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The Optimization of the Move of Robot Arm by Benders Decomposition

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Abstract. In this paper the move of a robot arm is optimized via Benders decomposition.

1 Introduction

The problem of this paper has been motivated by printed circuit assembly. A good survey on this topic is [Crama et al. 2002]. The results of the present paper can be applied at all of the cases where a robot assemblies a product and the objective is the minimize the length of the arm of the robot.

2 Technological arrangement

The task being assembled by the robot is in a fixed position. The components are in a sequence of cells. Each cell contains a different type of component. Each component has a well-defined position on the task where to be assembled. The duration of the assembly of a component is an a priori given fixed value. The only possibility to save some time, i.e. to accelerate the production, is to minimize the total move of the arm of the robot.

When the assembly of a component is finished then the arm goes for the next component to the appropriate cell and from there it goes to the position of the next component on the task. Hence it follows that the total move of the arm depends on both *(i) the assignment of the components to cells*, and *(ii) the order of the components in which they are assembled.* Therefore the whole problem is the "direct product" of the assignment problem of (i), and the TSP of (ii). The distances among cells and positions are supposed to be symmetric.

In this paper we shall suppose that the following two assumption are valid:

$$\text{The number of cells is equal to the number of components.} \tag{A1}$$

$$\text{Each component is used on the part only ones.} \tag{A2}$$

These assumption are simplifying the problem which still remains difficult enough to be solved.

3 Problem formulation

To describe the problem mathematically the following notation are introduced:

n	the number of cells and components
i	the index of cells
j, k, l	indices of components and positions
d_{ik}	the symmetric distance of cell i and position k
x_{ij}	is 1 if component j is assigned to cell i, otherwise is 0
y_{kl}	is 1 if component l is assembled immediately after component k

The system of x's and y's are the decision variables. They must satisfy the following sets of constraints.

Each component is assigned to exactly one cell and vice versa:

$$\sum_{j=1}^{n} x_{ij} \;=\; 1, \quad i = 1, ..., n, \tag{1}$$

and

$$\sum_{i=1}^{n} x_{ij} \;=\; 1, \quad j = 1, ..., n. \tag{2}$$

Each component is assembled exactly ones, i.e. the order of the components is a Hamiltonian circuit:

$$\sum_{k=1}^{n} y_{kl} \;=\; 1, \quad l = 1, ..., n, \tag{3}$$

and

$$\sum_{l=1}^{n} y_{kl} \;=\; 1, \quad k = 1, ..., n, \tag{4}$$

and

$$\forall H \subset \{1, 2, ..., n\}, n >\mid H \mid \geq 2 : \quad \sum_{k \in H} \sum_{l \in \overline{H}} y_{kl} \;\geq\; 1. \tag{5}$$

Formally we repeat that all variables are binary:

$$x_{ij},\, y_{kl} \;\in\; \{0,\, 1\}, \quad i, j, k, l = 1, 2, ..., n. \tag{6}$$

The arm moves from position k of the task to cell i only if cell i contains the component of the immediate successor position. Assume that the index of it is l. Then until the next position the length of the move is $d_{ik} + d_{il}$. These terms of the distance function can be selected by the decision variables and the total distance of the move of the arm to be minimized is:

$$\min \sum_{k=1}^{n} \sum_{l=1}^{n} \sum_{i=1}^{n} d_{ik} y_{kl} x_{il} \;+\; \sum_{i=1}^{n} \sum_{l=1}^{n} d_{il} x_{il}. \tag{7}$$

Thus the mathematical problem to be solved is to optimize (7) under the conditions (1-6). This problem formulation has two drawbacks. At first there are exponential many constraints in (5). We shall see that only those of them will be used, which are violated. Secondly the objective function (7) is nonlinear. It can be linearized with the usual method. New variables, say w_{ikl}'s, are introduced as follows:

$$w_{ikl} \;=\; x_{il} y_{kl} \quad i, k, l = 1, 2, ..., n. \tag{8}$$

If both x_{il} and y_{kl} are zero-one variables then w_{ikl} is zero-one as well. It is well-known that equation (8) is equivalent to the inequalities

$$w_{ikl} \ \geq \ x_{il} + y_{kl} - 1 \quad i, k, l = 1, 2, ..., n \tag{9}$$

and

$$2w_{ikl} \ \leq \ x_{il} + y_{kl} \quad i, k, l = 1, 2, ..., n \tag{10}$$

assuming that

$$w_{ikl} \ \in \ \{ 0, \, 1 \}, \quad i, k, l = 1, 2, ..., n. \tag{11}$$

Thus the new form of the objective function when it is multiplied by (-1) is

$$\max \sum_{i=1}^{n} \sum_{l=1}^{n} (-d_{il}) x_{il} \ + \ \sum_{k=1}^{n} \sum_{l=1}^{n} \sum_{i=1}^{n} (-d_{ik}) w_{ikl}. \tag{12}$$

Thus the final form is to optimize (12) under the conditions (1)-(6), and (10)-(11).

For the sake of convenience we need a compact form of the constraints, too. Inequalities (1)-(2) and (3)-(5), respectively, contain only the variables $\mathbf{x}$ and $\mathbf{y}$, respectively. Thus these sets can be written separately. If it is necessary the inequalities are multiplied by (-1). Finally the form

$$\begin{array}{lccccccc} \max & \mathbf{c}^T\mathbf{x} & + & \mathbf{0}^T\mathbf{y} & + & \mathbf{f}^T\mathbf{w} & & \\ & \mathbf{A}_1\mathbf{x} & + & \mathbf{O}\mathbf{y} & + & \mathbf{O}\mathbf{w} & = & \mathbf{e}_{2n} \\ & \mathbf{O}\mathbf{x} & + & \mathbf{B}_2\mathbf{y} & + & \mathbf{O}\mathbf{w} & = (\leq) & \mathbf{b}_2 \\ & \mathbf{A}_3\mathbf{x} & + & \mathbf{B}_3\mathbf{y} & + & \mathbf{C}_3\mathbf{w} & \leq & \mathbf{b}_3 \\ & \multicolumn{7}{l}{\mathbf{x}, \mathbf{y} \in \{ 0, \, 1 \}^{n^2}, \ \ \mathbf{w} \in \{ 0, \, 1 \}^{n^3},} \end{array} \tag{13}$$

where the vectors $\mathbf{c}$, and $\mathbf{f}$ are formed from the distances according to (12) and all components of the $2n$-dimensional vector $\mathbf{e}_{2n}$ are 1and finally $\mathbf{O}$ is a zero matrix of appropriate size.

4 The Benders decomposition in the general case

The Benders decomposition [Benders 1962] is summarized in this section as it is the main tool to develop our algorithm and it is referred very rarely in the literature. The section does not contain new results.

The Benders decomposition is actually the dual of the Dantzig-Wolfe decomposition. Here not the constraints but the variables are divided into two parts. The first one represents a linear programming part while the second one is arbitrary. The problem to solve is

$$\begin{array}{lccccc} \max & \mathbf{c}^T\mathbf{p} & + & f(\mathbf{r}) & & \\ & \mathbf{A}\mathbf{p} & + & \mathbf{F}(\mathbf{r}) & \leq & \mathbf{b} \\ & \multicolumn{5}{l}{\mathbf{p} \geq \mathbf{0}, \ \mathbf{r} \in \mathcal{S},} \end{array} \tag{14}$$

where $\mathbf{c}$, and $\mathbf{p}$ are s-dimensional vectors, $\mathbf{A}$ is a real matrix of size $m \times s$, $f : \mathbf{R}^t \to \mathbf{R}$, and $\mathbf{F} : \mathbf{R}^t \to \mathbf{R}^m$ are arbitrary functions, $\mathcal{S}$ is an arbitrary subset of $\mathbf{R}^t$ and $\mathbf{r}$ is a t-dimensional vector.

For a fixed $\hat{\mathbf{r}}$ the problem (14) becomes the linear programming problem

$$\begin{array}{rrcl} \max & \mathbf{c}^T\mathbf{p} & + & f(\hat{\mathbf{r}}) \\ & \mathbf{A}\mathbf{p} & \leq & \mathbf{b} - \mathbf{F}(\hat{\mathbf{r}}) \\ & \mathbf{p} \geq \mathbf{0}, & & \end{array} \tag{15}$$

where the term $f(\hat{\mathbf{r}})$ in the objective function is only an additive constant. The dual of (15) is

$$\begin{array}{rrcl} \min & (\mathbf{b} - \mathbf{F}(\hat{\mathbf{r}}))^T\mathbf{u} & & \\ & \mathbf{A}^T\mathbf{u} & \geq & \mathbf{c} \\ & \mathbf{u} & \geq & \mathbf{0}. \end{array} \tag{16}$$

If Problem (16) has no feasible solution then Problem (15) is either unbounded or has no feasible solution for each particular $\hat{\mathbf{r}}$. Hence the original problem has no optimal solution. Therefore in the rest of the paper it is assumed that Problem (16) has at least one feasible solution.

An equivalent form of (14) can be obtained by introducing an objective function variable, say z, and slack variables, say v_0 and $\mathbf{v}$, to obtain equations instead of the inequalities. The new form of (14) is

$$\begin{array}{rrcrcrcrcrcl} \max & \mathbf{0}^T\mathbf{p} & + & \mathbf{0}^T\mathbf{r} & + & z & + & 0v_0 & + & \mathbf{0}^T\mathbf{v} & & \\ & -\mathbf{c}^T\mathbf{p} & - & f(\mathbf{r}) & + & z & + & v_0 & + & \mathbf{0}^T\mathbf{v} & = & 0 \\ & \mathbf{A}\mathbf{p} & + & \mathbf{F}(\mathbf{r}) & + & \mathbf{0}z & + & \mathbf{0}v_0 & + & \mathbf{v} & = & \mathbf{b} \\ & \multicolumn{11}{c}{\mathbf{p} \geq \mathbf{0},\ \mathbf{r} \in \mathcal{S},\ v_0 \geq 0,\ \mathbf{v} \geq \mathbf{0}.} \end{array} \tag{17}$$

The following theorem is an immediate consequence of the Farkas theorem taking into account that the variables in Problem (17) with the possible exception of $\mathbf{r}$ are nonnegatives.

Theorem 4.1 *For a given pair $(\hat{\mathbf{r}}, \hat{z})$, where $\hat{\mathbf{r}} \in \mathcal{S}$ and $\hat{z} \in \mathbb{R}$ there exist a vectors $\hat{\mathbf{p}}$, and $\hat{\mathbf{v}}$ and a number $\hat{v}_0$ such that the 5-tuple $(\hat{\mathbf{p}}, \hat{\mathbf{r}}, \hat{z}, \hat{v}_0, \hat{\mathbf{v}})$ is a feasible solution of Problem (17) if and only if the inequality*

$$u_0(f(\hat{\mathbf{r}}) - \hat{z}) + \mathbf{u}^T(\mathbf{b} - \mathbf{F}(\hat{\mathbf{r}})) \ \geq\ 0 \tag{18}$$

holds for every real number u_0 and vector $\mathbf{u} \in \mathbb{R}^m$ such that

$$\mathbf{A}^T\mathbf{u} \ \geq\ \mathbf{c}u_0, \quad u_0 \ \geq\ 0, \quad \mathbf{u} \ \geq\ \mathbf{0}. \tag{19}$$

The set of the $m+1$-dimensional vectors $(u_0, \mathbf{u}^T)^T$ satisfying (19) is obviously a pointed and polyhedral cone denoted by $\mathcal{C}$. It is well-known that it is spanned by the finite set of its extremal directions, say $\mathcal{Q}$. If inequality (18) holds for all elements of $\mathcal{Q}$ then it holds for all of the elements of $\mathcal{C}$. The problem is from computational point of view that the set $\mathcal{Q}$ may have too many elements to explore all of them as an initial step of the algorithm. Therefore a "column generation" type algorithm should be developed, which uses only those elements of $\mathcal{Q}$, which are really required. As it will be seen this type of algorithm is of "row generation" in the case of Benders decomposition.

Furthermore if $(\hat{u}_0, \hat{\mathbf{u}}) \in \mathcal{Q}$ and $\hat{u}_0 \neq 0$ then without loss of generality we may assume that $\hat{u}_0 = 1$. Hence one can conclude that to test if a given pair $(\hat{\mathbf{r}}, \hat{z})$ is a part of an optimal solution it is enough to solve Problem (16). Let $(\hat{u}_0^*, \hat{\mathbf{u}}^*)$ be the optimal solution. Only the following cases exist:

(i) optimal solution: If an optimal solution of Problem (16) exists and the equation

$$\hat{z} \;=\; (\mathbf{b} - \mathbf{F}(\hat{\mathbf{r}}))^T \hat{\mathbf{u}^*} + f(\hat{\mathbf{r}}) \tag{20}$$

holds then the pair is optimal and the missing part $\hat{\mathbf{p}}^*$ of the optimal solution can be obtained by solving Problem (15).

(ii) a new element of the set $\mathcal{Q}$ is explored: Assume that an optimal solution of Problem (16) exists and the inequality

$$\hat{z} \;>\; (\mathbf{b} - \mathbf{F}(\hat{\mathbf{r}}))^T \hat{\mathbf{u}^*} + f(\hat{\mathbf{r}}) \tag{21}$$

holds. Let $\hat{\mathbf{u}}^*$ the optimal solution of Problem (16). Then Inequality (18) does not hold for the vector $\left(1, \hat{\mathbf{u}}^{*T}\right)^T$. Therefore a new candidate for being $(\hat{\mathbf{r}}, \hat{z})$ must be generated by taking into account even this inequality.

(iii) two new elements of the set $\mathcal{Q}$ is explored: Assume that no optimal solution of Problem (16) exists but the objective function is unbounded. Assume that Problem (16) is solved by the simplex method. At the very moment when the unboundedness of the problem is recognized there are a current basic solution and a direction of the unboundedness, say $\hat{\mathbf{u}}$ and $\hat{\mathbf{t}}$, respectively. Then Inequality (18) must be satisfied for the vectors $(1, \hat{\mathbf{u}})$, and $(0, \hat{\mathbf{t}})$, too.

A candidate $(\hat{\mathbf{r}}, \hat{z})$ can be generated as follows. Let $\hat{\mathcal{Q}}$ be the subset of $\mathcal{Q}$ consisting of the explored elements. Then the new candidate is an optimal solution of the problem

$$\begin{array}{c} \max z \\ \forall\, (u_0, \mathbf{u}) \in \hat{\mathcal{Q}} :\; u_0(f(\mathbf{r}) - z) + \mathbf{u}^T(\mathbf{b} - \mathbf{F}(\mathbf{r})) \;\geq\; 0 \\ \mathbf{r} \in \mathcal{S}. \end{array} \tag{22}$$

Thus the Benders decomposition solves the difficult Problem (14) by a finite alternating sequence of Problems (16), and (22), which are of type linear programming and a pure not linear, e.g. in our particular case integer programming.

It is worth to note that as there is no restriction on the set $\mathcal{S}$, if there are any constraints containing only the variables $\mathbf{r}$, then the satisfaction of these constraints can be included in the definition of the set $\mathcal{S}$.

5 The frame of the Benders decomposition in the particular case

This section consists of two parts. First the special structures of the coefficient matrices of problem (13) are explored. Based on that the form of the Benders decomposition is described in this particular case. Further special properties are discussed in the next section.

In what follows $\mathbf{e}_t$ is again the t-dimensional vector of which all of its components are 1.

Assume that the order of the components in vector $\mathbf{w}$ is $w_{111}, w_{112}, ..., w_{11n}, w_{121}, ..., w_{nnn}$. Similarly let $d_{11}, d_{12}, ..., d_{1n}, d_{21}, d_{22}, ..., d_{nn}$ be the order of the components in vector $\mathbf{d}$ formed from the distances. It is also assumed that the order of the components in $\mathbf{x}$ is $x_{11}, x_{12}, ..., x_{1n}, x_{21}, x_{22}, ..., x_{nn}$. Then vector $\mathbf{f}$, which is the vector of the objective function coefficients of $\mathbf{w}$, is obtained by the following matrix multiplication:

$$\mathbf{f} = \begin{pmatrix} -\mathbf{e}_n & \mathbf{0} & ... & \mathbf{0} \\ \mathbf{0} & -\mathbf{e}_n & ... & \mathbf{0} \\ & & ... & \\ \mathbf{0} & \mathbf{0} & ... & -\mathbf{e}_n \end{pmatrix} \mathbf{d}. \tag{23}$$

The structure of the coefficient matrices of the constraints are as follows. $\mathbf{A}_1$ is the matrix of an $n \times n$ assignment problem, i.e. its structure is this:

$$\mathbf{A}_1 = \begin{pmatrix} \mathbf{A}_{11} \\ \mathbf{A}_{12} \end{pmatrix}, \mathbf{A}_{11} = \begin{pmatrix} \mathbf{e}_n^T & \mathbf{0}^T & ... & \mathbf{0}^T \\ \mathbf{0}^T & \mathbf{e}_n^T & ... & \mathbf{0}^T \\ & & ... & \\ \mathbf{0}^T & \mathbf{0}^T & ... & \mathbf{e}_n^T \end{pmatrix}, \mathbf{A}_{12} = \begin{pmatrix} \mathbf{I}_n & \mathbf{I}_n & ... & \mathbf{I}_n \end{pmatrix}, \tag{24}$$

where $\mathbf{I}_n$ is the $n \times n$ unit matrix. Constraints (3)-(5) describe the feasible set of a TSP. Therefore $\mathbf{B}_2$ consists of three parts, i.e.

$$\mathbf{B}_2 = \begin{pmatrix} \mathbf{B}_{21} \\ \mathbf{B}_{22} \\ \mathbf{B}_{23} \end{pmatrix}, \tag{25}$$

and $\begin{pmatrix} \mathbf{B}_{21} \\ \mathbf{B}_{22} \end{pmatrix}$ is again the matrix of an $n \times n$ assignment problem, i.e.

$$\mathbf{B}_{21} = \mathbf{A}_{11}, \mathbf{B}_{22} = \mathbf{A}_{12}. \tag{26}$$

Constraint (5) excludes short circuits. In principle it excludes all of them, in practice only the explored ones. Therefore its row are the negative characteristic vectors of sets of components containing at least 2 and at most $n-1$ components. The following notation is used:

$$\mathbf{B}_{23} = \begin{pmatrix} -\mathbf{v}_1^T \\ ... \\ -\mathbf{v}_m^T \end{pmatrix}, \tag{27}$$

where

$$\mathbf{v}_i \in \{0,1\}^n, \quad 2 \leq \sum_{k=1}^{n} v_{ik} \leq n-1, \quad i = 1, ..., m.$$

The appropriate right-hand side vector, i.e. $\mathbf{b}_2$, is partitioned accordingly, i.e.

$$\mathbf{b}_2 = \begin{pmatrix} \mathbf{b}_{21} \\ \mathbf{b}_{22} \\ \mathbf{b}_{23} \end{pmatrix} = \begin{pmatrix} \mathbf{e}_n \\ \mathbf{e}_n \\ -\mathbf{e}_m \end{pmatrix}. \tag{28}$$

The third set of constraints, i.e. Inequalities (10), and (9), describe the linearization of the $x_{il}y_{kl}$ products. Notice that (9) must be multiplied by -1 to obtain the form used in Problem (13). All the matrices $\mathbf{A}_3$, $\mathbf{B}_3$, and $\mathbf{C}_3$ and the vector $\mathbf{b}_3$ are partitioned according to the two sets of constraints, i.e.

$$\mathbf{A}_3 = \begin{pmatrix} \mathbf{A}_{31} \\ \mathbf{A}_{32} \end{pmatrix}, \quad \mathbf{B}_3 = \begin{pmatrix} \mathbf{B}_{31} \\ \mathbf{B}_{32} \end{pmatrix}, \quad \mathbf{C}_3 = \begin{pmatrix} \mathbf{C}_{31} \\ \mathbf{C}_{32} \end{pmatrix}, \quad \text{and} \quad \mathbf{b}_3 = \begin{pmatrix} \mathbf{b}_{31} \\ \mathbf{b}_{32} \end{pmatrix},$$

where the sizes of $\mathbf{A}_{31}$, and $\mathbf{B}_{31}$ are $n^3 \times n^2$, the size of $\mathbf{C}_{31}$ is $n^3 \times n^3$ and the structure of these matrices is as follows:

$$\mathbf{A}_{31} = \begin{pmatrix} \mathbf{I}_n & \mathbf{0} & \mathbf{0} & ... & \mathbf{0} \\ & & & ... & \\ \mathbf{I}_n & \mathbf{0} & \mathbf{0} & ... & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_n & \mathbf{0} & ... & \mathbf{0} \\ & & & ... & \\ \mathbf{0} & \mathbf{I}_n & \mathbf{0} & ... & \mathbf{0} \\ & & & ... & \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & ... & \mathbf{I}_n \\ & & & ... & \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & ... & \mathbf{I}_n \end{pmatrix}, \tag{29}$$

$$\mathbf{B}_{31} = \begin{pmatrix} \mathbf{I}_{n^2} \\ ... \\ \mathbf{I}_{n^2} \end{pmatrix}, \tag{30}$$

$$\mathbf{C}_{31} = -\mathbf{I}_{n^3}, \tag{31}$$

$$\mathbf{b}_{31} = \mathbf{e}_{n^3}. \tag{32}$$

Furthermore

$$\mathbf{A}_{32} = -\mathbf{A}_{31}, \quad \mathbf{B}_{32} = -\mathbf{B}_{31}, \quad \mathbf{C}_{32} = -2\mathbf{C}_{31}, \quad \text{and} \quad \mathbf{b}_{32} = \mathbf{0}. \tag{33}$$

The Benders decomposition is applied with the following "casting" of the variables. The role of the linear continuous variables, i.e. the role of $\mathbf{p}$ variables, is given to the vector $\mathbf{x}$ and the pair $(\mathbf{y}, \mathbf{w})$ plays the role of the vector $\mathbf{r}$.

As it has great importance the set $\mathcal{S}$ is given in a separated definition.

Definition 5.1 *The set $\mathcal{S}$ is defined such that both $\mathbf{y}$, and $\mathbf{w}$ are binary vectors and their are not contradiction with each other. This requirement means that:*

- $w_{ikl} = 1$ *only if* $y_{kl} = 1$ *and*
- *the vector* $\mathbf{y}$ *describes a Hamiltonian circuit.*

During the algorithm the requirement that $\mathbf{y}$ must define a Hamiltonian circuit is handled dynamically, i.e. only those constraints are required, which exclude a potential non-Hamiltonian solution.

It is supposed that the vector $\mathbf{u}$ of the variables of Problem (16) is partitioned into six parts of the constraints. The six parts are the two parts of the assignment problem, i.e. 1 and 1, the assignment part of the TSP and the exclusion of the small circuits, i.e. (3) and (4) together and (5), finally (9) and (10). Then the particular form of Problem (16) is for a fixed pair $(\hat{\mathbf{y}}, \hat{\mathbf{w}})$:

$$\begin{array}{ll} \min & \left(\begin{pmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \\ \mathbf{b}_3 \end{pmatrix} - \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{B}_2 & \mathbf{0} \\ \mathbf{B}_3 & \mathbf{C}_3 \end{pmatrix} \begin{pmatrix} \hat{\mathbf{y}} \\ \hat{\mathbf{w}} \end{pmatrix} \right)^T \mathbf{u} \\ \text{subject to} & \\ & \begin{array}{rcl} (\mathbf{A}_1^T, \mathbf{0}, \mathbf{A}_3^T)\mathbf{u} & \geq & \mathbf{d} \\ \mathbf{u}_{23} & \geq & \mathbf{0} \\ \mathbf{u}_3 & \geq & \mathbf{0} \end{array} \end{array} \tag{34}$$

The particular form of the individual linear inequalities is

$$u_{11i} + u_{12l} + \sum_{k=1}^{n} u_{31ikl} - \sum_{k=1}^{n} u_{32ikl} \geq -d_{il}. \tag{35}$$

Hence $\mathbf{u} = \mathbf{0}$ is always a feasible solution of (34), as the distances are nonnegatives. It means that the assumption that (16) has a feasible solution is automatically satisfied.

In the description of the algorithm the following notations are used:

- $\mathcal{R}$ is the set of pairs of binary vectors $(\mathbf{y}, \mathbf{w})$ satisfying that $y_{kl} = 0$ implies that $\forall\, i : w_{ikl} = 0$.
- If (T) denotes an optimization problem then let $\text{OPT}(T)$ denote an optimal solution of (T) provided by any algorithm used to solve the problem.
- Similarly if (T) is a linear programming problem then $\text{EXTR}(T)$ is the last extremal point visited by the simplex method and

- DIRECTION(T) is the direction such that the value of the objective function improves by a step started from EXTR(T) and following direction DIRECTION(T).
- The set $\mathcal{H}$ consists of smaller circuits, which appeared in a vector $\mathbf{y}$, i.e. the appropriate constraints (5) must be required for each $H \in \mathcal{H}$.
- For a given vector $\mathbf{y}$ SUBCIRCUIT($\mathbf{y}$) is one small, i.e. non-Hamiltonian circuit appears in $\mathbf{y}$ and
- CIRCUIT($\mathbf{y}$) is the number of circuits represented by $\mathbf{y}$. It is 1 if and only if $\mathbf{y}$ represents a Hamiltonian circuit.
- The variables depending on the iteration are these:
  - β the index of the iteration,
  - $\mathcal{Q}_\beta$ the set of explored extremal points and directions,
  - $\mathcal{C}$ the set of explored subarcuate,
  - z_β the optimal objective function value of linear programming subproblem,
  - s_β the optimal value of the integer programming subproblem,
  - $(\mathbf{y}_\beta, \mathbf{w}_\beta)$ the optimal solution of the integer programming subproblem denoted by INTEGER$_\beta$,
  - $(\mathbf{x}^*, \mathbf{y}^*, \mathbf{w}^*)$ the optimal solution of the original problem,
  - $\mathbf{u}_\beta$ the extremal point obtained in the k-th iteration,
  - $\mathbf{v}_\beta$ the extremal direction obtained in the k-th iteration.

Algorithm 5.1

1. **Begin**
2. $\mathcal{Q}_0 := \emptyset$
3. $\mathcal{C} := \emptyset$
4. $s_0 := +\infty$
5. $z_0 := 0$
6. $(\mathbf{y}_0, \mathbf{w}_0) \in \mathcal{S}$ {An arbitrary element}
7. $\beta := 0$
8.

$$z_\beta = \min \left(\begin{pmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \\ \mathbf{b}_3 \end{pmatrix} - \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{B}_2 & \mathbf{0} \\ \mathbf{B}_3 & \mathbf{C}_3 \end{pmatrix} \begin{pmatrix} \mathbf{y}_\beta \\ \mathbf{w}_\beta \end{pmatrix} \right)^T \mathbf{u}$$

$$\text{SUBJECT TO} \quad (\mathbf{A}_1^T, \mathbf{0}, \mathbf{A}_3^T)\mathbf{u} \geq \mathbf{d} \quad \mathbf{u}_{23} \geq \mathbf{0} \quad \mathbf{u}_3 \geq \mathbf{0} \qquad (34_\beta)$$

9. **if** $z_\beta \geq s_\beta - \mathbf{f}^T \mathbf{w}_\beta$

10. **then**

11. **begin**

12. **goto 36**.

13. **end**

14. **if** $-\infty < z_\beta < s_\beta - \mathbf{f}^T \mathbf{w}_\beta$

15. **then**

16. **begin**

17. $\mathbf{u}_\beta := \textsc{opt}(34_\beta)$

18. $\mathcal{Q}_{k+1} := \mathcal{Q}_\beta \cup \left\{ \begin{pmatrix} 1 \\ \mathbf{u}_\beta \end{pmatrix} \right\}$

19. **end**

20. **else**

21. **if** $z_\beta = -\infty$

22. **then**

23. **begin**

24. $\mathbf{u}_\beta := \textsc{extr}(34_\beta)$

25. $\mathbf{v}_\beta := \textsc{direction}(34_\beta)$

26. $\mathcal{Q}_{\beta+1} := \mathcal{Q}_\beta \cup \left\{ \begin{pmatrix} 1 \\ \mathbf{u}_\beta \end{pmatrix}, \begin{pmatrix} 0 \\ \mathbf{v}_\beta \end{pmatrix} \right\}$

27. **end**

28. **repeat**

29. $s_{\beta+1} := \max\ s$

$$\forall (u_0, \mathbf{u}) \in \mathcal{Q}_\beta : \ u_0(\mathbf{f}^T\mathbf{w} - s) + (\mathbf{u}_1^T, \mathbf{u}_2^T, \mathbf{u}_3^T) \left(\begin{pmatrix} \mathbf{e}_{2n} \\ \mathbf{b}_2 \\ \mathbf{b}_3 \end{pmatrix} - \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{B}_2 & \mathbf{0} \\ \mathbf{B}_3 & \mathbf{C}_3 \end{pmatrix} \begin{pmatrix} \mathbf{y} \\ \mathbf{w} \end{pmatrix} \right) \geq 0 \qquad (\textsc{integer}_\beta)$$

$$(\mathbf{y}, \mathbf{w}) \in \mathcal{R}$$

$$\forall H \in \mathcal{H} : \ \textstyle\sum_{k \in H} \sum_{l \in \overline{H}} y_{kl} \ \geq \ 1$$

30. $(\mathbf{y}_{\beta+1}, \mathbf{w}_{\beta+1})$:=OPT(INTEGER$_\beta$)

31. **if** CIRCUIT$(\mathbf{y}_{\beta+1}) > 1$

32. **then** $\mathcal{H} := \mathcal{H} \cup \{$SUBCIRCUIT$(\mathbf{y}_\beta)\}$

33. **until** CIRCUIT$(\mathbf{y}_{\beta+1}) > 1$

34. $\beta := \beta + 1$

35. **goto 8**.

36. $\mathbf{y}^* := \mathbf{y}_\beta$

37. $\mathbf{w}^* := \mathbf{w}_\beta$

38. $\mathbf{x}^* :=$ OPT$(\max\{(-\mathbf{d})^T\mathbf{x} \mid \mathbf{A}_1\mathbf{x} = \mathbf{e}_{2n},\ \mathbf{A}_3\mathbf{x} \leq \mathbf{b}_3 - \mathbf{B}_3\mathbf{y}^* - \mathbf{C}_3\mathbf{w}^*,\ \mathbf{x} \in \{0,1\}^{n\times n}\})$.

39. **end**

The correctness of the algorithm has not been proved yet as the problem in Row 35 giving the optimal $\mathbf{x}$ part of the solution is a combinatorial optimization problem instead of a pure linear programming one. The aim of the next section is to prove that the current version of (15) reserves its combinatorial nature.

6 The combinatorial nature of the Benders decomposition in the particular case

The particular form of (15) is

$$\begin{array}{rl} \max & (-\mathbf{d})^T\mathbf{x} \\ & \mathbf{A}_1\mathbf{x} = \mathbf{e}_{2n} \\ & \mathbf{A}_3\mathbf{x} \leq \mathbf{b}_3 - \mathbf{B}_3\hat{\mathbf{y}} - \mathbf{C}_3\hat{\mathbf{w}} \\ & \mathbf{x} \in \{0,1\}^{n^2}. \end{array} \tag{36}$$

Without the inequalities in the third row Problem (36) is an assignment problem. The objective of this section is just to show that (36) behaves in the frame of the Benders decomposition like an assignment problem. For the possible values of each pair of (y_{kl}, w_{ikl}) there are the following cases considering the appropriate constraints (9), (10) and the fact that $(\mathbf{y}, \mathbf{w}) \in \mathcal{R}$.

y_{kl}	w_{ikl}	x_{il}	the binding constraint
1	1	1	(10)
1	0	0	(9)
0	1	–	the case cannot occur
0	0	0,1	no binding constraint

These constraints may cause two types of infeasibilities. Then the appropriate sample of Problem (16) is unbounded. It is shown below that in both cases it is possible to give a direction such that the objective function of (16) is unbounded along it. To do so the following form of Farkas lemma is used.

Lemma 6.1 *Let* $\mathbf{G}$ *be an* $m \times n$ *matrix and* $\mathbf{g}$ *an* m*-dimensional vector. The* n*-dimensional vector of variables is denoted by* $\mathbf{t}$*. If the system*

$$\mathbf{G}\mathbf{t} \ \leq \ \mathbf{g} \tag{37}$$

has no solution then there is nonnegative m*-dimensional vector* $\underline{\lambda}$ *such that*

$$\mathbf{G}^T\underline{\lambda} \ = \ \mathbf{0} \quad \textit{and} \quad \mathbf{g}^T\underline{\lambda} \ < \ 0. \tag{38}$$

Assume that the linear programming problem

$$\begin{gathered} \min \mathbf{g}^T\underline{\mu} \\ \mathbf{G}^T\underline{\mu} = \mathbf{d} \end{gathered}$$

is to be solved, where $\mathbf{d}$ is a fixed n-dimensional vector. Then if system (38) has a solution then the linear programming problem either has no feasible solution, or is unbounded. In the latter case the vector $\underline{\lambda}$ gives a direction such that starting from any feasible solution the objective function is unbounded along this direction.

When lemma 6.1 is applied then the particular form of the system (37) is in case:

$$\mathbf{A}_1\mathbf{x} = \mathbf{e}_{2n}, \quad \mathbf{A}_3\mathbf{x} \leq \mathbf{b}_3 - \mathbf{B}_3\hat{\mathbf{y}} - \mathbf{C}_3\hat{\mathbf{w}}, \quad -\mathbf{x} \leq \mathbf{0}, \tag{39}$$

i.e. the matrix $\mathbf{G}$ is in this particular case

$$\mathbf{G} \ = \ \begin{pmatrix} \mathbf{A}_1 \\ \mathbf{A}_3 \\ -\mathbf{I}_{n\times n} \end{pmatrix}. \tag{40}$$

Similarly

$$\mathbf{g} \ = \ \begin{pmatrix} \mathbf{e}_{2n} \\ \mathbf{b}_3 - \mathbf{B}_3\hat{\mathbf{y}} - \mathbf{C}_3\hat{\mathbf{w}} \\ \mathbf{0} \end{pmatrix}. \tag{41}$$

As the first set of constraints is an equation system, it is allowed that their multipliers take negative values, too.

Case 1. Too many 1's are required. Assume that there are indices i_1, i_2, k_1, k_2, l with $i_1 \neq i_2$ such that $\hat{y}_{k_1l} = \hat{y}_{k_2l} = \hat{w}_{i_1k_1l} = \hat{w}_{i_2k_2l} = 1$ then the sum

$$\sum_{i=1}^{n} x_{il}$$

is at least 2 contradicting to the appropriate constraint (2). Then the appropriate inequalities of type (9) are

$$-x_{i_1k_1} - \hat{y}_{k_1l} + 2\hat{w}_{i_1k_1l} \leq 0, \quad -x_{i_2k_2} - \hat{y}_{k_2l} + 2\hat{w}_{i_2k_2l} \leq 0,$$

which are equivalent to

$$-x_{i_1k_1} \leq -1, \quad -x_{i_2k_2} \leq -1 \tag{42}$$

according to the current value of $\hat{\mathbf{y}}$, and $\hat{\mathbf{w}}$. Then the non-zero components of the appropriate $\underline{\lambda}$ vector are as follows. The weight of the equation

$$\sum_{i=1}^{n} x_{il} \ = \ 1,$$

and of the two inequalities of (42), and finally of the nonnegativity constraints

$$-x_{il} \ \leq \ 0 \quad i = 1, 2, ..., n, \ i \neq i_1, i_2$$

are 1. The weight of all other constraint is 0. With this weight the relation (38) is obtained. At the same time a direction of unboundedness of Problem (16) is determined, which (with a zero first component) must be added to the set $\mathcal{Q}$. It is easy to check if this case occurs. If the answer is yes then command in row 22 of the algorithm can be executed without applying any linear programming solver. The command in row 21 can be temporarily omitted as the extremal point can be added later to the set $\mathcal{Q}$ (with the supplementary component 1). The explanation is this. The scheme of Benders decomposition does not determine that what is the order in which the constraints of type (18) must be claimed. The only point is that in each iteration at least one new constraint must be added to Problem (22).

If there are indices i, k_1, k_2, l_1, l_2 such that $\hat{y}_{k_1l_1} = \hat{y}_{k_2l_2} = \hat{w}_{ik_1l_1} = \hat{w}_{ik_2l_2} = 1$ one can get the relation (38) in a similar way. It worth to note that this type of infeasibility does not exists with $k = k_1 = k_2$ because then the vector $\mathbf{y}$ is not a characteristic vector of a Hamiltonian circuit.

Case 2. Too few 1's are allowed. Here it is supposed that Case 1 does not occur. Not all x_{il} might be 1 as in the case $y_{kl} = 1$, $w_{ikl} = 0$ x_{il} must be 0. Let $\mathcal{P}$ be the set of such pairs. The elements of $\mathcal{P}$ are called *prohibited pair of indices.* In any feasible solution of the original problem the matrix $\mathbf{x}$ must be such that it contains in each row and in each column exactly one 1 and all other elements are 0. This requirement can be satisfied only if the maximal solution of the following matching problem consists of n edges. Let $\mathcal{V} = \{1, 2, ..., n\} \cup \{\hat{1}, \hat{2}, ..., \hat{n}\}$ be the set of vertices. The set of edges is $\mathcal{E} = \{(i, \hat{j}) \mid 1 \leq i, j \leq n\} \setminus \mathcal{P}$. König's theorem says that a matching of n edges exist if and only if for every nonempty subset $\mathcal{S}$ of $\{1, 2, ..., n\}$ the relation

$$|\mathcal{S}| \ \leq \ |\{\hat{j} \mid \exists i \in \mathcal{S} : \ (i, \hat{j}) \in \mathcal{E}\}|$$

holds.

The matching problem can be solved by a polynomial algorithm. If the optimal value is n then Problem (15) has an optimal solution. If Case 1 does not occur then still some variables x_{il} might be fixed to 1 but no other variable is fixed to 1 in their row, and column. These fixings must be taken into consideration when the matching problem is solved.

If the optimal value of the matching problem is less than n then the multipliers in (38) are these. Then there is a nonempty index set $\mathcal{S} \subset \{1, 2, ..., n\}$ and another set $\mathcal{T} \subset \{\hat{1}, \hat{2}, ..., \hat{n}\}$ such that $|\mathcal{S}| > |\mathcal{T}|$ and $\forall i \in \mathcal{S} \forall \hat{j} \in \{\hat{1}, \hat{2}, ..., \hat{n}\} \setminus \mathcal{T} : (i, \hat{j}) \notin \mathcal{E}$. The multipliers of the equations (1) belonging to indices $i \notin \mathcal{S}$ are 1. The multipliers of equations (2) belonging to an index $\hat{j} \notin \mathcal{T}$ are -1. The current form of the constraints (9) for prohibited pair, i.e. if $y_{kl} = 1$ and $w_{ikl} = 0$, is $x_{il} \leq 0$. As all pair with $(i, \hat{j})$ with $i \in \mathcal{S}$ and $\hat{j} \notin \mathcal{T}$ are prohibited, therefore the multipliers of all such constraints of type (9) are 1. Finally the multipliers of the nonnegativity constraints of variables $x_{i\hat{j}}$ with $i \notin \mathcal{S}$ and $\hat{j} \in \mathcal{T}$ are 1. The multipliers of all other constraints are 0.

Thus Problem (15) or equivalently (34_β) can be handled during the algorithm as follows:

- At first Case 1 type of infeasibilities are eliminated.

-Then Problem (36) is reduced according to which x_{il}'s must be 1. The reduced problem is solved with the following modified objective function $(-\hat{\mathbf{d}}^T)\mathbf{x}$, where

$$-\hat{d_{il}} = \begin{cases} -d_{il} & \text{if } (i,l) \notin \mathcal{P} \\ -\infty & \text{if } (i,l) \in \mathcal{P}. \end{cases}$$

Thus an assignment problem is obtained and it can be solved by the some combinatorial algorithm. If the optimal value is finite then it is the optimal value of the current Problem (36). Optimal solution can be generated e.g. via complementary slackness. If the optimal value is $-\infty$ then Case 2 type of infeasibility occurs and a direction of unboundedness is obtained.

References

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