu_2 - \sum_{t=k+1}^{n} z_t^2] - \sigma^2_{k+1,n}[(n-k)\mu_1 - \sum_{t=k+1}^{n} z_t]}{\sigma_{k+1,n}\sigma^2_{k+2,n} - \sigma^2_{k+1,n}\sigma_{k+2,n}} \left[f_{k+1} - \frac{\sum_{t=k+1}^{n} f_t}{n-k}\right] . \tag{3.46}$$

In all formulas given above, $\Sigma_{i,j}$, $\Sigma^2_{i,j}$, $\sigma_{i,j}$, $\sigma^2_{i,j}$, $\gamma_{i,j}$, $\gamma^2_{i,j}$, $\alpha_{i,j}$ and $\alpha^2_{i,j}$ are defined as in (3.10).

4 The Case of the Binomial Moment Problem

In case of the binomial moment problem (1.2) we look at the special case, where

$$z_i = i, \quad i = 0,\ ...,\ n\ , \quad f_0 = 0, \quad f_1 = ... = f_n = 1\ .$$

In case of $m = 2$ we give the sharp lower and upper bounds for the probability that at least $r = 1$ out of n events occur. We look at the problem (1.2) but the constraints are supplemented by shape constraints of the unknown probability distribution $p_0, ..., p_n$.

In the following three subsections we us the same shape constraints that we have used in Section 3.1-3.3.

4.1 TYPE 1: $p_0 \geq p_1 \geq ... \geq p_n$

Let $m = 2$. If we introduce the variables

$$v_0 = p_0 - p_1,\ \dots,\ v_{n-1} = p_{n-1} - p_n,\ v_n = p_n,$$

then problem (1.2) can be written as

$$min(max)\{v_1 + 2v_2 + ... + nv_n\}$$

subject to

$$v_0 + 2v_1 + 3v_2 + 4v_3 + ... + (n+1)v_n = 1$$

$$\binom{2}{2} v_1 + \binom{3}{2} v_2 + \binom{4}{2} v_3 + ... + \binom{n+1}{2} v_n = S_1$$

$$\binom{2}{2} v_2 + \left[\binom{2}{2} + \binom{3}{2}\right] v_3 + ... + \left[\binom{2}{2} + ... + \binom{n}{2}\right] v_n = S_2$$

$$v_0,\ \dots,\ v_n \geq 0\ . \tag{4.1}$$

Taking into account the equation:

$$1 + \binom{3}{2} + ... + \binom{k}{2} = \frac{(k-1)k(k+1)}{6}\ ,$$

the problem is the same as the following:

$$min(max) \sum_{i=1}^{n} = iv_i$$

subject to

$$\sum_{i=0}^{n} (i+1)v_i = 1$$

$$\sum_{i=0}^{n} (i+1)iv_i = 2S_1 \tag{4.2}$$

$$\sum_{i=0}^{n} (i+1)i(i-1)v_i = 6S_2$$

$$v_0, ..., v_n \geq 0\ .$$

Problem (4.2) is equivalent to the following:

$$min(max)\{v_1 + 2v_2 + ... + nv_n\}$$

subject to

$$
\begin{aligned}
v_0 + 2v_1 + 3v_2 + 4v_3 + ... + (n+1)v_n &= 1 \\
2v_1 + 6v_2 + 12v_3 + ... + (n+1)nv_n &= 2S_1 \\
6v_2 + 24v_3 + ... + (n+1)n(n-1)v_n &= 6S_2 \\
v_0, ..., v_n &\geq 0 \ .
\end{aligned}
\tag{4.3}
$$

Let A be the coefficient matrix of equality constraints in (4.3). One can easily show that any 3×3 matrix taken from A has a positive determinant. In fact, any 3×3 determinant of the form

$$
\begin{vmatrix} i+1 & j+1 & k+1 \\ (i+1)i & (j+1)j & (k+1)k \\ (i+1)i(i-1) & (j+1)j(j-1) & (k+1)k(k-1) \end{vmatrix}
$$

is positive. Similarly, one can easily show that all minors of order 4 from $\begin{pmatrix} f^T \\ A \end{pmatrix}$ are positive, where $f = (0, 1, 2, ..., n)^T$.

By Theorem 1, the dual feasible bases of the minimization and maximization problems in (4.3) are in the form

$$
B_{min} = (a_0, a_i, a_{i+1}) \quad \text{and} \quad B_{max} = (a_j, a_{j+1}, a_n),
$$

respectively, where $1 \leq i \leq n-1$ and $0 \leq j \leq n-2$.

The basis B_{min} is also primal feasible if

$$
i - 1 \leq \frac{3S_2}{S_1} \leq i, \qquad \left(i = \left\lceil \frac{3S_2}{S_1} \right\rceil \right) \ . \tag{4.4}
$$

The basis B_{max} is also primal feasible if

$$
2(n+j)S_1 - n(j+1) \leq 6S_2 \leq 2(n+j-1)S_1 - nj \ . \tag{4.5}
$$

Since a both primal and dual feasible basis is optimal, it follows that the objective function values corresponding to these bases are sharp lower and upper bounds for the probability of $\xi \geq 1$.

Thus, we have the following sharp lower and upper bounds for $P(\xi \geq 1)$:

$$
\frac{2(2i+1)S_1 - 6S_2}{(i+1)(i+2)} \leq P(\xi \geq 1) \leq \frac{n}{n+1} + \frac{2(2j+n+1)S_1 - 6S_2 - 2n(j+1)}{(n+1)(j+1)(j+2)} \ . \tag{4.6}
$$

The bounds in (4.6) hold for any i, j, but they are sharp if (4.4) and (4.5) are satisfied.

Remark. We consider the special case

$$
z_i = i, \quad i = 0, \ ..., \ n \ .
$$

Since

$$\binom{\nu}{1} = \nu \quad \text{and} \quad \binom{\nu}{2} = \frac{\nu(\nu-1)}{2} = \frac{\nu^2-\nu}{2} \ ,$$

the first and the second binomial moments are equal to

$$S_1 = \mu_1 \quad \text{and} \quad S_2 = \frac{\mu_2 - \mu_1}{2} \ .$$

If we substitute

$$\mu_1 = S_1 \quad \text{and} \quad \mu_2 = 2S_2 + S_1$$

in the closed bound formulas in (3.12) in Section 3, then we obtain the sharp bounds in (4.6).

4.2 TYPE 2: $p_0 \le p_1 \le ... \le p_n$

Let us introduce the variables $v_0 = p_0,\ v_1 = p_1 - p_0,\ ...,\ v_n = p_n - p_{n-1}$. In this case problem (1.2) can be written as

$$min(max)\{nv_0 + nv_1 + (n-1)v_2 + ... + v_n\}$$

subject to

$$(n+1)v_0 + nv_1 + (n-1)v_2 + ... + v_n = 1$$

$$\binom{n+1}{2}(v_0+v_1) + \left[\binom{n+1}{2} - 1\right] v_2 + ... + \left[\binom{n+1}{2} - \binom{n}{2}\right] v_n = S_1$$

$$\left[\binom{2}{2} + ... + \binom{n}{2}\right](v_0+v_1+v_2) + \left[\binom{3}{2} + ... + \binom{n}{2}\right] v_3 + ... + \binom{n}{2} v_n = S_2$$

$$v \ge 0 \ . \tag{4.7}$$

Taking into account the equations:

$$\binom{n+1}{2} - \binom{i}{2} = \frac{(n+i)(n-i+1)}{2}, \qquad 2 \le i \le n$$

and

$$\binom{i}{2} + ... + \binom{n}{2} = \frac{(n+1)n(n-1) - (i-2)(i-1)i}{6}, \qquad 2 \le i \le n$$

problem (4.7) can be written as

$$min(max)\{nv_0 + nv_1 + (n-1)v_2 + .. + v_n\}$$

subject to

$$(n+1)v_0 + nv_1 + (n-1)v_2 + ... + v_n = 1$$

$$(n+1)n(v_0+v_1)+(n+2)(n-1)v_2+...+(n+i)(n-i+1)v_i+...+2nv_n=2S_1$$
$$(n+1)n(n-1)(v_0+v_1+v_2)+...+[(n+1)n(n-1)-(i-2)(i-1)i]v_i+...+3n(n-1)v_n=6S_2$$
$$v_0,...,v_n \geq 0\ . \qquad (4.8)$$

We consider the following 4×4 matrix.

$$B=\begin{pmatrix} n-i+1 & n-j+1 & n-k+1 & n-l+1 \\ n-i+1 & n-j+1 & n-k+1 & n-l+1 \\ (n+i)(n-i+1) & (n+j)(n-j+1) & (n+k)(n-k+1) & (n+l)(n-l+1) \\ n^3-n-i^3+3i^2-2i & n^3-n-j^3+3j^2-2j & n^3-n-k^3+3k^2-2k & n^3-n-l^3+3l^2-2l \end{pmatrix}$$

for all $0\leq i<j<k<l\leq n$. Since $detB=0$ we cannot use Theorem 1.

Let A be the coefficient matrix of equality constraint in (4.8). By the use of Theorem 2, dual feasible bases in the minimization and the maximization problems in (4.8) are in the form

$$B_{min}=(a_0,a_i,a_{i+1}) \quad \text{and} \quad B_{max}=\begin{cases}(a_0,a_1,a_n) & \text{if } n-1\geq 1\\ (a_1,a_j,a_{j+1}) & \text{if } n\geq 1\\ (a_1,a_s,a_t), & \text{if } n-1\geq 2\end{cases},$$

respectively, where $1\leq i\leq n-1$, $2\leq j$, $s\leq n-1$ and $s+1\leq t\leq n$.

The basis B_{min} is also primal feasible if i is determined by the following inequalities:

$$2(n+i-1)S_1-6S_2\geq in, \qquad (4.9)$$

$$2[(n-i)(n+i-1)+2(i-1)]S_1-6(n-i+1)S_2\leq n(i-1)(n-i+1)\ . \qquad (4.10)$$

First, we notice that the basis $B_{max}=(a_0,a_1,a_n)$ is also primal feasible. The basis $B_{max}=(a_1,a_j,a_{j+1})$ is also primal feasible if j is determined by the following inequalities:

$$6j(n-j)S_2-2(n^3-n^2-n-j^3+j+1)S_1\leq (n+1)[(1-j)(n^2-1)(n-j-1)-j(1-j^2)]\ , \qquad (4.11)$$

$$6j(n-j+1)S_2-2(n^3-n^2-n-j^3+3j^2-2j+1)S_1\geq (n+1)[n^3+n-j^3+4j^2-4j+2-nj(n-j+3)]\ . \qquad (4.12)$$

We also remark that if $B_{max}=(a_1,a_j,a_{j+1})$ or $B_{max}=(a_1,a_s,a_t)$, then the objective function and the left hand side of the first constraint are the same and therefore the upper bound is equal to 1.

Thus, we have the following lower and upper bounds for $P(\xi\geq 1)$.

$$\frac{2(-2i^2+n^2+3i+ni-1)S_1-6(n-i+1)S_2}{(n+1)i(i+1)(n-i+1)}+\frac{n(i-1)}{(n+1)(i+1)(n-i)}$$
$$\leq \quad P(\xi\geq 1) \quad \leq \qquad (4.13)$$
$$min\{1,\frac{2(2n-1)S_1-6S_2}{n(n+1)}\}\ .$$

The bounds in (4.14) are sharp if i satisfies (4.9) and (4.10).

4.3 TYPE 3: $p_0 \le ... \le p_k \ge ... \ge p_n$

Now we assume that the distribution is unimodal with a known modus.

If we introduce the variables $v_i,\ i = 0, 1, ..., n$:

$$v_0 = p_0,\ v_1 =,\ ...\ ,\ v_k = p_k - p_{k-1}\ ,$$

$$v_{k+1} = p_{k+1} - p_{k+2},\ ...\ ,\ v_{n-1} = p_{n-1} - p_n,\ v_n = p_n\ ,$$

then problem (1.2) can be written as

$$min(max)\{kv_0 + kv_1 + (k-1)v_2 + ... + v_k + v_{k+1} + 2v_{k+2} + ... + (n-k)v_n\}$$

subject to

$$(k+1)v_0 + kv_1 + (k-1)v_2 + ... + v_k + v_{k+1} + ... + (n-k)v_n = 1$$

$$\binom{k+1}{2}(v_0+v_1) + \left[\binom{k+1}{2} - \binom{i}{2}\right] v_i + ... + kv_k + (k+1)v_{k+1}$$

$$+ \left[\binom{k+3}{2} - \binom{k+1}{2}\right] v_{k+2} + ... + \left[\binom{n+1}{2} - \binom{k+1}{2}\right] v_n = S_1$$

$$\left[\binom{2}{2} + ... + \binom{k}{2}\right](v_0+v_1+v_2) + \left[\binom{3}{2} + \binom{k}{2}\right] v_3 + \binom{k}{2} v_k + \binom{k+1}{2} v_{k+1} +$$

$$\left[\binom{k+1}{2} + \binom{k+2}{2}\right] v_{k+1} + ... + \left[\binom{k+1}{2} + ... + \binom{n}{2}\right] v_n = S_2$$

$$v \ge 0\ . \tag{4.14}$$

This is equivalent to

$$min(max)\{kv_0 + kv_1 + (k-1)v_2 + ... + v_k + v_{k+1} + 2v_{k+2} + ... + (n-k)v_n\}$$

subject to

$$(k+1)v_0 + kv_1 + (k-1)v_2 + ... + v_k + v_{k+1} + ... + (n-k)v_n = 1$$

$$(k+1)kv_0 + (k+1)kv_1 + ... + (k+i)(k-i+1)v_i + ... + 2kv_k + 2(k+1)v_{k+1} + ... + (n-k)(n+k+1)v_n = 2S_1$$

$$(k+1)k(k-1)(v_0+v_1+v_2) + ... + [(k^3-k) - (i-2)(i-1)i]v_i + ... + 3k(k-1)v_k$$

$$+3(k+1)kv_{k+1} + ... + (n-k)(n^2+nk+k^2-1)v_n = 6S_2$$

$$v \ge 0\ . \tag{4.15}$$

Let A be the coefficient matrix of equality constraints in (4.15). By Theorem 2, a dual feasible basis for minimization problem in (4.15) is of the form

$$B_{min} = (a_0, a_i, a_{i+1}),$$

where $1 \leq i \leq n-1$ and a dual feasible basis for maximization problem in (4.1) is of the form

$$B_{max} = \begin{cases} (a_0, a_1, a_n) & \text{if } \ n-1 \geq 1 \\ (a_1, a_j, a_{j+1}) & \text{if } \ n \geq 1 \\ (a_1, a_s, a_t), & \text{if } \ n-1 \geq 2 \end{cases} ,$$

where $2 \leq j \ , \ s \leq n-1$ and $s+1 \leq t \leq n$.

The basis $B_{min} = (a_0, a_i, a_{i+1})$ is also primal feasible if one of the following conditions is satisfied:

$$2(k+i-1)S_1 - 6S_2 \geq ik \ ,$$

$$2[(k-i)(k+i-1) + 2(i-1)]S_1 - 6(k-i+1)S_2 \leq k(i-1)(k-i+1)$$

$$\text{if} \quad i+1 \leq k \ , \qquad (4.16)$$

$$2(i+k-1)S_1 - ik \leq 6S_2 \leq 2(i+k)S_1 - k(i+1) \qquad \text{if} \quad i \geq k+1 \ , \qquad (4.17)$$

$$4(k-1)S_1 - k(k-1) \leq 6S_2 \leq 4kS_1 - k(k+1) \qquad \text{if} \quad i = k \ . \qquad (4.18)$$

The basis $B_{max} = (a_1, a_j, a_{j+1})$ is also primal feasible if one of the following conditions is satisfied:

$$6j(k-j)S_2 - 2(k^3 - k^2 - k - j^3 + j + 1)S_1 \leq (k+1)[(1-j)(k^2-1)(k-j-1) - j(1-j^2)]$$

$$6j(k-j+1)S_2 - 2(k^3 - k^2 - k - j^3 + 3j^2 - 2j + 1)S_1 \geq (k+1)[k^3 + k - j^3 + 4j^2 - 4j + 2 - kj(k-j+3)]$$

$$\text{if} \quad j+1 \leq k \ , \qquad (4.19)$$

$$\frac{2(j-k)(j+k)S_1 - 6(j-k)S_2}{(j+1)(j-k)} \leq k+1 \leq \frac{2(j+k+1)S_1 - 6S_2}{j+2}$$

$$\text{if} \quad j \geq k+1 \ , \qquad (4.20)$$

$$2(2k-1)S_1 - k(k+1) \leq 6S_2 \leq 2(2k+1)S_1 - (k+1)(k+2)$$

$$\text{if} \quad j = k \ . \qquad (4.21)$$

The basis $B_{max} = (a_0, a_1, a_n)$. We remark that $B_{min} = (a_0, a_k, a_{k+1})$ and $B_{max} = (a_1, a_k, a_{k+1})$ are also primal feasible since k is known.

Using the same reasoning that we have used in Section 4.2, one can easily show that the upper bound is equal to 1 if $B_{max} = (a_1, a_j, a_{j+1})$ or $B_{max} = (a_1, a_s, a_t)$.

Three possible cases for the lower and upper bounds for $P(\xi \geq 1)$ can be given as follows:

***Case* 1.** $i+1 \le k$

$$\frac{2(-2i^2+k^2+3i+ki-1)S_1-6(k-i+1)S_2}{(k+1)i(i+1)(k-i+1)}+\frac{k(i-1)}{(k+1)(i+1)(k-i)}$$

$$\le \quad P(\xi \ge 1) \quad \le$$

$$min\{1, \frac{2(n+k)S_1-6S_2}{(n+1)(k+1)}\} \ . \tag{4.22}$$

***Case* 2.** $i \ge k+1$

$$\frac{k}{k+1}+\frac{2k(2i+k+1)S_1-6kS_2-2k^2(i+1)}{(k+1)(i+1)(i+2)}$$

$$\le \quad P(\xi \ge 1) \quad \le$$

$$min\{1, \frac{2(n+k)S_1-6S_2}{(n+1)(k+1)}\} \ . \tag{4.23}$$

***Case* 3.** $i=k$

$$\frac{6kS_1-6S_2+k^3-k}{k(k+1)(k+2)} \quad \le \quad P(\xi \ge 1) \quad \le \quad min\{1, \frac{2(n+k)S_1-6S_2}{(n+1)(k+1)}\} \ . \tag{4.24}$$

The above inequalities are sharp if the bases are primal feasible too, i.e., the inequalities (4.16), ... , (4.21) are satisfied.

5 Examples

We present four examples to show that if the shape of the distribution is given, then by the use of our bounding methodology, we can obtain tighter bounds for $P(\xi \ge 1)$.

First, let us consider the following binomial moment problem where the shape information is not given.

$$min(max)\{p_1+...+p_n\}$$

subject to

$$\sum_{i=0}^{n} p_i = 1$$

$$\sum_{i=1}^{n} ip_i = S_1 \tag{5.1}$$

$$\sum_{i=2}^{n} \binom{i}{2} p_i = S_2$$

$$p_i \geq 0 \ , \quad i = 0, ..., n \ .$$

Let A be the coefficient matrix of the equality constraints in (5.1). By Theorem 2, a dual feasible basis for the minimization problem in (5.1) is of the form:

$$B_{min} = (a_0, a_i, a_{i+1}),$$

where $1 \leq i \leq n-1$. Similarly, a dual feasible basis for the maximization problem in (5.1) is given as follows:

$$B_{max} = \begin{cases} (a_0, a_1, a_n) & \text{if} \ \ n-1 \geq 1 \\ (a_1, a_j, a_{j+1}) & \text{if} \ \ n \geq 1 \\ (a_1, a_s, a_t), & \text{if} \ \ n-1 \geq 2 \end{cases} ,$$

where $2 \leq j, \ s \leq n-1$ and $s+1 \leq t \leq n$.

The primal feasibility of the basis B_{min} is ensured if the following condition is satisfied:

$$i - 1 \leq \frac{2S_2}{S_1} \leq i \ . \tag{5.2}$$

First, we remark that the basis $B_{max} = (a_0, a_1, a_n)$ is primal feasible. We also note that the upper bound for $P(\xi \geq 1)$ is equal to 1 if $B_{max} = (a_0, a_1, a_n)$ or $B_{max} = (a_1, a_s, a_t)$.

The basis $B_{max} = (a_1, a_j, a_j + 1)$ is also primal feasible if j is determined by the following condition:

$$j \leq \frac{2S_2}{S_1 - 1} \leq j + 1 \ . \tag{5.3}$$

Similarly, the basis $B_{max} = (a_1, a_s, a_t)$ is also primal feasible if

$$s \leq \frac{2S_2}{S_1 - 1} \leq t \ . \tag{5.4}$$

The sharp lower and upper bounds for $P(\xi \geq 1)$ can be given as follows:

$$\frac{2iS_1 - 2S_2}{i(i+1)} \leq \ P(\xi \geq 1) \ \leq min\{1, S_1 - \frac{2S_2}{n}\} \ , \tag{5.5}$$

where i satisfies (5.2).

Example 1. In order to create example for $S_1, \ S_2$ we take the following probability distribution $p_0^* = 0.4, \ p_1^* = 0.3, \ p_2^* = 0.25, \ p_3^* = 0.03, \ p_4^* = 0.02$. With these probabilities the binomial moments are

$$S_1 = \sum_{i=1}^{4} i p_i^* = 0.97 \quad \text{and} \quad S_2 = \sum_{i=2}^{4} \left(\begin{array}{c} i \\ 2 \end{array} \right) p_i^* = 0.46 \ .$$

We have the following binomial moment problem:

$$min(max)\{p_1 + p_2 + p_3 + p_4\}$$

subject to

$$p_0 + p_1 + p_2 + p_3 + p_4 = 1$$

$$\sum_{i=1}^{4} ip_i = 0.97$$

$$\sum_{i=2}^{4} \binom{i}{2} p_i = 0.46$$

$$p_0, ..., p_4 \geq 0 \ . \tag{5.6}$$

Let A be the coefficient matrix of the equality constraints in (5.6). By Theorem 2, we have $B_{min} = (a_0, a_1, a_2)$, $B_{max} = (a_0, a_1, a_4)$ for problem (5.1).

By the use of (5.5), we obtain the following lower and upper bounds:

$$0.51 \leq \ P(\xi \geq 1) \leq 0.74 \ . \tag{5.7}$$

Now, we assume that the probability distribution in (5.6) is decreasing, i.e., $p_0 \geq ... \geq p_4$. In this case problem (5.6) can be transformed to Type 1 problem in Section 4.1 as follows:

$$min(max)\{v_1 + 2v_2 + 3v_3 + 4v_4\}$$

subject to

$$v_0 + 2v_1 + 3v_2 + 4v_3 + 5v_4 = 1$$

$$v_1 + 3v_2 + 6v_3 + 10v_4 = 0.97$$

$$v_2 + 4v_3 + 10v4 = 0.46$$

$$v \geq 0 \ . \tag{5.8}$$

Now let A be the coefficient matrix of the equality constraints in (5.8). By using (4.4) and (4.5), the optimal bases for problem (5.8) are $B_{min} = (a_0, a_2, a_3)$ and $B_{max} = (a_1, a_2, a_4)$.

The following are the lower and upper bounds obtained from (4.6):

$$0.5783 \leq P(\xi \geq 1) \leq 0.6273. \tag{5.9}$$

One can easily see that these bounds are the optimum values of problem (5.6) together with the shape constraint $p_0 \geq ... \geq p_4$.

Example 2. Let $n = 5$, $S_1 = 3.95$, $S_2 = 7$. The corresponding binomial problem is

$$min(max)\{p_1 + p_2 + p_3 + p_4 + p_5\}$$

subject to

$$p_0 + p_1 + p_2 + p_3 + p_4 + p_5 = 1$$

$$\sum_{i=1}^{5} i p_i = 3.95$$

$$\sum_{i=2}^{5} \binom{i}{2} p_i = 7$$

$$p_0, ..., p_5 \geq 0 \ . \tag{5.10}$$

From (5.5) we obtain

$$0.88 \leq \ P(\xi \geq 1) \leq 1 \ . \tag{5.11}$$

Now we assume that the distribution is increasing. Problem (5.10) together with the shape constraint $p_0 \leq ... \leq p_5$ can be transformed to Type 2 problem in Section 4.2 as follows:

$$min(max)\{5v_0 + 5v_1 + 4v_2 + 3v_3 + 2v_4 + v_5\}$$

subject to

$$6v_0 + 5v_1 + 4v_2 + 3v_3 + 2v_4 + v_5 = 1$$

$$15v_0 + 15v_1 + 14v_2 + 12v_3 + 9v_4 + 5v_5 = 3.95$$

$$20v_0 + 20v_1 + 20v_2 + 19v_3 + 16v4 + 10v_5 = 7$$

$$v \geq 0 \ . \tag{5.12}$$

The optimal bases for (5.12) are $B_{min} = (a_0, a_4, a_5)$ and $B_{max} = (a_0, a_1, a_5)$. By the use of Problem (5.12) and the formulas given in (4.13), the sharp lower and upper bounds for $P(\xi \geq 1)$ can be found as follows:

$$0.94 \leq P(\xi \geq 1) \leq 0.97 \ . \tag{5.13}$$

Example 3. Let $S_1 = 8.393, \ \ S_2 = 34.625$. The corresponding binomial problem is

$$min(max) \sum_{i=1}^{10} p_i$$

subject to

$$\sum_{i=0}^{10} p_i = 1$$

$$\sum_{i=1}^{10} i p_i = 8.393$$

$$\sum_{i=2}^{10} \binom{i}{2} p_i = 34.625$$

$$p_0, ...p_{10} \geq 0 \ . \tag{5.14}$$

$B_{min} = (a_0, a_9, a_{10})$ and $B_{max} = (a_1, a_9, a_{10})$ are the optimal bases for (5.14) and from (5.5) we obtain

$$0.909 \leq \ P(\xi \geq 1) \leq 1 \ . \tag{5.15}$$

By adding the shape constraint $p_0 \leq ... \leq p_{10}$, problem (5.14) can be transformed to Type 2 problem in Section 4.2 as follows:

$$min(max)\{10v_0 + 10v_1 + 9v_2 + 8v_3 + ... + v_{10}\}$$

subject to

$$\sum_{i=0}^{10}(11-i)v_i = 1$$

$$\sum_{i=0}^{10} \frac{(10+i)(11-i)v_i}{2} = 8.393$$

$$\sum_{i=0}^{10} \frac{(990-(i-2)(i-1)i)v_i}{6} = 34.625$$

$$v \geq 0 \ . \tag{5.16}$$

By the use of Problem (5.16) and the sharp bound formulas in (4.13), the sharp lower and upper bounds for $P(\xi \geq 1)$ can be given as follows:

$$0.975 \leq P(\xi \geq 1) \leq 1 \ . \tag{5.17}$$

Let $n = 5, \quad S_1 = 3.95, \quad S_2 = 7$. The corresponding binomial problem is

$$min(max)\{p_1 + p_2 + p_3 + p_4 + p_5\}$$

subject to

$$p_0 + p_1 + p_2 + p_3 + p_4 + p_5 = 1$$

$$\sum_{i=1}^{5} ip_i = 3.95$$

$$\sum_{i=2}^{5} \binom{i}{2} p_i = 7$$

$$p_0, ..., p_5 \geq 0 \ . \tag{5.18}$$

From (5.5) we obtain

$$0.88 \leq \ P(\xi \geq 1) \leq 1 \ . \tag{5.19}$$

Now we assume that the distribution is increasing. Problem (5.10) together with the shape constraint $p_0 \leq ... \leq p_5$ can be transformed to Type 2 problem in Section 4.2 as follows:

$$min(max)\{5v_0 + 5v_1 + 4v_2 + 3v_3 + 2v_4 + v_5\}$$

subject to

$$6v_0 + 5v_1 + 4v_2 + 3v_3 + 2v_4 + v_5 = 1$$

$$15v_0 + 15v_1 + 14v_2 + 12v_3 + 9v_4 + 5v_5 = 3.95$$

$$20v_0 + 20v_1 + 20v_2 + 19v_3 + 16v4 + 10v_5 = 7$$

$$v \geq 0 \ . \qquad (5.20)$$

The optimal bases for (5.12) are $B_{min} = (a_0, a_4, a_5)$ and $B_{max} = (a_0, a_1, a_5)$. By the use of Problem (5.12) and the formulas given in (4.13), the sharp lower and upper bounds for $P(\xi \geq 1)$ can be found as follows:

$$0.94 \leq P(\xi \geq 1) \leq 0.97. \qquad (5.21)$$

Example 4. Let $n = 4, \quad S_1 = 1.93, \quad S_2 = 1.27$. The corresponding binomial problem is

$$min(max) \sum_{i=1}^{4} p_i$$

subject to

$$\sum_{i=0}^{4} p_i = 1$$

$$\sum_{i=1}^{4} ip_i = 1.93$$

$$\sum_{i=2}^{4} \binom{i}{2} p_i = 1.27$$

$$p_0, ...p_4 \geq 0 \ . \qquad (5.22)$$

By the use of (5.5) and Problem (5.22) we have

$$0.8633 \leq \ P(\xi \geq 1) \leq 1 \ . \qquad (5.23)$$

Now we assume that the distribution is unimodal and $k = 2$. By adding the shape constraint $p_0 \leq p_1 \leq p_2 \geq p_3 \geq p_4$, problem (5.22) can be transformed to Type 3 problem in Section 4.3 as follows:

$$min(max)\{2v_0 + 2v_1 + v_2 + v_3 + 2v_4\}$$

subject to

$$3v_0 + 2v_1 + v_2 + v_3 + 2v + 4 = 1$$

$$3v_0 + 3v_1 + 2v_2 + 3v_3 + 7v_4 = 1.93$$
$$v_0 + v_1 + v_2 + 3v_3 + 9v_4 = 1.27$$
$$v \geq 0 \; . \tag{5.24}$$

By the use of Problem (5.24) and the closed bound formulas in (4.24), the sharp lower and upper bounds for $P(\xi \geq 1)$ can be given as follows:

$$0.8975 \leq P(\xi \geq 1) \leq 1 \; . \tag{5.25}$$

6 Applications

In this section we present two examples for the application of our bounding technique, where shape information about the unknown probability distribution can be used.

Example 1. *Application in PERT.*

In PERT we frequently concerned with the problem to approximate the expectation or the values of the probability distribution of the length of the critical path.

In the paper by Prékopa et al. (2004) a bounding technique is presented for the c.d.f. of the critical, i.e., the largest path under moment information. In that paper first an enumeration algorithm finds those paths that are candidates to become critical. Then probability distribution of the path lengths are approximated by a multivariate normal distribution that serves a basis for the bounding procedure.

In the present example we are concerned with only one path length but drop the normal approximation to the path length. Instead, we assume that the random length of each arc follows beta distribution, as it is usually assumed in PERT. Arc lengths are assumed to be independent, thus the probability distribution of the path length is the convolution of beta distribution with different parameters.

The p.d.f. of the beta distribution in the interval $(0, 1)$ is defined as

$$f(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}, \qquad 0 < x < 1, \tag{6.1}$$

where $\Gamma(.)$ is the gamma function,

$$\Gamma(p) = \int_0^\infty x^{p-1} e^{-x} dx, \quad p > 0.$$

The kth moment of this distribution can easily be obtained by the use of the fact that

$$\int_0^1 \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1} dx = 1.$$

In fact,

$$\int_0^1 x^k f(x)dx = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 x^{k+\alpha-1}(1-x)^{\beta-1}dx$$

$$= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \frac{\Gamma(k+\alpha)\Gamma(\beta)}{\Gamma(k+\alpha+\beta)} \tag{6.2}$$

$$= \frac{\Gamma(\alpha+k)\Gamma(\alpha+k-1)...\Gamma(\alpha+1)}{\Gamma(\alpha+\beta+k)\Gamma(\alpha+\beta+k-1)...\Gamma(\alpha+\beta+1)}$$

If $\alpha,\ \beta$ are integers, then using the relation: $\Gamma(m) = (m-1)!$ the above expression takes a simple form.

The beta distribution in PERT is defined over a more general interval (a, b) and we define its p.d.f. as the p.d.f. of $a + (b-a)X$, where X has p.d.f. given by (6.1).

In practical problems the values $a,\ b,\ \alpha,\ \beta$ are obtained by the use of expert estimations of the shortest largest and most probable times of accomplishing the job represented by the arc (see, e.g., Battersby, 1970).

Let n be the number of arcs in a path and assume that each arc length ξ_i has beta distribution with known parameters $a_i,\ b_i,\ \alpha_i,\ \beta_i,\ \ i = 1, ..., n$. Assume that $\alpha_i \geq 1,\ \beta_i \geq 1,\ i = 1, ..., n$. We are interested to approximate the values of the c.d.f. of the path length, i.e., $\xi = \xi_1 + ... + \xi_n$.

The analytic form of the c.d.f. cannot be obtained in closed form but we know that the p.d.f. of ξ is unimodal. In fact, each ξ_i has logconcave p.d.f. hence the sum ξ also has logconcave p.d.f. (for the proof of this assertion see, e.g., Prékopa 1995) and any logconcave function is also unimodal.

In order to apply our bounding methodology we discretize the distribution of ξ, by subdividing the interval $(\sum_{i=1}^n a_i,\ \sum_{i=1}^n b_i)$ and handle the corresponding discrete distribution as unknown, but unimodal such that some of its first m moments are also known. In principle any order moment of ξ is known but for practical calculation it is enough to use the first few moments, at least in many cases, to obtain good approximation to the values of the c.d.f. of ξ.

The probability functions obtained by the discretizations, using equal length subintervals, are logconcave sequences. Convolution of logconcave sequences are also logconcave and any logconcave sequence is unimodal in the sense of Section 3.3.

In order to apply our methodology we need to know the modus of the distribution of ξ. A heuristic method to obtain it is the following. We take the sum of the modi of the terms in $\xi = \xi_1 + ... + \xi_n$ and the compute a few probability around it.

Example 2. *Application in Reliability.*

Let $A_1, ...,\ A_n$ be independent events and define the random variables $X_1, ..., X_n$ as the characteristic variables corresponding to the above events, respectively, i.e.,

$$X_i = \begin{cases} 1 & \text{if } A_i \text{ occurs} \\ 0 & \text{otherwise} \end{cases}$$

Let $p_i = P(X_i = 1),\ i = 1, ..., n$. The random variables $X_1, ..., X_n$ have logconcave discrete distributions on the nonnegative integers, consequently the distribution of

$$X = X_1 + ... + X_n$$

is also logconcave on the same set.

In many applications it is an important problem to compute, or at least approximate, e.g., by the use of probability bounds the probability

$$X_1 + ... + X_n \geq r, \quad 0 \leq r \leq n. \tag{6.3}$$

If $I_1, ..., I_{\binom{n}{k}}$ designate the k-element subsets of the set $\{1, ..., n\}$ and $J_l = \{1, ..., n\} \backslash I_l,\ l = 1, ..., \binom{n}{k}$, then we have the equation

$$P(X_1 + ... + X_n \geq r) = \sum_{k=r}^{n} \sum_{l=1}^{\binom{n}{k}} \prod_{i \in I_l} p_i \prod_{j \in J_l} (1 - p_j), \qquad 0 < r \leq n \tag{6.4}$$

If n is large, then the calculation of the probabilities on the right hand side of (6.4) may be hard, even impossible. However, we can calculate lower and upper bounds for the probability on the left hand side of (6.4) by the use of the sums:

$$S_k = \sum_{1 \leq i_1 < ... < i_k \leq n} p_{i_1} ... p_{i_k} = \sum_{l=1}^{\binom{n}{k}} \prod_{i \in I_l} p_i, \qquad k = 1, ..., m, \tag{6.5}$$

where m may be much smaller than n. Since the random variable $X_1 + ... + X_n$ has logconcave, hence unimodal distribution, we can impose the unimodality condition on the probability distribution:

$$P(X_1 + ... + X_n = k), \quad k = 0, ..., n. \tag{6.6}$$

Then we solve both the minimization and maximization problems considered in Section 4.2 to obtain the bounds for the probability (6.3). If m is small the bounds can be obtained by formulas. Note that the largest probability (6.5) corresponds to

$$k_{max} = \left\lfloor (n+1) \frac{p_1 + ... + p_n}{n} \right\rfloor.$$

Note that a formula first obtained by C. Jordan (1867) provides us with the probability (6.3), in terms of the binomial moments $S_r, ..., S_n$:

$$P(X_1 + ... + X_n \geq r) = \sum_{k=r}^{n} (-1)^{k-r} \binom{k-1}{r-1} S_k. \tag{6.7}$$

However, to compute the binomial moments involved may be extremely difficult, if not impossible. The advantage of our approach is that we use the first few binomial moments $S_1, ..., S_m$, where m is relatively small and we can obtain very good bounds, at least in many cases.

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