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ON DOMINATION IN CUBIC GRAPHS

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Abstract. Let $v(G)$ and $\gamma(G)$ denote the number of vertices and the domination number of a graph G, respectively, and let $\rho(G) = \gamma(G)/v(G)$. In 1996 B. Reed conjectured that if G is a cubic graph, then $\gamma(G) \le \lceil v(G)/3 \rceil$. In 2005 A. Kostochka and B. Stodolsky disproved this conjecture for cubic graphs of connectivity one and maintained that the conjecture may still be true for cubic 2-connected graphs. Their minimum counterexample C has 4 bridges, $v(C) = 60$, and $\gamma(C) = 21$. In this paper we disprove Reed's conjecture for cubic 2-connected graphs by providing a sequence $(R_k : k \ge 3)$ of cubic graphs of connectivity two with $\rho(R_k) = \frac{1}{3} + \frac{1}{60}$, where $v(R_{k+1}) > v(R_k) > v(R_3) = 60$ for $k \ge 4$, and so $\gamma(R_3) = 21$ and $\gamma(R_k) - \lceil v(R_k)/3 \rceil \to \infty$ with $k \to \infty$. We also provide a sequence of $(L_r : r \ge 1)$ of cubic graphs of connectivity one with $\rho(L_r) > \frac{1}{3} + \frac{1}{60}$. The minimum counterexample $L = L_1$ in this sequence is 'better' than C in the sense that L has 2 bridges while C has 4 bridges, $v(L) = 54 < 60 = v(C)$, and $\rho(L) = \frac{1}{3} + \frac{1}{54} > \frac{1}{3} + \frac{1}{60} = \rho(C)$. We also give a construction providing for every $r \in \{0, 1, 2\}$ infinitely many cubic cyclically 4-connected Hamiltonian graphs G_r such that $v(G_r) = r \bmod 3$, $r \in \{0, 2\} \Rightarrow \gamma(G_r) = \lceil v(G_r)/3 \rceil$, and $r = 1 \Rightarrow \gamma(G_r) = \lfloor v(G_r)/3 \rfloor$. At last we suggest a stronger conjecture on domination in cubic 3-connected graphs. **Keywords**: cubic graph, domination set, domination number, connectivity.

1 Introduction

We consider simple undirected graphs. All notions on graphs that are not defined here can be found in [5].

Let G be a graph, $V(G)$ and $E(G)$ the sets of vertices and edges of G, respectively, $v(G) = |V(G)|$ and $e(G) = |E(G)|$. Let $N(v, G)$ denote the set of vertices in G adjacent to a vertex v. Let $\kappa(G)$ denote the vertex connectivity of G. A vertex subset X of G is called *dominating* if every vertex in $G - X$ is adjacent to a vertex in X. Let $\gamma(G)$ denote the size of a minimum dominating set in G; $\gamma(G)$ is called the *dominating number* of G. We call $\rho(G) = \gamma(G)/v(G)$ the *dominating ratio* of G. A graph G is called *cubic* if every vertex of G has degree three.

Quite a few papers (e.g. [1, 2, 4, 7, 9, 10, 11, 12]), a survey paper [4], and a book [6] are devoted to various problems related to the domination number and its relations with some other parameters of graphs.

In 1996 [12], B. Reed proved that if the minimum vertex degree in G is at least three, then $\gamma(G) \le 3v(G)/8$ and conjectured that if in addition G is cubic, then $\gamma(G) \le \lceil v(G)/3 \rceil$. In 2005 [9] A. Kostochka and B. Stodolsky disproved Reed's conjecture for cubic graphs of connectivity one by presenting a sequence of cubic graphs G of connectivity one with $\rho(G) > \frac{1}{3} + \frac{1}{69}$ and maintained that Reed's conjecture may still be true for cubic 2–connected graphs. Let C and H be the minimum counterexample and another counterexample in [9], respectively. Then C has four bridges, $v(C) = 60$, and $\rho(C) = \frac{7}{20} = \frac{1}{3} + \frac{1}{60} > \rho(H) > \frac{1}{3} + \frac{1}{69}$.

In this paper we disprove Reed's conjecture for cubic 2-connected graphs by giving several constructions (see **2.5**, **2.8**, and **2.12**) that provide infinitely many counterexamples of connectivity two. One of our constructions (see **2.5**) provides a sequence $(R_k : k \ge 3)$ of cubic graphs of connectivity two with $\rho(R_k) = \frac{1}{3} + \frac{1}{60}$, where $v(R_{k+1}) > v(R_k) > v(R_3) = 60$ for $k \ge 4$, and so $\gamma(R_3) = 21$ and $\gamma(R_k) - v(R_k)/3 \to \infty$ with $k \to \infty$. Thus the violation $\gamma(G) - \lceil v(G)/3 \rceil$ of the inequality in the Reed's conjecture may be arbitrarily large. Graph R_3 is the minimum 2-connected counterexample we have found.

We also present (see **2.6**) a sequence $(L_r : r \ge 1)$ of 'better' counterexamples of connectivity one than those in [9]. Namely, L_1 has two bridges, $v(L_1) = 54$, $v(L_r) < v(L_{r+1})$, and $\rho(L_r) = \frac{7}{20} + \frac{1}{200r+340} \to \frac{7}{20}$ with $r \to \infty$, and so $\rho(C) = \frac{1}{3} + \frac{1}{60} < \rho R_k) < \rho(L_{r+1}) < \rho(L_1) = \frac{1}{3} + \frac{1}{54}$. Therefore every counterexample in this construction has larger domination ratio than every counterexample in [9]. Moreover, L_1 has less vertices, larger domination ratio, and less bridges than C.

We give constructions (see **3.1** and **3.3**) that for every $r \in \{0, 1, 2\}$ provide infinitely many cubic 3-connected and cyclically 4-connected graphs G_r such that $v(G_r) = r \bmod 3$, $r \in \{0, 2\} \Rightarrow \gamma(G_r) = \lceil v(G_r)/3 \rceil$, and $r = 1 \Rightarrow \gamma(G_r) = \lfloor v(G_r)/3 \rfloor$.

At last we suggest a stronger conjecture (see **3.5**) on domination in cubic 3-connected graphs.

The results of this paper were discussed in the Department of Mathematics, UPR, in February 2006.

2 Constructions of counterexamples

We start with the following easy observation.

2.1 *Let G be a graph, H an induced subgraph of G, and X the set of vertices in H adjacent to some vertices in $G - V(H)$. Suppose that $\gamma(H - V) = \gamma(H)$ for every $V \subseteq X$. If D is a dominating set of G, then $|D \cap V(H)| \geq \gamma(H)$.*

Let H be a graph, $\{h_1, h_2\} \subseteq V(H)$, and $\dot{H} = (H, \{h_1, h_2\})$. Let G and H be disjoint graphs and $e = v_1v_2 \in E(G)$. If G' is obtained from $G - e$ and H by identifying h_1 with v_1 and h_2 with v_2, then we say that *G' is obtained from G by replacing edge e by $\dot{H}$*.

Let U be a graph, $\{u_1, u_2, u_3\} \subseteq V(U)$, and $\dot{U} = (U, \{u_1, u_2, u_3\})$. Let G and U be disjoint graphs, $v \in V(G)$, and $N(v, G) = \{v_1, v_2, v_3\}$. If G' is obtained from $G - v$ and U by adding three new edges u_iv_i, $i \in \{1, 2, 3\}$, then we say that *G' is obtained from G by replacing vertex v by $\dot{U}$*.

Let $(X, \{x_1, x_2\}$ and $(Y, \{y_1, y_2\})$ be two disjoint copies of $(H, \{h_1, h_2\})$ and let F' (F'') be obtained from $X \cup Y \cup \{x_1y_1, x_2y_2\}$ by subdividing edge x_1y_1 with a new vertex z_1 (respectively, by subdividing each edge x_iy_i with a new vertex z_i, $i \in \{1, 2\}$).

Let F_2 be the graph obtained from $F'' \cup z_1z_2$ by subdividing two edges x_1z_1 and y_1z_1 with new vertices x and y, respectively. Let F_3 be the graph obtained from F_2 by subdividing edge z_1z_2 with a new vertex z. Let $\mathcal{T}_1(\dot{H}) = (F', z_1)$, $\mathcal{T}_2(\dot{H}) = (F'', \{z_1, z_2\}))$, $\mathcal{F}_2(\dot{H}) = (F_2, \{x, y\}))$, and $\mathcal{F}_3(\dot{H}) = (F_3, \{x, y, z\}))$.

Let e_1, e_2, and e_3 be three edges in $K_{3,3}$ incident to the same vertex. Let A be the graph obtained from $K_{3,3}$ by subdividing e_i with a new vertex a_i for every $i \in \{1, 2\}$. Similarly, let B be the graph obtained from $K_{3,3}$ by subdividing e_i with a new vertex b_i for every $i \in \{1, 2, 3\}$.

Let $\dot{A} = (A, \{a_1, a_2\})$, $\dot{B} = (B, \{b_1, b_2, b_3\})$, $\mathcal{T}_1(\dot{A}) = \dot{S} = (S, s)$, $\mathcal{T}_2(\dot{A}) = \dot{T} = (T, \{t_1, t_2\})$, $\mathcal{F}_2(\dot{A}) = \dot{P} = (P, \{p_1, p_2\})$, and $\mathcal{F}_3(\dot{A}) = \dot{Q} = (Q, \{q_1, q_2, q_3\})$.

It is easy to see the following.

2.2 [9] *$v(A) = 8$, $\gamma(A) = \gamma(A - a_i) = 3$ for every $i \in \{1, 2\}$, and $\gamma(A - \{a_1, a_2\}) = 2$.*

It is also easy to see the following.

2.3 *$v(B) = 9$ and $\gamma(B - V) = 3$ for every $V \subseteq \{b_1, b_2, b_3\}$.*

From **2.2** we have:

2.4 *Obviously $v(S) = 17$, $v(T) = 18$, $v(P) = 20$, and $v(Q) = 21$. Moreover,*
$(a1)$ $\gamma(S) = \gamma(S - s) = \gamma(T) = \gamma(T - t_1) = \gamma(T - t_2) = \gamma(T - \{t_1, t_2\}) = 6$,
$(a2)$ $\gamma(P) = \gamma(P - p_1) = \gamma(P - p_2) = \gamma(P - \{p_1, p_2\}) = 7$, and
$(a3)$ $\gamma(Q - V) = 7$ for every $V \subseteq \{q_1, q_2, q_3\}$.

Let R_k be a graph obtained from a $2k$-vertex cycle $(v_0, \ldots, v_{2k-1}, v_{2k})$ with $v_{2k} = v_0$ by replacing each edge $v_{2i}v_{2i+1}$ by a copy $(P_i, \{p^i_1, p^i_2\}$ of $(P, \{p_1, p_2\})$.

2.5 *Let* $k \ge 3$*. Then* R_k *is a cubic graph,* $\kappa(R_k) = 2$*,* $v(R_k) = 20k$*, and* $\gamma(R_k) = 7k$*, and so* $\rho(R_k) = \frac{7}{20} = \frac{1}{3} + \frac{1}{60}$ *and* $\gamma(R_k) - v(R_k)/3 = k/3 \to \infty$ *with* $k \to \infty$*.*

Proof Since $v(P) = 20$, clearly $v(R_k) = 20k$. By **2.1** and **2.4** $(a2)$, $\gamma(R_k) = 7k$.

Let T_r be obtained from a $2r$-vertex path $(v_1, \ldots, v_{2r})$ by replacing each edge $v_{2i-1}v_{2i}$ by a copy $(P_i, \{p^i_1, p^i_2\}$ of $(P, \{p_1, p_2\})$. Let $L_r = T_r \cup S_1 \cup S_2 \cup \{s_1v_1, s_2v_{2r}\}$, where (S_1, s_1) and (S_2, s_2) are two copies of (S, s) and T_r, S_1, and S_2 are disjoint.

From **2.1** and **2.4** $(a1)$,$(a2)$ we have:

2.6 *Let* $r \ge 1$*. Then* L_r *is a cubic graph,* L_r *has exactly* $r + 1$ *bridges (and so* $\kappa(L_r) = 1$*)* $v(L_r) = 20r + 34$*, and* $\gamma(L_r) = 7r + 12$*, and so* $\rho(L_r) = \frac{7}{20} + \frac{1}{200r+340} \to \frac{7}{20}$ *with* $r \to \infty$ *and* $\rho(C) = \frac{1}{3} + \frac{1}{60} < \rho(L_{r+1}) < \rho(L_r) \le \rho(L_1) = \frac{1}{3} + \frac{1}{54}$*.*

Let P' be the graph obtained from P by adding two new vertices p'_1, p'_2 and two new edges $p_1p'_1$, $p_2p'_2$ and let $\dot{P}' = (P', \{p'_1, p'_2\})$. Let $G(P)$ be a graph obtained from a graph G by replacing each edge e by a copy $\dot{P}'_e$ of $\dot{P}'$.

From **2.1** and **2.4** $(a2)$ we have:

2.7 *Let* G *be a graph. If* $\kappa(G) = 1$*, then also* $\kappa(G(P)) = 1$*. If* G *is 2–connected, then* $\kappa(G(P)) = 2$*. Also* $v(G(P)) = v(G) + 20e(G)$ *and* $\gamma(G(P)) = 7e(G)$*.*

From **2.7** we have:

2.8 *Let* G *be a connected cubic graph with* $2k$ *vertices and possible parallel edges. Then* $v(G(P)) = 62k$*,* $\gamma(G(P)) = 21k$*, and so* $\rho(G(P)) = \frac{1}{3} + \frac{1}{186}$*. If* $\kappa(G) = 1$*, then also* $\kappa(G(P)) = 1$*. If* G *is 2–connected, then* $\kappa(G(P)) = 2$*.*

Given a cubic graph G, let $G(P, B)$ be a graph obtained from G by replacing each vertex v of G by a copy $\dot{B}_v$ of $\dot{B}$ and each edge e of G by a copy $\dot{P}'_e$ of $\dot{P}'$.

From **2.1**, **2.3**, and **2.4** $(a2)$ we have:

2.9 *Let* G *be a cubic graph with possible parallel edges and with* $2k$ *vertices. Let* $G' = G(P, B)$*. Then* $v(G') = 78k$*,* $\gamma(G') = 27k$*, and so* $\rho(G') = \frac{1}{3} + \frac{1}{78}$*. If* $\kappa(G) = 1$*, then also* $\kappa(G') = 1$*. If* G *is 2–connected, then* $\kappa(G') = 2$*.*

Let us define $\dot{P}^i$ recursively. Let $\dot{P}^1 = \dot{P}$ and $\dot{P}^{i+1} = \mathcal{F}_2(\dot{P}^i)$. Let $\mathcal{P} = \{\dot{P}^i : i \ge 1\}$.

2.10 *Let* $\dot{P}^i = (P^i, \{p_1, p_2\})$*,* $i \ge 1$*. Then*
(a) $\gamma(P^{i+1}) = 2\gamma(P^i) + 1$ *and* $\gamma(P^i) = \gamma(P^i - p_1) = \gamma(P^i - p_2) = \gamma(P^i - \{p_1, p_2\})$ *and*
(b) $v(P^i) = 2^{i+2}3 - 4$ *and* $\gamma(P^i) = 2^{i+2} - 1$*, and so* $\rho(P^i) = \frac{1}{3} + \frac{1}{12(2^i3-1)}$*.*

Proof (uses **2.4**). Claim (a) can be easily proved by induction using **2.4**. We prove (b). Obviously $v(P^1) = 20$ and by **2.4**, $\gamma(P^1) = 7$. By the definition of $\dot{P}^i$, $v(P^{i+1}) = 2v(P^i) + 4$. Now (b) follows from the above recursions for $v(P^{i+1})$ and $\gamma(P^{i+1})$.

Let $\dot{Q}^i = \mathcal{F}_3(P^i)$ and $\mathcal{Q} = \{\dot{Q}^i : i \geq 1\}$. ¿From **2.4** $(a3)$ and **2.10** we have:

2.11 *Let* $\dot{Q}^i = (Q^i, \{q_1, q_2, q_3\})$. *Then*
(a) $\gamma(Q^i) = \gamma(Q^i - V)$ *for every* $V \subseteq \{q_1, q_2, q_3\}$ *and*
(b) $v(Q^i) = v(P^i) + 1 = 3(2^{i+2} - 1)$ *and* $\gamma(Q^i) = 2^{i+2} - 1$, *and so* $v(Q^i) = 3\gamma(Q^i)$.

From **2.10** and **2.11** we have:

2.12 *Let* G *be either* R_k *or* L_r *or* $H(P)$ *or* $H(P, B)$ *for some connected cubic graph* H. *Let* G' *be obtained from* G *by replacing some copies of* $\dot{P}$ *and/or* $\dot{Q}$ *in* G *by copies of some members of* $\mathcal{P}$ *and some copies of* $\dot{B}$ *by some copies of members of* $\mathcal{Q}$. *Then* G' *is a cubic graph,* $\gamma(G') > \lceil v(G')/3 \rceil$, *and if* G *is 2-connected, then* G' *is also 2-connected.*

3 Cubic 3-connected graphs G with $\gamma(G) = \lceil v(G)/3 \rceil$

Let G be a cubic graph and $G[\dot{B}]$ be a graph obtained from G by replacing every vertex v in G by a copy $\dot{B}_v$ of $\dot{B}$. Let K_2^3 be the graph with two vertices and three parallel edges. We assume that K_2^3 is 3-connected by definition.

From **2.3** we have:

3.1 *Let* G *be a cubic graph with possible parallel edges and* $G' = G[\dot{B}]$. *Then* $v(G') = 9v(G)$, $\gamma(G') = 3v(G)$, $\kappa(G') = \kappa(G)$, *and* G' *is not cyclically 4-connected.*

The minimum cubic 3-connected graph provided by the above construction is $K_2^3[B]$. Obviously $v(K_2^3[B]) = 18$, $\gamma(K_2^3[B]) = 6$, and $K_2^3[B]$ is obtained from two disjoint copies $(B', \{b'_1, b'_2, b'_3\})$ and $(B'', \{b''_1, b''_2, b''_3\})$ of $(B, \{b_1, b_2, b_3\})$ by adding three new edges $b'_i b''_i$, $i \in \{1, 2, 3\}$.

Let P_7^2 be the Petersen (7,2)-graph. Obviously P_7^2 is a cubic cyclically 4-connected graph with 14 vertices. It can be checked that $\gamma(P_7^2) = 5 = \lceil v(P_7^2)/3 \rceil$ and P_7^2 is Hamiltonian.

Below (see **3.3**) we give constructions that for every $r \in \{0, 1, 2\}$ provide infinitely many cubic 3-connected and cyclically 4-connected graphs G_r such that $v(G_r) = r \bmod 3$, $r \in \{0, 2\} \Rightarrow \gamma(G_r) = \lceil v(G_r)/3 \rceil$, and $r = 1 \Rightarrow \gamma(G_r) = \lfloor v(G_r)/3 \rfloor$.

Let S be square $(t_1 s_1 t_2 s_2 t_1)$, P be 4-vertex path $P = (q_1 p_1 p_2 q_2)$. Let W be the graph obtained from disjoint S and P by identifying q_1 with s_1 and q_2 with s_2. Obviously $T = \{t_1, t_2, p_1, p_2\}$ is the set of degree two vertices in W.

It is easy to prove the following.

3.2 *Let $\dot{W} = (W, T)$ and $V \subseteq T$. Then $\gamma(W - V) = 1$ if $V = \{p_1, p_2, t_i\}$ for some $i \in \{1, 2\}$, and $\gamma(W - V) = 2$, otherwise.*

Let $k \geq 1$ be an integer, $X = (x_0 \cdots x_{3k})$ and $Y = (y_0 \cdots y_{3k})$ be two disjoint cycles, and $M_k^2 = X \cup Y \cup \{x_0y_0, x_1y_1\} \cup \{x_iy_i : 1 \leq i \leq 3k-2, i = 1 \bmod 3\} \cup \{x_iy_{i+1}, x_{i+1}y_i : 2 \leq i \leq 3k-1, i = 1 \bmod 3\}$. Let $M_k^0 = (M_k^2 - \{x_0, y_0\}) \cup \{x_1x_{3k}, y_1y_{3k}\}$, and $M_k^1 = (M_k^2 - \{x_0, y_0, x_1, y_1\}) \cup \{x_2x_{3k}, y_2y_{3k}\}$. Obviously $v(M_k^i) = i \bmod 3$.

3.3 *Each M_k^i is a cubic cyclically 4-connected Hamiltonian graph and*
$(a0)$ $v(M_k^0) = 6k$ and $\gamma(M_k^0) = 2k$,
$(a1)$ $v(M_k^1) = 6k - 2$ and $\gamma(M_k^1) = 2k - 1$, and
$(a2)$ $v(M_k^2) = 6k + 2$ and $\gamma(M_k^0) = 2k + 1$.

Proof (uses **3.2**). It is easy to see that each M_k^i, $i \in \{0, 1, 2\}$, is cyclically 4-connected and has a Hamiltonian cycle. We prove $(a2)$. Claims $(a0)$ and $(a1)$ can be proved similarly. Obviously $v(M_k^2) = 6k + 2$.

Since M_k^2 is Hamiltonian, it has a dominating set with $2k+1$ vertices, and so $\gamma(M_k^2) \leq 2k+1$. Thus it is sufficient to show that if D is a dominating set in M_k^2, then $|D| \geq 2k+1$. We prove our claim by induction on k. It is easy to check that our claim is true for $k \in \{1, 2\}$. So let $k \geq 3$.

Let R_{3i+r} be the subgraph of M_k^2 induced by the vertex subset $\{x_{3i+r}, x_{3i+r+1}, x_{3i+r+2}, y_{3i+r}, y_{3i+r+1}, y_{3i+r+2}\}$, where $i \in \{0, \ldots k-1\}$ and $r \in \{1, 2\}$. Then each R_{3i+r} is isomorphic to W in **3.2** with $\{x_{3i+r+1}, y_{3i+r+1}\}$ corresponding to $\{s_1, s_2\}$, $V(R_{3i+r}) \cap V(R_{3j+r}) = \emptyset$ for $i \neq j$, $V(R_{3i+r}) \cap \{x_{r-1}, y_{r-1}\} = \emptyset$, and $V(M_k^2) = \{x_{r-1}, y_{r-1}\} \cup \{V(R_{3i+r}) : i \in \{0, \ldots, k-1\}\}$. Let $M = M_k^2$ and $R = R_{3i+1}$.

(**p1**) Suppose that M has a minimum dominating set containing $Z = \{x_{3i+r}, y_{3i+r}\}$ for some $i \in \{0, \ldots, k-1\}$ and $r \in \{2, 3\}$. By symmetry of M, we can assume that $r = 2$. Obviously Z is a dominating set of R and every degree two vertex in R is adjacent to exactly one vertex in $M - R$. Therefore $\gamma(M) = \gamma(M - R) + |Z|$. Let $M' = (M - R) \cup \{x_{3i}x_{3i+4}, y_{3i}y_{3i+4}\}$. Then $\gamma(M - R) \geq \gamma(M')$. By the induction hypothesis, $\gamma(M') = 2k - 1$. Thus $\gamma(M) = \gamma(M - R) + |Z| \geq \gamma(M') + |Z| = (2k-1) + 2 = 2k + 1$.

(**p2**) Suppose that M has a minimum dominating set D containing one of the sets $\{x_{3i+r}, y_{3i+r+2}\}$, $\{y_{3i+r}, x_{3i+r+2}\}$, $\{y_{3i+r}, y_{3i+r+2}\}$, $\{y_{3i+r}, y_{3i+r+2}\}$ for some $i \in \{0, \ldots, k-1\}$ and $r \in \{1, 2\}$. By symmetry of M, we can assume that D contains $\{x_{3i+1}, y_{3i+3}\}$ from $V(R)$. If there is $z \in D \cap \{x_{3i+2}, y_{3i+2}\}$, then $D - z + x_{3i+3}$ is also a minimum dominating set of M. Therefore we are done by **(p1)**. If $y_{3i+1} \in D$, then $D - y_{3i+1} + x_{3i}$ is also a minimum dominating set of M. Thus we can assume that $D \cap V((R) = \{x_{3i+1}, y_{3i+3}\}$. Then $D' = D \setminus \{x_{3i+1}, y_{3i+3}\}$ dominates $V(M) \setminus (\{x_{3i}, y_{3i+4}\} \cup V(R - x_{3i+3}))$. Since D' dominates x_{3i+3}, clearly $x_{3i+4} \in D'$. Let $M' = (M - R) \cup \{x_{3i}x_{3i+4}, y_{3i}y_{3i+4}\}$. Then M' is isomorphic to M_{k-1}^2 and since x_{3i+4} dominates $\{x_{3i}, y_{3i+4}\}$, clearly D' dominates M'. Therefore $|D'| \geq \gamma(M')$. By the induction hypothesis, $\gamma(M') = 2k-1$. Therefore $2k+1 \geq |D| = |D'| + |\{x_{3i+1}, y_{3i+3}\}| = (2k-1) + 2 = 2k + 1$.

(**p3**) Suppose that M has a minimum dominating set D containing one of the sets $\{x_{3i+r}, y_{3i+r+1}\}$, $\{y_{3i+r}, x_{3i+r+1}\}$ for some $i \in \{0, \dots, k-1\}$ and $r \in \{0, 1\}$. By symmetry of M, we can assume that D contains $\{x_{3i+1}, y_{3i+2}\}$ from $V(R)$. By (**p1**) and (**p2**), we can assume that $D \cap \{x_{3i+2}, x_{3i+3}, x_{3i+4}, y_{3i+3}, y_{3i+4}\} = \emptyset$. Therefore $\{x_{3i+5}, y_{3i+5}\} \subseteq D$. If $x_{3i+5}y_{3i+5} \notin E(M)$, then we are done by (**p1**). Therefore $x_{3i+5}y_{3i+5} \in E(M)$. If $y_{3i+1} \in D$, then $D - y_{3i+1} + y_{3i}$ is also a minimum dominating set of M. Thus we can assume that $D \cap V(R) = \{x_{3i+1}, y_{3i+2}\}$. Then $D' = D \setminus \{x_{3i+1}, y_{3i+2}\}$ dominates $V(M - x_{3i}) \setminus V(R)$. Let M' be as in (**p2**). If $D' \cap \{x_{3i-1}, x_{3i}, y_{3i-1}\} \neq \emptyset$, then D' dominates M', and we are done by the arguments similar to those in (**p2**). If $D' \cap \{x_{3i-1}, x_{3i}, y_{3i-1}\} = \emptyset$, then $y_{3i} \in D'$. By (**p2**), we can assume that $D' \cap \{x_{3i-2}, y_{3i-2}\} = \emptyset$. Then $\{x_{3i-3}, y_{3i-3}\} \subseteq D'$. Since $k \geq 3$, clearly $x_{3i-3}y_{3i-3} \notin E(M)$. Therefore we are done by (**p1**).

(**p4**) Suppose that M has a minimum dominating set D that has exactly one vertex z in R_{3i+r} for some $i \in \{0, \dots, k-1\}$ and $r \in \{1, 2\}$. By symmetry of M, we can assume that $r = 1$. Then by **3.2**, $z \in \{x_{3i+3}, y_{3i+3}\}$. By symmetry of M, we can assume that $z = x_{3i+3}$, and so by **3.2**, $y_{3i+4} \in D$. Since $\{x_{3i+3}, y_{3i+4}\} \subseteq D$, we are done by (**p3**).

(**p5**) Now suppose that for some $s \in \{0, \dots, k-1\}$,

($d1$) a minimum dominating set D contains exactly one of the four sets $\{x_{3s+2}, x_{3s+3}\}$, $\{x_{3s+2}, y_{3s+3}\}$, $\{x_{3s+3}, y_{3s+2}\}$, and $\{y_{3s+2}, y_{3s+3}\}$.

We can also assume by (**p1**) and (**p2**) that

($d2$) $D \cap \{x_{3s+1}, y_{3s+1}, x_{3s+4}, y_{3s+4}\} = \emptyset$.

Then ($d1$) and ($d2$) hold for every $i \in \{0, \dots, k-1\}$. Hence $D \cap \{x_0, x_1, y_0, y_1\} \neq \emptyset$ because D is a dominating set of G. Thus $|D| \geq 2k + 1$.

Let $N^r_k(i) = (M^2_k - \{x_{3i+1}x_{3i+2}, y_{3i+1}y_{3i}) \cup \{x_{3i+1}y_{3i}, y_{3i+1}x_{3i+2}\}$, where $1 < i < k$ and $r \in \{0, 1, 2\}$. One can also prove the following.

3.4 *Each $N^r_k(i)$ is a cubic 3-connected (but not cyclically 4-connected) Hamiltonian graph and*

($a0$) $v(N^0_k(i)) = 6k$ *and* $\gamma(N^0_k(i)) = 2k$,

($a1$) $v(N^1_k(i)) = 6k - 2$ *and* $\gamma(N^1_k(i)) = 2k - 1$, *and*

($a2$) $v(N^2_k(i)) = 6k + 2$ *and* $\gamma(N^0_k(i)) = 2k + 1$.

We believe that the following is true.

3.5 Conjecture *Let G be a cubic 3-connected graph. If $v(G) \neq 1 \bmod 3$, then $\gamma(G) \leq \lceil v(G)/3 \rceil$. If $v(G) = 1 \bmod 3$, then $\gamma(G) \leq \lfloor v(G)/3 \rfloor$.*

From **3.1**, **3.3**, and **3.4** it follows that Conjecture **3.5** is best possible for both 3-connected and cyclically 4-connected cubic graphs.

From the results in [8] it follows that if G is a Hamiltonian cubic graph with $v(G) = 1 \bmod 3$, then $\gamma(G) \leq \lfloor v(G)/3 \rfloor$. Therefore Conjecture **3.5** is true for Hamiltonian cubic graphs.

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