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Neighborhood hypergraphs of bipartite graphs

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Abstract. Matrix symmetrization and several related problems have an extensive literature, with a recurring ambiguity regarding their complexity status and their relation to graph isomorphism. We present a short survey of these problems to clarify their status. In particular we recall results from the literature showing that matrix symmetrization is in fact NP-hard, while graph isomorphism is still an open problem. One of the equivalent formulations of matrix symmetrization is the problem of recognizing if a given hypergraph can be realized as the neighborhood hypergraph of a graph. While all variants of this problem are NP-hard, in general, the restriction of one of the variants for bipartite graphs is known to be equivalent with graph isomorphism. Generalizing these results, we consider several other variants of the bipartite neighborhood recognition problem, and show that all these variants are either polynomial-time solvable, or are equivalent with graph isomorphism.
Finally, we consider the uniqueness of bipartite neighborhood realizations of hypergraphs. In general, a realization, if exists, may not be unique for any variants of the problem, even when realizations are restricted to bipartite graphs. Settling an open problem, we prove uniqueness of bipartite realizations for two variants, for the so called open and closed neighborhood hypergraphs of bipartite graphs.
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1 Introduction

In this paper we consider variants of matrix symmetrization (that is the problem of deciding if by row and column permutations a matrix can be brought to a symmetric form), matrix transposition (deciding if by row and column permutations we can derive the transpose of a given matrix), and graph isomorphism. Numerous papers appear in the literature addressing these and other related problems, sometimes with seemingly contradictory complexity claims. We devote the first part of our paper to a short survey on matrix symmetrization and transposition, clarifying the relations between the various definitions and clarifying the complexity status of these problems.

One can observe three distinct "waves" in the related literature, with some good results getting forgotten between the waves. In 1979–1981 a series of papers was published by McCarthy and McKay [35, 36], Colbourn [17], Colbourn and McKay [18], Lalonde [31, 30], and Bhat [7] on matrix symmetrization and transposition. On the one hand, a nice result of Lalonde implies that it is NP-hard to recognize neighborhood hypergraphs (see Subsection 2.2), while on the other hand, McCarthy, McKay, and Colbourn noted that recognizing neighborhood hypergraphs is equivalent to matrix symmetrization (see Subsection 2.3), and that matrix transposition is computationally equivalent to graph isomorphism (see Subsection 2.5). Later, in 1993–1994 Aigner and Triesch [3, 2] rediscovered the result of Lalonde, yet noted that "we do not know whether matrix symmetrization is NP-complete" (see Subsection 2.3). Furthermore, they derived some additional related problems equivalent to graph isomorphism (see Section 2). In 2003 Bondy, Durán, Lin, and Szwarcfiter [10] showed that another problem is equivalent with graph isomorphism, which later turned out to be equivalent with matrix transposition (see Subsection 2.5). Furthermore, they claimed the computational equivalence of matrix symmetrization and graph isomorphism, based on a statement which unfortunately is not true (see Subsection 2.4 for details and a counterexample).

The second part of this paper presents some new results. One of the equivalent formulations of matrix symmetrization is the problem of recognizing if a given hypergraph can be realized as the neighborhood hypergraph of a graph. While all variants of this problem are known to be NP-hard in general (see Lalonde [31, 30]), a related problem for bipartite graphs is known to be equivalent with graph isomorphism (see Aigner and Triesch [2, 3]). Generalizing this result, and other variants of the bipartite neighborhood recognition problem, we show that all these problems are either polynomial-time solvable, or are equivalent with graph isomorphism. Finally, we consider the uniqueness of bipartite neighborhood realizations of hypergraphs. In general, a realization, if exists, may not be unique for any variants of the problem, even when realizations are restricted to bipartite graphs. Settling an open problem, we prove uniqueness of bipartite realizations for two variants, for the so called open and closed neighborhood hypergraphs of bipartite graphs.

We use standard graph-theoretic terminology, see for example Berge [6], Melnikov, Sarvanov, Tyshkevich, Yemelichev, and Zverovich [37], or West [42].

2 A survey

2.1 Matrix symmetrization and bigraphs

An $n \times n$ matrix $A = (a_{ij})$ is *symmetric* if $a_{ij} = a_{ji}$ for all $i, j \in \{1, 2, \ldots, n\}$. It is called *symmetrizable* if its rows and columns can be permuted independently such that the resulting matrix is symmetric. The *matrix symmetrization* problem (or MS, in short) is to decide whether a given square matrix A is symmetrizable. In fact, there are several equivalent definitions for matrix symmetrization.

Fact 1 *The following claims are equivalent for an $n \times n$ matrix A:*

(i) a symmetric matrix can be obtained from A by permuting its rows,

(ii) a symmetric matrix can be obtained from A by permuting its columns,

(iii) matrix A is symmetrizable, and

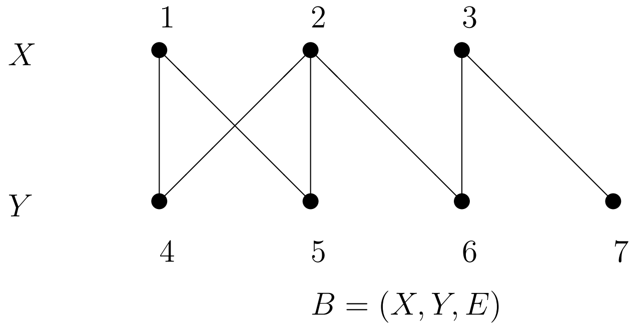
(iv) there exists a permutation σ such that $a_{\sigma(i)j} = a_{\sigma(j)i}$ holds for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, n$.

Implications (i) $\Longrightarrow$ (iii), and (ii) $\Longrightarrow$ (iii) are obvious, by the definitions. To see (iii) $\Longrightarrow$ (i), (ii), assume that we can produce a symmetric matrix from A by applying a permutation π to its rows and a permutation ρ to its columns, then a symmetric matrix can also be obtained from A by applying $\rho^{-1}\pi$ to its rows, or $\pi^{-1}\rho$ to its columns. Finally, (iv) is clearly equivalent to (i).

2.2 Bigraphs and semi-induced pairings

The matrix symmetrization problem has an interesting graphical interpretation. We define a *bigraph* $B = (X, Y, E)$ as a bipartite graph on vertex-set $V = X \cup Y$ with a fixed order (X, Y) of its parts, i.e., where $X \cap Y = \emptyset$ and $E \subseteq X \times Y$. To a bigraph $B = (X, Y, E)$ we can associate its *X-Y-adjacency* matrix $A(B) = (a_{ij}) \in \{0, 1\}^{X \times Y}$ defined by $a_{ij} = 1$ if and only if $(i, j) \in E$. Conversely, any $0 - 1$ matrix $A = (a_{ij})$ can be viewed as the X-Y adjacency matrix $A = A(B)$ of a corresponding bigraph $B = (X, Y, E)$, where X is the set of row indices of A, Y is the set of column indices of A, and $(i, j) \in E$ if and only if $a_{ij} = 1$, see an example in Figure 1.

The Matrix Symmetrization Problem can be reformulated in terms of special automorphisms of bigraphs. Consider a bigraph $B = (X, Y, E)$, and an *automorphism* $\alpha : (X \cup Y) \to (X \cup Y)$ of the underlying bipartite graph B, that is for which $(i, j) \in E$ if and only if $(\alpha(i), \alpha(j)) \in E$. We call α *involutory* if $\alpha(i) = j$ implies $\alpha(j) = i$ (i.e., if α^2 is identity), and it is called *switching* if $\alpha(X) = Y$ and $\alpha(Y) = X$.



$$\begin{matrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{matrix}$$

$A(B)$

Figure 1: A bigraph $B = (X, Y, E)$ and its adjacency matrix $A(B)$.

Fact 2 *A bigraph $B = (X, Y, E)$ has a switching and involutory automorphism if and only if its adjacency matrix $A = A(B)$ is symmetrizable.*

Let $A = A(B) = (a_{ij})$ be a square $0-1$ matrix, and assume first that it is symmetrizable. According to Fact 1, this is equivalent with the existence of a permutation σ of the rows of A satisfying the equalities in (iv) of Fact 1. Let us then define

$$\alpha(i) = \sigma(i) \text{for} i \in X \qquad \text{and} \qquad \alpha(j) = \sigma^{-1}(j) \qquad \text{for} \qquad j \in Y. \tag{1}$$

Clearly, for every permutation σ the corresponding mapping α will be switching and involutory, furthermore, it is an automorphism of B, by (iv) of Fact 1. Conversely, every switching and involutory automorphism α of B defines a unique permutation σ of the rows of $A = A(B)$ by (1).

Given a bigraph $B = (X, Y, E)$, let us call a set $S \subseteq X \times Y$ a *semi-induced pairing* if it satisfies the following properties:

(a) S consists of pairwise disjoint pairs,

(b) for any two distinct pairs $(i, j) \in S$ and $(i', j') \in S$ the two pairs (i, j') and (i', j) are simultaneously edges or non-edges of B.

Furthermore, S is called a *perfect* semi-induced pairing of B, if in addition it satisfies

(c) $|S| = |X| = |Y|$.

For a given bigraph $B = (X, Y, E)$ let us consider the problem of *finding a perfect semi-induced pairing* in B, and let us call it in short problem PSP.

Fact 3 *Given a bigraph $B = (X, Y, E)$ there is a one-to-one correspondence between its switching and involutory automorphisms, and its perfect semi-induced pairings.*

To a switching and involutory automorphism α of $B = (X, Y, E)$ let us associate the pairing $S = \{(i, \alpha(i)) \mid i \in X\}$. Then properties (a), (b) and (c) follow from the facts that α is a switching and involutory automorphism of B. Conversely, given a perfect semi-induced pairing $S \subseteq X \times Y$, we can define a mapping α by setting $\alpha(i) = j$ and $\alpha(j) = i$ for all $(i, j) \in S$. Then, properties (a), (b) and (c) above will imply that α is a switching and involutory automorphism of B.

Fact 2 and Fact 3 immediately imply another equivalent reformulation of the matrix symmetrization problem.

Fact 4 *Let A be a square $0-1$ matrix, and let $B = (X, Y, E)$ be the corresponding bigraph, for which $A = A(B)$. Then, A is symmetrizable if and only if B has a perfect semi-induced pairing.*

Fact 4 implies that problems MS and PSP are polynomial-time equivalent. A *semi-induced matching* of a bigraph $B = (X, Y, E)$ is a semi-induced pairing S for which $S \subseteq E$, see Figure 2.

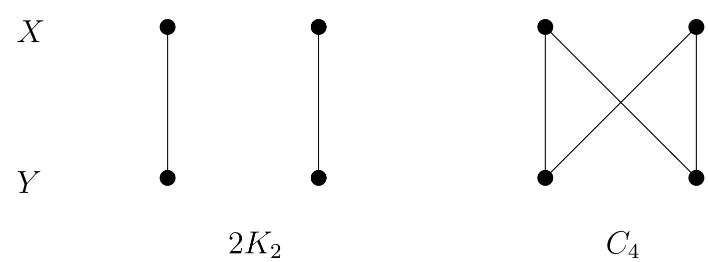


Figure 2: Two possibilities for two edges of a semi-induced matching.

If additionally we have $|S| = |X| = |Y|$, then S is called a *perfect semi-induced matching*, see Figure 3.

The above definitions are similar to the well-known notion of an *induced matching*, i.e., a matching M in a graph G such that, for every pair of edges of M, their end-vertices induce a $2K_2$ in G.

Theorem 1 (Cameron [11]) *Finding a maximum size induced matching in a bigraph is NP-hard.*

Let us observe that the same construction, as in Cameron [11], can be used straightforwardly to prove an analogous claim for semi-induced matchings.

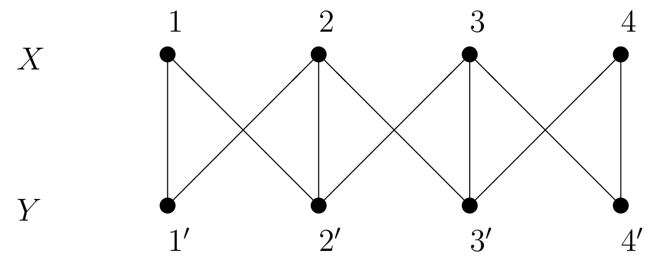


Figure 3: A perfect semi-induced matching $S = \{11', 22', 33', 44'\}$.

Fact 5 *Finding a maximum size semi-induced matching in a bigraph is NP-hard.*

However, the above results do not imply the hardness of the problem of finding perfect semi-induced matchings in bigraphs. Let us note that, analogously to Fact 4, the latter problem corresponds to a special version of matrix symmetrization.

Fact 6 *Let A be a square $0-1$ matrix, and let $B = (X, Y, E)$ be the corresponding bigraph, for which $A = A(B)$. Then there are row and column permutation which change A into a symmetric matrix with all-1 main diagonal if and only if B has a perfect semi-induced matching.*

Let us also consider the problem of finding a perfect semi-induced matching in a given bigraph, and let us call it on short problem PSM. According to the above claim, problem PSM is computationally equivalent with a restricted version of problem MS.

By a reduction form the 3-Satisfiability Problem Lubiw [32] showed that it is NP-hard to recognize whether a given graph has an involutory automorphism without fixed points. Using this result, Lalonde [30, 31] clarified the complexity of problem PSP.

Theorem 2 (Lalonde [30, 31]) *The* Perfect Semi-induced Pairing Problem *is NP-hard.*

Given a bigraph $B = (X, Y, E)$ let us introduce its *bi-complement* $\overline{B} = (X, Y, \overline{E})$ defined by $\overline{E} = \{(i, j) \mid i \in X, j \in Y, (i, j) \notin E\}$. Let us then observe the following easy relations between perfect semi-induced pairings of B and $\overline{B}$.

Fact 7 *Bigraphs B and $\overline{B}$ have the same perfect semi-induced pairings. Furthermore, M is a perfect semi-induced matching in $\overline{B}$ if and only if it is perfect semi-induced pairing of B satisfying $M \cap E = \emptyset$.*

Lalonde [30, 31] also studied the problem of finding a perfect semi-induced pairing S in a given bigraph $B = (X, Y, E)$ for which $S \cap E = \emptyset$. According to Fact 7, this problem is equivalent with PSM, and we can restate his result as follows.

Theorem 3 (Lalonde [30, 31]) *From any bigraph B one can construct another bigraph B', in linear time, such that B has a perfect semi-induced pairing if and only if B' has a perfect semi-induced matching.*

These results and the equivalences of the considered problems then readily settle the complexity status of those problems.

Corollary 1 *The* Matric Symmetrization*, the* Perfect Semi-induced Pairing *and the* Perfect Semi-induced Matching *are all NP-hard problems.*

Let us remark finally that by Fact 7 problem PSM corresponds to two special variants of matrix symmetrization, which are also NP-hard problems.

Corollary 2 *The* Matric Symmetrization Problem *remains NP-hard even if prescribe the main diagonal of the resulting symmetric matrix either as $a_{ii} = 1$ for $i = 1, 2, \ldots, n$, or as $a_{ii} = 0$ for $= 1, 2, \ldots, n$.*

2.3 Neighborhood hypergraphs

A further reformulation of matrix symmetrization is in terms the so-called neighborhood hypergraphs. Given a graph $G = (V, E)$, the *neighborhood* of a vertex $x \in V$ is defined as $N_G(x) = N(x) = \{y \in V \mid (x, y) \in E\}$. The set $N(x)$ is frequently referred as the *open neighborhood* of x, emphasizing the fact that $x \notin N(x)$ in a loopless graph G. Correspondingly, the set $N_G[x] = N[x] = N(x) \cup \{x\}$ is called the *closed neighborhood* of x. We associate two hypergraphs to G, defined on the same set V of vertices, by setting

$$\mathcal{N}(G) \;=\; \{N(x) \mid x \in V\} \quad \text{and} \quad \mathcal{N}[G] \;=\; \{N[x] \mid x \in V\}.$$

Let us note that these hypergraphs are in fact multi-hypergraphs, since different vertices may have the same neighborhood. All hypergraphs mentioned in our paper will be multi-hypergraphs, and to simplify our writing, we will just refer to them as hypergraphs.

The hypergraph $\mathcal{N}(G)$ is called the *open neighborhood hypergraph* of G, while $\mathcal{N}[G]$ is called the *closed neighborhood hypergraph* of G. Let us remark that a characterizing feature of neighborhoods is the following obvious relation:

$$x \in N(y) \quad \text{if and only if} \quad y \in N(x). \tag{2}$$

Given a hypergraph $\mathcal{H}$ on vertex V, the *open* (respectively, *closed*) *neighborhood realization* problem is asking to find a graph $G = (V, E)$ for which $\mathcal{N}(G) = \mathcal{H}$ (respectively, $\mathcal{N}[G] = \mathcal{H}$). We shall refer to these problems respectively, as problems ON and CN. Problem ON was proposed by Sós [41]. Let us observe that in fact problems ON and CN are equivalent.

Fact 8 *Given a graph $G = (V, E)$, let us denote its complementary graph by $\overline{G}$. Then we have the following relation between their neighborhood hypergraphs:*

$$\mathcal{N}(\overline{G}) \;=\; \{V \setminus X \mid X \in \mathcal{N}[G]\} \;\; \textit{and} \;\; \mathcal{N}[\overline{G}] \;=\; \{V \setminus X \mid X \in \mathcal{N}(G)\}.$$

Straightforward by the definitions.

Let us note next that problem CN is equivalent with finding perfect semi-induced matchings in bigraphs. To see this, let us associate to a hypergraph $\mathcal{H}$ on vertex-set V a bigraph $B_{\mathcal{H}} = (X, Y, E)$ by setting $X = V$, $Y = \mathcal{H}$, and defining $E = \{(x, H) \mid x \in H \in \mathcal{H}\}$. We call $B = B_{\mathcal{H}}$ the *bigraph* of $\mathcal{H}$.

Fact 9 *Given a hypergraph $\mathcal{H}$ on vertex-set V, there exists a graph $G = (V, E)$ such that $\mathcal{H} = \mathcal{N}[G]$ if and only if the bigraph representation $B_{\mathcal{H}}$ of $\mathcal{H}$ has a perfect semi-induced matching.*

Assume first that $\mathcal{H} = \mathcal{N}[G]$ for a graph $G = (V, E)$, and consider the set $M = \{(x, N[x]) \mid x \in V\}$. It follows by the definitions and by (2) that M is a perfect semi-induced matching in the bigraph $B_{\mathcal{H}}$. Conversely, let us assume that M is a perfect semi-induced matching in the bigraph $B_{\mathcal{H}}$, and define $E = \{(x, y) \mid y \in H, y \neq x, (x, H) \in M\}$. Then it is again straightforward to see that for the graph $G = (V, E)$ we have $\mathcal{H} = \mathcal{N}[G]$.

More generally, given a graph $G = (V, E)$ and a mapping $\mathbf{b} : V \to \{0, 1\}$, let us define

$$\mathcal{N}(G, \mathbf{b}) \;=\; \{N(x) \mid x \in V, \mathbf{b}(x) = 0\} \cup \{N[x] \mid x \in V, \mathbf{b}(x) = 1\}.$$

We call $\mathcal{N}(G, \mathbf{b})$ the *mixed neighborhood hypergraph* of G. Clearly, if $\mathbf{b}(x) = 0$ for all $x \in V$ then $\mathcal{N}(G, \mathbf{b}) = \mathcal{N}(G)$, while if $\mathbf{b}(x) = 1$ for all $x \in V$, then we have $\mathcal{N}(G, \mathbf{b}) = \mathcal{N}[G]$. Fact 8 can be generalized as follows.

Fact 10 *Given a graph $G = (V, E)$, let $\overline{G}$ denote is complementary graph. For a mapping $\mathbf{b} : V \to \{0, 1\}$ let $\overline{\mathbf{b}} = \mathbf{1} - \mathbf{b}$. Then we have*

$$\mathcal{N}(\overline{G}, \overline{\mathbf{b}}) \;=\; \{V \setminus X \mid X \in \mathcal{N}(G, \mathbf{b})\}.$$

Straightforward by the definitions.

The *Mixed Neighborhood Realization* problem (MN in short) asks, if for a given hypergraph $\mathcal{H}$ on vertex-set V there exists a labelling $\mathbf{b} : V \to \{0, 1\}$ and a graph $G = (V, E)$ such that $\mathcal{H} = \mathcal{N}(G, \mathbf{b})$.

Let us note that problem MN is quite obviously related to perfect semi-induced pairings in bigraphs.

Fact 11 *Given a hypergraph $\mathcal{H}$ on vertex-set V, there exist a labelling $\mathbf{b} : V \to \{0,1\}$ and a graph $G = (V, E)$ for which $\mathcal{N}(G, \mathbf{b}) = \mathcal{H}$ hold if and only if the bigraph representation $B_{\mathcal{H}}$ of $\mathcal{H}$ has a perfect semi-induced pairing.*

The proof is perfectly analogous to that of Fact 9

Since an arbitrary bigraph $B = (X, Y, E)$ defines uniquely a hypergraph $\mathcal{H}$ on vertex-set $V = X$ for which $B = B_{\mathcal{H}}$, the above claim together with Fact 4 readily implies the following result.

Fact 12 *The* Matrix Symmetrization Problem *and the* Mixed Neighborhood Realization Problem *are equivalent.*

A further consequence of Lalonde's earlier cited results, summarized in Corollary 1, Fact 9 and Fact 12 is the following fact.

Corollary 3 *The* Open Neighborhood Realization*, the* Closed Neighborhood Realization *and the* Mixed Neighborhood Realization *are all NP-hard problems.*

Let us remark that the hardness of problem MN was rediscovered by Aigner and Triesch [3, 2], without realizing that it is equivalent with problem MS.

Finally, consider a further restricted version of problem MN, called the *Fixed Neighborhood Realization* problem (or FN in short), in which the input consists of a hypergraph $\mathcal{H}$ on vertex-set V and a labelling $\mathbf{b} : V \to \{0,1\}$, and the task is to find a graph $G = (V, E)$ for which $\mathcal{H} = \mathcal{N}(G, \mathbf{b})$. Clearly, problem FN is hard for certain input labellings $\mathbf{b}$, e.g., when $\mathbf{b} \equiv \mathbf{0}$ or when $\mathbf{b} \equiv \mathbf{1}$. Consequently, problem FN is not easier than problem ON, when we do not restrict the input labellings. However, the complexity of its restricted versions is known only when the input labellings are restricted to the set $\{\mathbf{1}, \mathbf{0}\}$ of constant vectors.

2.4 Graph isomorphism

Several of the above considered problems, or their variants, are related frequently in the literature to graph isomorphism, sometimes incorrectly. To clarify these connections, we recall here a few basic facts about graph isomorphism, without any intention of providing a comprehensive survey of this large and interesting area.

Given two hypergraphs, $\mathcal{H}$ on vertex-set V and $\mathcal{H}'$ on vertex-set V', we say that they are *isomorphic*, if there exists a bijection $\alpha : V \leftrightarrow V'$ such that $\alpha(S) = \{\alpha(x) \mid x \in S\} \in \mathcal{H}'$ if and only if $S \in \mathcal{H}$. The problem of *hypergraph isomorphism* (or in short problem HI) consists in recognizing if two given hypergraphs are isomorphic.

In the special case when the hypergraphs $\mathcal{H}$ and $\mathcal{H}'$ are the edge-sets of the graphs $G = (V, E)$ and $G = (V', E')$, respectively, we call the graphs G and G' isomorphic if the hypergraphs $\mathcal{H} = E$ and $\mathcal{H}' = E'$ are isomorphic. The corresponding restricted version of problem HI is well-known as *graph isomorphism* (or in short, problem GI).

We shall consider a further restriction to bigraphs. Given two bigraphs, $G = (X, Y, E)$ and $G' = (X', Y', E')$ we say that they are *isomorphic*, if there are bijections $\alpha : X \leftrightarrow X'$ and $\beta : Y \leftrightarrow Y'$ such that $(i, j) \in E$ if and only if $(\alpha(i), \beta(j)) \in E'$. The corresponding recognition problem we call *bigraph isomorphism* (or in short, problem BI).

Let us note that bigraph isomorphism is a proper restriction of graph isomorphism, in the sense that for some bigraphs $G = (X, Y, E)$ and $G' = (X', Y', E')$ we may have an isomorphism $\alpha : X \cup Y \leftrightarrow X' \cup Y'$ such that $(i, j) \in E$ if and only if $(\alpha(i), \alpha(j)) \in E'$, and still G and G' may not be isomorphic as bigraphs. Despite this, it is well-known and easy to see that these three problems are in fact equivalent.

Fact 13 *Problems* Hypergraph Isomorphism*,* Graph Isomorphism *and* Bigraph Isomorphism *are polynomial-time equivalent.*

It is immediate by the definitions that hypergraphs $\mathcal{H}$ and $\mathcal{H}'$ are isomorphic if and only if their bigraphs $B_{\mathcal{H}}$ and $B_{\mathcal{H}'}$ are isomorphic (as bigraphs). Problem GI is clearly a special case of problem HI. Finally, it is easy to associate to bigraphs $B = (X, Y, E)$ and $B' = (X', Y', E')$ graphs, by introducing new vertices connected to all elements of X (respectively of X') such that the resulting graphs will be isomorphic if and only if B and B' are isomorphic, as bigraphs.

It was noted by many authors that problem MS is at least as hard as problem GI. For completeness, we include a short proof here.

Fact 14 *The* Graph Isomorphism Problem *is polynomial-time reducible to the* Matrix Symmetrization Problem*.*

Let us consider bigraphs $B = (X, Y, E)$ and $B' = (X', Y', E')$. We use their adjacency matrices $A(B)$ and $A(B')$ to construct a new matrix M shown in Figure 4, where $\mathbf{1}$ (respectively, $\mathbf{0}$) is all-1 $|X| \times |X'|$ submatrix (respectively, all-0 $|Y'| \times |Y|$ submatrix).

We claim that M is symmetrizable if and only if B and B' are isomorphic (as bigraphs), from which the statement follows readily. To see the claim, observe that if a permutation π of rows of M produces a symmetric matrix, then π is a union of π_X that permutes the first $|X|$ rows and $\pi_{Y'}$ that permutes the last $|Y'|$ rows. Indeed, otherwise there is an all-1 row in the $A(B')^T$ part, but there is no all-1 column in the $A(B)$ part, since an all-0 row arises in this part.

In a recent publication Bondy, Durán, Lin, and Szwarcfiter [10] claimed that there exists a polynomial-time algorithm for deciding isomorphism between arbitrary graphs if and only if there is a polynomial-time algorithm to decide whether a given bipartite graph admits a switching and involutory automorphism (see Theorem 12 in [10]). As we have seen earlier,

$$M = \begin{array}{c|c|c|l} \multicolumn{1}{c}{} & \multicolumn{1}{c}{X'} & \multicolumn{1}{c}{Y} & \\ \cline{2-3} & \mathbf{1} & A(B) & X \\ \cline{2-3} & A(B')^T & \mathbf{0} & Y' \\ \cline{2-3} \end{array}$$

Figure 4: The reduction of problem GI to problem MS.

Fact 2 and Corollary 1 then would imply that the Graph Isomorphism Problem is NP-hard, which however cannot (yet) be concluded, since the proof in [10] is not correct. Their proof relies on the following statement: *Given a bigraph $B = (X, Y, E)$ let us associate to it two split graphs G and H, by adding to E the edge-set of the complete graph on X for G, and on Y for H. Then, they claim, G and H are isomorphic if and only if the bigraph B admits a switching and involutory automorphism.* This latter claim however is not true, as the following example in Figure 5 shows.

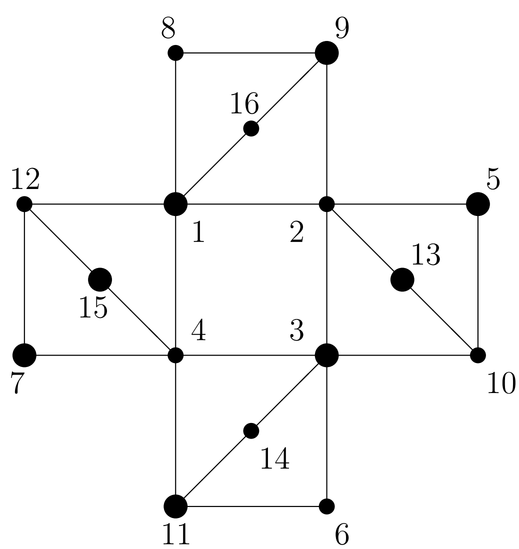


Figure 5: A counterexample – the bigraph $B = (X, Y, E)$.

In this example we have $X = \{1, 3, 5, 7, 9, 11, 13, 15\}$ and $Y = \{2, 4, 6, 8, 10, 12, 14, 16\}$, and edges are as indicated in Figure 5. This bigraph has an automorphism α that switches the parts X and Y, e.g., the one induced by a 90° rotation of the figure. This shows that the corresponding split graphs G and H are isomorphic. However, no involutory automorphism of $B = (X, Y, E)$ switches the parts. Lalonde [31] mentioned another example for such a bigraph, attributed to Babai.

Let us remark finally that there are graphs G and G' which are non-isomorphic, but have isomorphic open neighborhood hypergraphs. For instance, let $V = \{1, 2, 3, 4, 5, 6\}$ and $E_1 = \{(1, 2), (2, 3), (3, 4), (4, 5), (5, 6), (6, 1)\}$ and $E_2 = \{(1, 3), (3, 5), (5, 1), (2, 4), (4, 6), (6, 4)\}$. Then $G_1 = (V, E_1)$ is a 6-cycle and $G_2 = (V, E_2)$ is the graph formed by two disjoint triangles. They are obviously not isomorphic, still $\mathcal{N}(G_1) = \mathcal{N}(G_2) = E_2$. See Catlin [13] for similar examples.

2.5 Transposable matrices

Another problem frequently related (and/or mixed up) with matrix symmetrization is matrix transposition. A matrix A is called *transposable* if it is possible to independently permute its rows and columns so that the resulting matrix A' is the transpose of A, i.e., $A' = A^T$. The *Matrix Transposition Problem* (or in short, problem MT) is to decide whether a given $0 - 1$ matrix A is transposable or not. To illustrate relation between transposability and symmetrizability, consider the matrix

$$A = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix}$$

which is symmetrizable. Indeed, switching rows 1 and 3 produces a symmetric matrix

$$A' = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix}.$$

If we apply the same permutation to the columns of A', then we obtain the transposed matrix of A:

$$A^T = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix}.$$

In fact, a symmetrizable matrix is always transposable, but not vice versa. Consider for instance the bigraph $B = (X, Y, E)$ given in Figure 5. As we observed in the previous section, this bigraph has an automorphism α which switches its parts X and Y, and therefore its adjacency matrix $A = A(B)$ is transposable. However, A is not symmetrizable by Fact 2, it has no switching and involutory automorphism.

Let us remark that such transposable but not symmetrizable matrices can also be obtained from partitionable graphs G, on $n = \alpha(G)\omega(G) + 1$ vertices. Minimal such example exists for $\alpha \times \omega = 5 \times 4$, i.e., for $n = 21$. Up to an isomorphism there are 8340 partitionable graphs on 21 vertices[1], and for two of them the corresponding $n \times n$ $0-1$ matrix is transposable but not symmetrizable. Transposability in fact means that the graph is isomorphic to its Tucker dual. We describe one of these examples in the appendix.

While matrix symmetrization is an NP-hard problem according to Corollary 1, the complexity of the Matrix Transposition Problem is still open.

Theorem 4 (Colbourn and McKay [18]) *The* Matrix Transposition Problem *and the* Graph Isomorphism Problem *are polynomial-time equivalent.*

This result was rediscovered in Bondy, Durán, Lin, and Szwarcfiter [10] in an equivalent form. A matrix A is called *quasi-symmetric* if the multi-set of its row vectors is the same as the multi-set of its column vectors. The problem of *matrix quasi-symmetrization* (or in short, problem MCQS) is to decide whether the columns of a given $0-1$ matrix A can be permuted such that the resulting matrix is quasi-symmetric.

Theorem 5 (Colbourn and McKay [18]) *The* Matrix Quasi-symmetrization Problem *is GI-complete.*

To see that problems MT and MCQS are in fact equivalent, let us first observe that we could consider six different problems, related to matrix transposition and quasi-symmetrization. Namely, given a matrix A we can consider the following questions:

MCQS Is it possible to permute the columns of A such that the resulting matrix is quasi-symmetric?

MRQS Is it possible to permute the rows of A such that the resulting matrix is quasi-symmetric?

MQS Is it possible to permute the rows and columns of A such that the resulting matrix is quasi-symmetric?

MCT Is it possible to permute the columns of A such that the resulting matrix is A^T?

MRT Is it possible to permute the rows of A such that the resulting matrix is A^T?

MT Is it possible to permute the rows and columns of A such that the resulting matrix is A^T?

[1] These computational results are due to Stefan Hougardy.

To clarify the relations between these problems, let us first recall a result from [10]:

Fact 15 (Bondy, Durán, Lin, and Szwarcfiter [10]) *For a matrix A, the following conditions are equivalent:*

(i) A is quasi-symmetric,

(ii) $A = A^T P$ for some permutation matrix P, and

(iii) $A = PA^T$ for some permutation matrix P.

Since checking if a matrix A is quasi-symmetric, can obviously be done in linear time in the size of A, the above claim implies readily the following result.

Fact 16 *Problems* MCT *and* MRT *can be solved in linear time.*

By definition, a matrix A is quasi-symmetric if and only if A^T is quasi-symmetric. Thus (ii) and (iii) of Fact 15 imply that the answer is "yes" in problems MRT and MCT if and only if A is quasi-symmetric, which then can be checked easily.

We claim that the remaining four problems are all equivalent.

Fact 17 *Problems* MCQS*,* MRQS*,* MQS *and* MT *are polynomial-time equivalent.*

Suppose that the answer for problem MCQS is "yes", so it is possible to permute columns of A in such a way that the resulting matrix is quasi-symmetric. It means that AQ is a quasi-symmetric matrix for some permutation matrix Q. Then we have by part (ii) of Fact 15 that $AQ = (AQ)^T P$ for some permutation matrix P, or equivalently that $A = Q^T A^T P Q^T$ (since $Q^{-1} = Q^T$ for a permutation matrix), implying that A is transposable. Consequently, the answer to problem MT is also "yes". Similarly, if the answer to either MRQS or MQS is "yes", then the answer to MT must also be "yes".

Now suppose that the answer to MT is "yes", that is $A^T = PAQ$ for some permutation matrices P and Q. We have

$$(AQ)^T = Q^{-1}A^T = Q^{-1}PAQ = (Q^{-1}P)(AQ),$$

and thus Fact 15 implies that AQ is a quasi-symmetric matrix, and consequently that the answer to MCQS is also "yes". Similarly,

$$(PA)^T = A^T P^{-1} = PAQP^{-1} = (PA)(QP^{-1}),$$

implying that PA is a quasi-symmetric matrix, and thus the answer to MRQS is also "yes". Obviously, any of these imply that the answer to MQS is also "yes".

Therefore, Theorem 4 and Theorem 5 are in fact equivalent, and imply the following result.

Corollary 4 *Problems* MCQS, MRQS, MQS, *and* MT *are all GI-complete.*

Let us close this subsection with a generalization of matrix transposition. Given two $0-1$ matrices A and B one may ask, whether B can be obtained from A with independent permutations of rows and columns of A. This *Matrix Equivalence Problem* (or in short problem ME) contains transposability, as a special case, when $B = A^T$. On the other hand, it is clear that the answer is "yes" if and only if the bigraphs corresponding to A and B are isomorphic, i.e., that problem ME is also GI-complete.

3 Neighborhood hypergraphs of bipartite graphs

In this section we consider neighborhood hypergraphs of bipartite graphs. The restricted versions of the realization problems ON, CN, MN and FN, in which neighborhood realization is sought within the family of bipartite graphs, will be denoted by ONB, CNB, MNB, and FNB, respectively. We shall represent hypergraphs by their bigraphs, as defined in Subsection 2.3, and let us recall from Facts 7, 8, 9, and 11 that problems ON, CN, and MN are respectively equivalent with finding a perfect semi-induced anti-matching, matching, and pairing in the bigraph representation of the given hypergraph. Analogously, problem FN is equivalent with finding a perfect semi-induced pairing in the bigraph of the given hypergraph, in which the pairs are edges or non-edges according to the given labelling **b**.

Let us recall that for a bigraph $B = (X, Y, E)$ we denoted by $\overline{B}$ its bi-complement. Let us further denote by $B^T = (Y, X, E)$ the bigraph obtained from B by flipping its parts (recall that a bigraph is a bipartite graph with a fixed order of its parts), called the *transpose* of B.

To be able to formulate the bipartite restricted versions of neighborhood realization problems, let us observe a few properties of perfect semi-induced pairings.

Lemma 1 *Given a bigraph $B = (X, Y, E)$ let us denote by $B_i = (X_i, Y_i, E_i)$, $i = 1, 2, \ldots, q$ its connected components, and assume that $S \subseteq X \times Y$ is a perfect semi-induced pairing of B. Then the following properties hold:*

(i) If for some $(x, y) \in S$ and index i we have $x \in X_i$ and $y \in Y_i$, then $S_i = S \cap (X_i \times Y_i)$ is a perfect semi-induced pairing of B_i.

(ii) If for some $(x, y) \in S$ and indices $i \neq j$ we have $x \in X_i$ and $y \in Y_j$, then $S_{ij} = S \cap ((X_i \times Y_j) \cup X_j \times Y_i))$ is a perfect semi-induced pairing of the bigraph $B_i \cup B_j$. In this case S_{ij} is an anti-matching, and furthermore, such a perfect semi-induced anti-matching S_{ij} exists if and only if the bigraphs B_i and B_j^T (or equivalently B_i^T and B_j) are isomorphic.

Let us first note that (ii) is the same claim as the first part of (ii), in case of $i = j$. Thus to prove them together, let us assume that i and j are (possibly equal) indices, and that for some $(x, y) \in S$ we have $x \in X_i$ and $y \in Y_j$. Then, the perfect semi-inducedness of S

implies that for all neighbors $y' \in N(x) \subseteq Y_i$ we must have a vertex $x' \in N(y) \subseteq X_j$ such that $(x', y') \in S$. Repeating then the same argument for all these (x', y') pairs, and so on, the connectedness of B_i and B_j will imply the claim.

For the second half of claim (ii), let us note that for every isomorphism $\alpha : X_i \cup Y_i \to X_j \cup Y_j$ between B_i and B_j the set $\{(x, \alpha(x)) \mid x \in X_i\} \cup \{(\alpha(y), y) \mid y \in Y_i\}$ is a perfect semi-induced anti-matching of $B_i \cup B_j$. Conversely, if $S_{ij} \subseteq ((X_i \times Y_j) \cup (X_j \times Y_i))$ is a perfect semi-induced anti-matching of $B_i \cup B_j$, then the mapping α defined by $\alpha(x) = y$ for $x \in X_i$, $(x, y) \in S_{ij}$, and $\alpha(y) = x$ for $y \in Y_i$ and $(x, y) \in S_{ij}$ is an isomorphism of B_i and B_j.

Let us note that finding a perfect semi-induced anti-matching as in (ii) above is equivalent with problem BI, according to Lemma 1, while finding a perfect semi-induced pairing in a connected bigraph (as in (i) above) is problem MN, which is NP-hard according to Corollary 3. We shall show in the sequel that this difference disappears when we consider bipartite neighborhood realizations.

Given a perfect semi-induced pairing $S \subseteq X \times Y$ of a bigraph $B = (X, Y, E)$, let us call it *partitionable* if $S = S^1 \cup S^2$ such that the vertices of both S^1 and S^2 induces a subgraph of B which does not contain any edge from the set $E \setminus S$. In other words, a partitionable perfect semi-induced anti-matching can be decomposed into two parts such that both parts induce an edgeless subgraph of the bigraph B, while a partitionable perfect semi-induced matching can be decomposed into two parts such that both parts are induced matchings of B.

Let us note that problems ONB, CNB, MNB and FNB are equivalent with finding in a given bigraph B respectively, a partitionable perfect semi-induced anti-matching, matching, pairing, or pairing with the prescribed edge/non-edge structure. Let us also note that the perfect semi-induced anti-matching S_{ij} in part (ii) of Lemma 1 is automatically partitionable, due the disjointness of the components B_i and B_j. Namely, $S^1_{ij} = S_{ij} \cap (X_i \times Y_j)$ and $S^2_{ij} = S_{ij} \cap (X_j \times Y_i)$ is such a partition.

Lemma 2 *A connected bigraph does not have a partitionable perfect semi-induced anti-matching.*

Assume indirectly that the bigraph $B = (X, Y, E)$ has a partitionable perfect semi-induced anti-matching S with partition $S = S^1 \cup S^2$. Let us denote by X^j and Y^j the vertex subsets of X and Y, respectively, covered by pairs of S^j, $j = 1, 2$. Then no edge of B can connect the non-empty sets $X^1 \cup Y^2$ and $X^2 \cup Y^1$, contradicting the connectivity of B.

Let us return to bipartite neighborhood realization of hypergraphs, and consider first problem ONB. In this case then input is a hypergraph $\mathcal{H}$ on vertex-set X, and the task is to find a bipartite graph $G = (U, V, \hat{E})$ such that $U \cup V = X$ and $\mathcal{N}(G) = \mathcal{H}$. If we represent the input by the bigraph of $\mathcal{H}$, $B_{\mathcal{H}} = B = (X, Y, E)$ where $Y = \mathcal{H}$ and E is defined as in Section 2.3, then problem ONB can equivalently be formulated as deciding the existence of a partitionable perfect semi-induced *anti-matching* M of B.

Let us point out that Aigner and Triesch [2] considered an easier looking version of this problem and proved that it is equivalent with graph isomorphism.

Theorem 6 (Aigner and Triesch [2]) *Given a hypergraph $\mathcal{H}$ on vertex-set X, and a partition $X = U \cup V$, the problem of deciding the existence of a bipartite graph $G = (U, V, \hat{E})$ for which $\mathcal{N}(G) = \mathcal{H}$ is polynomial-time equivalent with graph isomorphism.*

We show that the harder looking problem ONB is also equivalent computationally with graph isomorphism.

Theorem 7 *The* Bipartite Open Neighborhood Realization Problem *and the* Graph Isomorphism Problem *are polynomial-time equivalent.*

As we noted in Fact 13, problems GI and BI are equivalent, and thus it is enough to prove that ONB is polynomial-time equivalent with BI.

Let us first consider two bigraphs $B = (X, Y, E)$ and $B' = (X', Y', E')$ as an instance to problem BI, and consider the (disjoint) union of B^T and B', i.e., the bigraph $\hat{B} = (X' \cup Y, Y' \cup X, E \cup E')$, as an instance to problem ONB. Without any loss of generality we may assume that both B and B' are connected bipartite graphs. We claim that the hypergraph represented by $\hat{B}$ has a bipartite open neighborhood realization if and only if B and B' are isomorphic, as bigraphs, or in other words, solving problem ONB for $\hat{B}$ decides the isomorphism of B and B'. To see this claim, assume first that B and B' are isomorphic, and let $\alpha : X \cup Y \to Y \cup Y'$ be an isomorphism between them. Then, both $M^1 = \{(x, \alpha(x)) \mid x \in X\}$ and $M^2 = \{(y, \alpha(y)) \mid y \in Y\}$ are inducing edgeless subgraphs of $\hat{B}$ by its definition, and $M = M^1 \cup M^2$ is a perfect semi-induced anti-matching in $\hat{B}$.

Conversely, if $M = M^1 \cup M^2$ is a perfect semi-induced anti-matching of $\hat{B}$, such that both M^1 and M^2 induce edgeless subgraphs of $\hat{B}$, then by the connectivity of B and B' we must have $M^1 \subseteq X \times X'$ and $M^2 \subseteq \times Y'$. This and the perfect semi-inducedness of M then will imply that the mapping $\alpha : X \cup Y \to X' \cup Y'$ defined by $\alpha(x) = y$ for $(x, y) \in M$ is an isomorphism of bigraphs B and B'.

For the converse direction, suppose that we have an algorithm $\mathcal{A}$ to solve problem BI, and let us consider an input bigraph $B = (X, Y, E)$ for problem ONB. Let us first decompose B into connected subgraphs $B_i = (X_i, Y_i, E_i)$ for $i = 1, 2, \ldots, q$, which can be done in $O(|E|)$ time. According to Lemma 2 and part (ii) of Lemma 1, all partitionable perfect semi-induced anti-matchings of B formed as the (disjoint) union of perfect semi-induced anti-matchings $S_{ij} \subseteq ((X_i \times Y_j) \cup (X_j \times Y_i))$ for some pairs $1 \leq i < j \leq q$. Furthermore, the bigraphs B_i and B_j^T must then be isomorphic.

Let us then create an auxiliary graph $\hat{G}$ on vertex-set $V(\hat{G}) = \{1, 2, \ldots, q\}$ in which $(i, j) \in E(\hat{G})$ if and only if the bigraphs $B_i = (X_i, Y_i, E_i)$ and $B_j^T = (Y_j, X_j, E_j)$ are isomorphic. The edge-set $E(\hat{G})$ can be constructed by calling algorithm $\mathcal{A}$ $\binom{q}{2}$ times. Then, B has a partitionable perfect semi-induced anti-matching M, if and only if the auxiliary graph

$\hat{G}$ has a perfect matching. Let us also observe that $\hat{G}$ is the disjoint union of cliques and complete bipartite graphs, and consequently a perfect matching of it can be found in $O(q^2)$ time.

Before turning to problems CNB, MNB and FNB, let us analyze first the structure of partitionable perfect semi-induced pairings of connected bigraphs.

Let us denote by $N(U) = \{v \in X \cup Y \mid (u,v) \in E \quad \text{for some} \quad u \in U\}$ the set of neighboring vertices of a set $U \subseteq X \cup Y$ of the vertices of a bigraph $B = (X,Y,E)$, and introduce the notation $N^0(U) = U$ and $N^{k+1}(U) = N(N^k(U))$ for $k = 0,1,2,\ldots$. Note that we have $U \subseteq N^2(U) \subseteq N^4(U) \subseteq \cdots$, and $N^1(U) \subseteq N^3(U) \subseteq \cdots$.

Lemma 3 *Assume that $B = (X,Y,E)$ is a connected bigraph and $e = (x,y) \in E$ is an edge of B. Then, B has a partitionable perfect semi-induced pairing only if*

(i) $|X| = |Y|$,

(ii) for $k = |X| - 1$ the sets $X^1 = N^{2k}(x)$, $Y^1 = N^{2k}(y)$, $X^2 = N^{2k+1}(x)$ and $Y^2 = N^{2k+1}(y)$ form a partition of the vertex-set $X \cup Y$ of B, such that $X = X^1 \cup X^2$ and $Y = Y^1 \cup Y^2$, and

(iii) the edge-sets $M^1 = E \cap X^1 \times Y^1$ and $M^2 = E \cap X^2 \times Y^2$ are matchings (not necessarily perfect), induced respectively by the vertex-sets $X^1 \cup Y^1$ and $X^2 \cup Y^2$.

Furthermore, the above conditions can be tested and the sets X^j, Y^j and M^j for $j = 1,2$ can be constructed in $O(|E|)$ time.

Assume that S is a partitionable perfect semi-induced pairing of B such that $e \in S$, and denote by S^1 and S^2 the parts of its bipartition $S = S^1 \cup S^2$. Clearly, condition (i) is necessary for such a perfect pairing to exist.

Without any loss of generality, we can assume that $e \in S^1$, and let us denote by X^j and Y^j the subsets of X and Y covered by the pairs of S^j, for $j = 1,2$. Then, by the perfect semi-inducedness of S, we must have $X = X^1 \cup X^2$, $Y = Y^1 \cup Y^2$, and that the edge-sets $M^1 = S^1 \cap E$ and $M^2 = S^2 \cap E$ are induced matchings. Since $x \in X^1$ and $y \in Y^1$, the semi-inducedness of S implies that we must have $N^{2j}(x) \subseteq X^1$, $N^{2j+1}(x) \subseteq Y^2$, $N^{2j}(y) \subseteq Y^1$ and $N^{2j+1}(y) \subseteq X^2$ for all indices $j = 0,1,\ldots$. Since B is connected, we must have equalities in the above relations for some index $j \leq |X| - 1$, implying condition (ii). These also imply condition (iii), since S is assumed to be a partitionable perfect semi-induced pairing of B with partitions on $X^1 \cup Y^1$ and $X^2 \cup Y^2$.

Finally, the sets $N^{2k}(x)$, $N^{2k}(y)$, $N^{2k+1}(x)$, and $N^{2k+1}(y)$ can be determined by a simple labelling procedure starting first from x and second from y, running thus in linear time in the number of edges.

Let us next consider the case of closed neighborhoods in bipartite graphs. In problem CNB, the input is a hypergraph $\mathcal{H}$ on vertex-set X, and the task is to find a bipartite graph $G = (U,V,\hat{E})$ such that $U \cup V = X$ and $\mathcal{N}[G] = \mathcal{H}$. If we represent the input by the bigraph

$B = B_{\mathcal{H}} = (X, Y, E)$ of $\mathcal{H}$, where $Y = \mathcal{H}$ and E is defined as in Section 2.3, then the problem is to look for a partitionable perfect semi-induced matching M in B.

Theorem 8 *The* Bipartite Closed Neighborhood Realization Problem *can be solved in time $O(\delta m)$, where m is the number of edges, and δ is the minimum degree of a vertex in the bigraph representation of the input hypergraph.*

Given a hypergraph, let us consider its bigraph representation B. According to Lemma 1 any perfect semi-induced matching of B must induce perfect semi-induced matchings in each connected components of B. In other words, we can assume without any loss of generality that the input bigraph $B = (X, Y, E)$ is connected.

For an arbitrary edge $e = (x, y) \in E$, we can test if e may belong to a partitionable perfect semi-induced matching M with an induced partition $M = M^1 \cup M^2$, or not. Assume that $e \in M^\sigma$ for some $\sigma \in \{1, 2\}$, i.e. that $x \in X^\sigma$ and $y \in Y^\sigma$. Then by Lemma 3, there is a unique partition $X^1 \cup X^2 = X$ and $Y^1 \cup Y^2 = Y$ of the vertices, such that for all partitionable perfect semi-induced matchings containing e the sets $X^1 \cup Y^1$ and $X^2 \cup Y^2$ must induce matchings, which together form a perfect semi-induced matching of B. This can be checked in $O(|E|)$ time. Thus, by repeating the above for all edges incident with a fixed vertex x^* of B of minimum degree δ, in $O(\delta m)$ time we either arrive to a perfect semi-induced matching, or can conclude that B does not have one.

In fact the proof of Theorem 8 shows readily how to construct all closed bipartite neighborhood realizations of a given hypergraph.

Corollary 5 *It is possible to construct in $O(\delta m)$ time all closed bipartite neighborhood realizations of a given hypergraph with a connected bigraph representation B, where m is the number of edges, and δ is the minimum degree of a vertex of B.*

Note that a disconnected bigraph may have exponentially many different closed bipartite neighborhood realizations. However, we shall show in the next section that these realizations are isomorphic, i.e., up to isomorphism all hypergraphs have at most one bipartite neighborhood realization.

The two mixed problems MNB and FNB can be formulated analogously to ONB and CNB, by talking about pairings instead of matching. In these cases a perfect semi-induced pairing M may consist of edges and non-edges, and it must be partitionable into two parts $M = M^1 \cup M^2$ such that both M^1 and M^2 are induced pairings (i.e., the induced subgraph of B do not contain any other edges). Clearly, edges (respectively, non-edges) in M correspond to closed (respectively, open) neighborhoods in a realization.

Theorem 9 *The* Bipartite Mixed Neighborhood Realization Problem *is polynomial-time equivalent to the* Graph Isomorphism Problem.

Analogously to the proof of Theorem 7, let us show first that problem MNB is not easier than GI. For this let us consider two bigraphs $B = (X, Y, E)$ and $B' = (X', Y', E')$ as inputs for problem BI. We may assume without loss of generality that both B and B' are connected and that $|X| \neq |Y|$ and $|X'| \neq |Y'|$ (for the latter note that we can always add a vertex to B, connected to all vertices in X and a vertex to B' connected to all vertices in X' such that the resulting bigraphs will be isomorphic if and only if B and B' are isomorphic). Let us then consider the disjoint union of B' and B^T, i.e., $\hat{B} = (X' \cup Y, Y' \cup X, E' \cup E)$, as an instance for problem MNB. Now the assumptions $|X| \neq |Y|$ and $|X'| \neq |Y'|$ and Lemma 1 imply that all perfect semi-induced pairings S of $\hat{B}$ (if any) must be as in part (ii) of Lemma 1, consequently they are automatically partitionable, and the bigraphs B' and $(B^T)^T = B$ must be isomorphic. Furthermore, if B' and B are isomorphic, then the isomorphism between them will define a partitionable perfect semi-induced pairing of $\hat{B}$, just like it was constructed in the proof of Theorem 7.

For the converse direction, let us suppose that we have an algorithm $\mathcal{A}$ to solve the Bigraph Isomorphism Problem, and consider a bigraph $B = (X, Y, E)$ as input for problem MNB.

Let us decompose B into its connected components $B_i = (X_i, Y_i, E_i)$, $i = 1, 2, \ldots, q$. According to Lemma 1 every perfect semi-induced pairing of B must be a disjoint union of partitionable perfect semi-induced pairings of some of the B_i components and of some of the pairs $(X_i \cup Y_j, X_j \cup Y_i, E_i \cup E_j)$, $i \neq j$. By (ii) of Lemma 1 we also know that for such pairs $i \neq j$ we must have (X_i, Y_i, E_i) isomorphic to (Y_j, X_j, E_j). Thus, by calling algorithm $\mathcal{A}$ $\binom{q}{2}$ times, we can check which are those pairs $i \neq j$, for which the subgraph $B_i \cup B_j$ has such a perfect semi-induced pairing.

This is not enough however, because unlike in the case of Theorem 7, a partitionable perfect semi-induced pairing S may involve partitionable perfect semi-induced pairings of the individual connected components of B.

We can test each connected component for the existence of a partitionable perfect semi-induced pairing as follows. By Lemma 2 such a pairing, if exists, must include at least one edge (it cannot be an anti-matching). Thus, we can apply Lemma 3 for each edge $e \in E_i$. For a given edge $e \in E_i$, let us test in $O(|E|)$ time the conditions (i), (ii), and (iii) of Lemma 3. If these conditions do not hold, then we continue with the next edge of B_i. Otherwise, let us test if the matching $M = M^1 \cup M^2$ can be extended into a partitionable perfect semi-induced pairing by using algorithm $\mathcal{A}$, as follows. Let us assume that $M = \{(x_j, y_j) \mid j = 1, 2, \ldots, r\}$, delete these edges from B_i, add a vertex-set R to B_i of cardinality $|R| = 4|X| + r(r + 1)$, and add paths of length $2|X| + j$ starting at vertices x_j and y_j, $j = 1, 2, \ldots, r$ such that these paths are pairwise disjoint, and all but their starting vertices are from R. Let us denote by B_i^* the bigraph obtained in this way. Then any partitionable perfect semi-induced anti-matching of B_i^* must match vertices x_j and y_j, for $j = 1, 2, \ldots, r$. Furthermore, it is easy to see that B_i has a partitionable perfect semi-induced pairing if and only if B_i^* has a partitionable perfect semi-induced anti-matching. As we have shown in Theorem 7, this can be decided by algorithm $\mathcal{A}$. Repeating the above for all edges of B_i, either we construct a partitionable perfect semi-induced pairing of B_i, or we arrive to the conclusion that such a pairing does not exist.

Finally, consider an auxiliary graph $\hat{G}$ on vertex-set $V(\hat{G}) = \{1, 2, \ldots, q\}$. We include a loop $(i, i) \in E(\hat{G})$ if the connected component B_i has a partitionable perfect semi-induced pairing, and we also include edge $(i, j) \in E(\hat{G})$ for $i \neq j$ if the union of the components $B_i \cup B_j$ has a perfect semi-induced anti-matching (which is automatically partitionable). Then, the bigraph B has a partitionable perfect semi-induced pairing if and only if the vertices of $\hat{G}$ can be covered by pairwise disjoint edges (i.e., a matching and some loops), which can be tested in $O(q^3)$ time.

The proof of Theorem 9 can be adapted to Problem FNB.

Corollary 6 *The* Bipartite Fixed Neighborhood Realization Problem *is polynomial-time equivalent to the* Graph Isomorphism Problem*.*

The proof is essentially the same as that of Theorem 9. We just additionally have check that the partial matching constructed by Lemma 3 agrees with the given labels.

Our results have obvious algorithmic consequences, since polynomial-time algorithms for GI were constructed for many classes of graphs, including trees (Aho, Hopcroft, and Ullman), graphs with bounded vertex degrees (Luks [34], Furst, Hopcroft, and Luks [21] and Galil, Hoffmann, Luks, Schnorr, and Weber [22]), planar graphs (Hopcroft and Tarjan [26, 25], Gazit [23] and Ja'Ja' and Kosaraju [27]), interval graphs (Lueker and Booth [33] and Klein [28]), graphs with bounded eigenvalue multiplicities (Babai [4] and Fürer [20]), partial k-trees (Bodlaender [8, 9]), graphs with bounded average genus (Chen [14]), convex bipartite graphs (Chen [15]), circulant graphs (Codenotti, Gerace, and Vigna [16]), graphs of bounded distance width (Yamazaki, Bodlaender, de Fluiter, and Thilikos [43]), and others, see Hoffmann [24], Köbler, Schöning, and Torán [29], Babai [5], Campbell and Radford [12], Chen [15], Fukuda and Nakamori [19], Ponomarenko [38, 39], Rasin [40], etc.

4 Uniqueness

Here we consider the uniqueness problem for bipartite realizations. In general, a closed neighborhood hypergraph does not determine uniquely the underlying graph. As an example, we can consider the complements of the graphs at the end of Subsection 2.5, see Figure 6 that shows two non-isomorphic graphs with the same closed neighborhood hypergraph $\mathcal{H}$: $\mathcal{H} = \{1234, 1235, 1236, 1456, 2456, 3456\}$.

Here we show that an open/closed neighborhood realization is unique within bipartite graphs, if it exists. On the other hand, a mixed neighborhood realization may not be unique within bipartite graphs. Indeed, both C_4 and $2K_2$ are bipartite graphs, and $\mathcal{N}(C_4) = \mathcal{N}[2K_2]$.

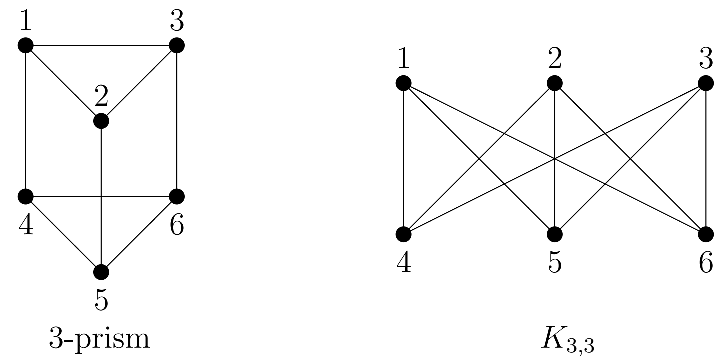


Figure 6: Non-isomorphic closed neighborhood realizations of the hypergraph $\mathcal{H} = \{\{1,2,3,4\},\{1,2,3,5\},\{1,2,3,6\}, \{1,4,5,6\},\{2,4,5,6\},\{3,4,5,6\}\}$

Theorem 10 *If a hypergraph $\mathcal{H} = \mathcal{N}(G)$ for a bipartite graph G, then G is unique within the family of bipartite graphs, up to isomorphism.*

Let $B = (X,Y,E)$ be the bigraph representation of $\mathcal{H}$. Without loss of generality we may assume that G is connected, and therefore B consists of two components, say $B_1 = (X_1, Y_1, E_1)$ and $B_2 = (X_2, Y_2, E_2)$ such that (X_1, Y_1, E_1) is isomorphic to (Y_2, X_2, E_2). Moreover, both B_1 and B_2 are isomorphic to G. Let a bipartite graph G' be another open neighborhood realization of $\mathcal{H}$. Since B has exactly two components, then G' must also be connected, and the above could be repeated for B', as well. Thus, we have $G \cong B_1 \cong G'$, implying the claim.

For closed neighborhood realizations, the uniqueness problem is more complicated.

Theorem 11 *If a hypergraph $\mathcal{H}$ has a bipartite graph G as its closed neighborhood realization, then G is unique within the family of bipartite graphs, up to isomorphism.*

Let $B = (X,Y,E)$ be the bigraph of $\mathcal{H}$. Without loss of generality we may assume that G is connected, and therefore B must also be connected. The graph G produces a perfect semi-induced matching M in B, and a partition $M^1 \cup M^2 = M$ such that each M^i is an induced matching. As before, the end-vertices of the edges of M^i in X (respectively, in Y) are denoted by X^i (respectively, Y^i), $i = 1,2$.

Let a bipartite graph G' be another closed neighborhood realization of $\mathcal{H}$. Accordingly, G' produces a perfect semi-induced matching M' in B, and a partition $M^{1'} \cup M^{2'} = M'$ such that each $M^{i'}$ is an induced matching. The end-vertices of the edges of $M^{i'}$ in X (respectively, in Y) are denoted by $X^{i'}$ (respectively, $Y^{i'}$), $i = 1,2$. Suppose that G' is not isomorphic to G.

Claim 1 *Every edge of M' either connects X^1 and Y^2 or X^2 and Y^1.*

If an edge e' of M' connects X^1 and Y^1, then $e = e'$ for some edge e of M, since the set $X^1 \cup Y^1$ induces the matching M^1. Connectedness of B implies that the edge $e = e'$ uniquely determines partitions of X and Y, as in Lemma 3, therefore $M = M'$ or $G \cong G'$, a contradiction. Similarly, e' cannot connect X^2 and Y^2.

Consider an arbitrary edge e' of M'. Using Claim 1 and symmetry, we may assume that $e' = (x_1, y_2)$ for some vertices $x_1 \in X^1$ and $y_2 \in Y^2$. Let $e_1 = (x_1, y_1)$ and $e_2 = (x_2, y_2)$ be the corresponding edges of M^1 and M^2, respectively.

Claim 2 *(i) If $e' \in M^{1'}$ then $(x_2, y_1) \in M^{2'}$.*

(ii) If $e' \in M^{2'}$ then $(x_2, y_1) \in M^{1'}$.

(i) First, (x_2, y_1) is an edge of B, since M is a semi-induced matching, and e' connects the end-vertices x_1 and y_2 of e_1 and e_2. Suppose that $e' \in M^{1'}$, but $(x_2, y_1) \notin M^{2'}$. Consider the edge f of M' which is incident to x_2. Clearly, $f = (x_2, y)$ for some vertex $y \in Y^1$ distinct from y_1. Since $e' \in M^{1'}$ and e' is incident to y_2, the vertex x_2 is in $X^{2'}$ and $f \in M^{2'}$. The edges e' and f are in different parts of the semi-induced matching M'. Their end-vertices x_2 and y_2 are adjacent. Hence the other two end-vertices, x_1 and y, must be adjacent. But it is impossible, since $x_1 \in X^1$, $y \in Y^1$ and the set $X^1 \cup Y^1$ induces the matching M^1.

(ii) It can be shown analogously to (i).

Claim 2 shows that all edges of $M \cup M'$ are contained in vertex-disjoint cycles C_4 so that two non-adjacent edges of each C_4, say e_1 and e_3, belong to M, and the two other edges, e_2 and e_4, are in M'. Moreover, e_1 and e_3 are in different parts of M, and e_2 and e_4 are in different parts of M', see Figure 7.

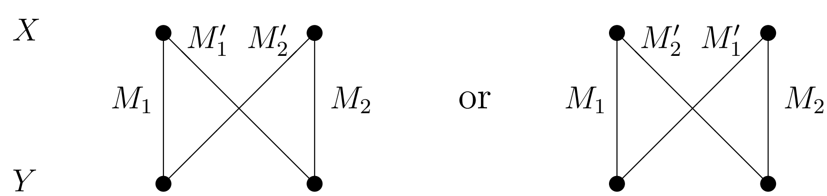


Figure 7:

Now we construct an isomorphism $V(G) \to V(G')$. The vertices of G (respectively, of G') bijectively correspond to the edges of M (respectively, of M'), therefore it is sufficient to construct a bijection $\psi : M \to M'$ such that edges $e_1 \in M^1$ and $e_2 \in M^2$ produce a C_4 in B if and only if $\psi(e_1) \in M^{1'}$ and $\psi(e_2) \in M^{2'}$ also produce a C_4. For an arbitrary edge $e_i \in M^i$, we define $\psi(e_i)$ to be the unique edge of $M^{i'}$ which has a common vertex with e_i. Claim 2 implies correctness of this definition.

Suppose that $a, b \in X^1$. Let us consider the corresponding edges (a, a') and (b, b') of M^1. They are contained in induced 4-cycles (a, d', d, a') and (b, c', c, b') such that (c, c'), $(d, d') \in M^2$, exactly one of (a, d'), (a', d) is in $M^{1'}$ while the other is in $M^{2'}$, and exactly one of (b, c'), (b', c) is in $M^{1'}$ while the other is in $M^{2'}$. By symmetry, let $(a, d') \in M^{1'}$ and $(a', d) \in M^{2'}$. Suppose that the set $\{a, a', c, c'\}$ induces C_4. The edge (b, c') cannot be in M^1, since $(a, d') \in M^{1'}$ and the vertices a and c' are adjacent. Hence $(b, c') \in M^{2'}$, $(b', c) \in M^{1'}$ and b is adjacent to d'. Now $(b, b') \in M^1$, $(d, d') \in M^2$ and b is adjacent to d'. It is then implied that b' is adjacent to d, see Figure 8. We have $\psi(a, a') = (a, d')$, $\psi(c, c') = (b, c')$, and both $\{a, a', c, c'\}$ and $\{a, d', b, c'\}$ induce C_4. The "only if" part is similar. Thus, ψ produces an isomorphism $V(G) \to V(G')$, completing the proof of the theorem.

The bigraph of Figure 8 has three bipartite perfect semi-induced matchings, they are isomorphic to C_4. However, the corresponding three bipartite closed neighborhood realizations are isomorphic.

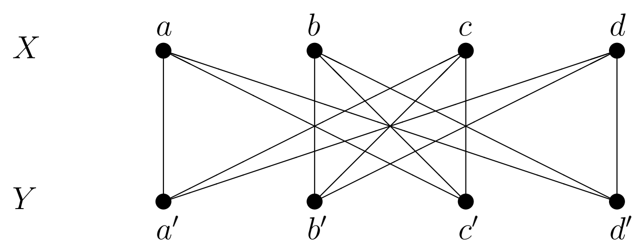


Figure 8:

Finally, we do not have a solution to the uniqueness problem within bipartite graphs for a hypergraph $\mathcal{H}$ and an arbitrary labelling $\mathbf{b} : V \to \{0, 1\}$. The following table summarizes our results on bipartite realizations and uniqueness within bipartite graphs.

PROBLEM	REALIZABILITY	UNIQUENESS
CNB	Polynomial-time solvable	Yes
ONB	Isomorphism-complete	Yes
MNB	Isomorphism-complete	No
FNB	Isomorphism-complete	Unknown

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